Research Article

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On intersections of two non-incident subgroups of finite p -groups

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Abstract: In this paper, we investigate finite p -groups G such that whenever $A, B < G$ are non-incident, then $A \cap B \trianglelefteq \langle A, B \rangle$. This partially solves a problem proposed by Y. Berkovich.

Keywords: finite p -group, metacyclic p -group, commutator subgroup

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1 Introduction

All groups considered in this paper are finite.

Let G be a group. If H and K are normal subgroups of G , then $H \cap K \trianglelefteq H$ and $H \cap K \trianglelefteq K$. It is interesting to investigate the structure of a group G by using the intersection of two subgroups of finite p -groups. For example, Janko in [1] determined completely the structure of 2-groups G in which any two distinct maximal abelian subgroups have cyclic intersection. Janko in [2] also classified non-abelian 2-groups such that any two distinct minimal non-abelian subgroups have cyclic intersection. We also remark that many authors investigate the structure of groups by using information about the intersection of the subgroups, for example [3,4].

In fact, it is easy to see that G is a Dedekind group if $H \cap K \trianglelefteq H$ and $H \cap K \trianglelefteq K$ for any two subgroups H and K of G . Now it is natural to ask the following question:

What can be said about the structure of a p -group G in which $H \cap K \trianglelefteq H$ and $H \cap K \trianglelefteq K$ for any two non-incident subgroups H and K of G ?

The question above was first posed by Berkovich (see Problem 1342 in [5] and Problem 1546 in [6]). In this paper, we hope to investigate the structure of a p -group G in which $H \cap K \trianglelefteq H$ and $H \cap K \trianglelefteq K$ for any two non-incident subgroups H and K of G (this means that $H \not\leq K$ and $K \not\leq H$). For convenience, we call this kind of p -groups Q - p -groups. In this paper, we get the following main results:

Main result 1.1. Suppose that G is a non-abelian 2-group of order 2^n and $d(G) = 2$. Then G is a Q -2-group if and only if G is one of the following groups:

- (1) G is a minimal non-abelian 2-group;
- (2) 2-groups of maximal class;
- (3) $\langle a, b | a^8 = b^{2^{n-3}} = 1, [a, b] = b^2 \rangle, n \geq 6$;
- (4) $\langle a, b | a^8 = b^{2^{n-3}} = 1, [a, b] = b^{2^{n-4}+2} \rangle, n \geq 6$;
- (5) $\langle a, b | a^4 = b^{2^{n-2}} = 1, [a, b] = b^2 \rangle, n \geq 5$;

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- (6) $\langle a, b | a^4 = b^{2^{n-2}} = 1, [a, b] = b^{2^{n-3}+2} \rangle, n \geq 5$;
 (7) $\langle a, b | a^8 = b^{2^{n-2}} = 1, a^4 = b^{2^{n-3}}, [a, b] = b^2 \rangle, n \geq 5$;
 (8) $\langle a, b, c | a^8 = c^2 = 1, a^4 = b^2, [a, b] = c, [c, a] = a^4, [c, b] = 1 \rangle$;
 (9) $\langle a, b, c | a^8 = 1, a^4 = b^2 = c^2, [b, a] = c, [c, a] = 1, [c, b] = c^2 \rangle$;
 (10) $\langle a, b, c | a^{2^{n-2}} = b^2 = 1, c^2 = a^{2^{n-3}}, [b, a] = c, [c, a] = 1, [c, b] = c^2 \rangle$.

Main result 1.2. Let G be a Q -2-group and $d(G) \geq 3$. Then $\exp(G') \leq 2$ and $c(G) \leq 2$.

Main result 1.3. Let G be a Q -2-group with $d(G) \geq 4$ and $|G| \geq 2^7$. Then G is Dedekind.

Main result 1.4. Let G be a Q - p -group and $p > 2$. Then $\exp(G') \leq p$ and $c(G) \leq 3$.

Main result 1.5. Let G be a Q - p -group with $d(G) \geq 4$ and $p > 2$. Then G is abelian.

2 Preliminaries

First we introduce some notions and notations.

Let G be a group. If every subgroup of G is normal, then G is called a Dedekind group.

Let G be a p -group. We use D_{2^n} , Q_{2^n} and SD_{2^n} to denote the dihedral group, the generalized quaternion group and the semi-dihedral group of order 2^n , respectively. We also use $d(G)$ and C_{p^m} to denote the minimal number of generators of G and the cyclic group of order p^m . Let $G_n(G) = \langle [g_1, g_2, \dots, g_n] | g_i \in G \rangle$. Then $c(G)$ denotes the nilpotency class of G . If H and K are groups, then $H * K$ denotes a central product of H and K . $H_G = \bigcap_{x \in G} x^{-1}Hx$ is the normal core of the subgroup H in G . $M \triangleleft G$ means M is a maximal subgroup of G . For other notations and terminologies the reader is referred to [7].

Now we list some known results which will be used later.

Lemma 2.1. [8, Section 36, Remark 1] *Let G be a p -group. If there is a normal subgroup $R < G'$ such that G/R is metacyclic, then G is also metacyclic.*

Lemma 2.2. [8, Section 1, Exercise 8a] *Let G be a minimal non-abelian p -group. Then G is one of the following groups:*

- (1) Q_8 ;
 (2) $M_p(n, m) = \langle a, b | a^{p^n} = b^{p^m} = 1, [a, b] = a^{p^{n-1}} \rangle$ is metacyclic with $n \geq 2$ and $m \geq 1$;
 (3) $M_p(n, m, 1) = \langle a, b, c | a^{p^n} = b^{p^m} = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1 \rangle$ is non-metacyclic with $n \geq m \geq 1$ and $m + n \geq 3$ when $p = 2$.

Lemma 2.3. [8, Section Appendix 1, Theorem A.1.4] *Let G be a p -group and $x, y \in G$.*

- (1) $(xy)^p \equiv x^p y^p \pmod{\mathcal{U}_1(G')G_p}$.
 (2) $[x^p, y] \equiv [x, y]^p \pmod{\mathcal{U}_1(N')N_p}$, where $N = \langle x, [x, y] \rangle$.

Lemma 2.4. [8, Section 1, Proposition 1.23] *If a p -group G is non-Dedekind, then there exists a normal subgroup K of index p in G' such that G/K is non-Dedekind.*

Lemma 2.5. [9, Lemma 2.2] *Suppose that G is a finite non-abelian p -group. Then the following conditions are equivalent:*

- (1) G is minimal non-abelian;
 (2) $d(G) = 2$ and $|G'| = p$;
 (3) $d(G) = 2$ and $\Phi(G) = Z(G)$.

Lemma 2.6. [10, Theorem 2] *Let G be a metacyclic 2-group. Then G has one presentation of the following three kinds:*

- (1) *G has a cyclic maximal subgroup.*
- (2) *Ordinary metacyclic 2-groups $G = \langle a, b | a^{2^{r+s+u}} = 1, b^{2^{r+s+t}} = a^{2^{r+s}}, a^b = a^{1+2^r} \rangle$, where r, s, t, u are non-negative integers with $r \geq 2$ and $u \leq r$.*
- (3) *Exceptional metacyclic 2-groups $G = \langle a, b | a^{2^{r+s+v+t'+u}} = 1, b^{2^{r+s+t}} = a^{2^{r+s+v+t'}}, a^b = a^{-1+2^{r+v}} \rangle$, where r, s, v, t, t', u are non-negative integers with $r \geq 2, t' \leq r, u \leq 1, tt' = sv = tv = 0$, and if $t' \geq r - 1$, then $u = 0$.*

Groups of different types or of the same type but with different values of parameters are not isomorphic to each other.

Lemma 2.7. [11, Theorem 3] *Let G be a 3-group in which $|\langle x \rangle / \langle x \rangle_G| \leq 3$, for any $x \in G$. Then $c(G) \leq 4$ if $d(G) = 2$ and $c(G) \leq 5$ if $d(G) > 2$.*

Lemma 2.8. [11, Theorem 6] *Let G be a 3-group in which $|\langle x \rangle / \langle x \rangle_G| \leq 3$, for any $x \in G$. If $d(G) = 2$, then $\exp(G') \leq 9$ and $\exp(G_3) \leq 3$.*

Lemma 2.9. [12, Corollary 2] *Let G be a p -group with $G \neq 1$ and $p > 2$. Then $d(G) \leq \log_p |\Omega_1(G)|$.*

3 Some properties of Q - p -groups

In this section, we discuss the properties of Q - p -groups which will be used later.

Lemma 3.1. *Let G be a Q - p -group. If H is a subgroup of G and N is a normal subgroup of G , then H and G/N are Q - p -groups.*

Proof. Let A and B be subgroups of H and $A \not\leq B, B \not\leq A$. Then, by the hypotheses of the lemma, $A \cap B \trianglelefteq \langle A, B \rangle$.

Set $\bar{G} = G/N$. Let $\bar{K} = K/N$ and $\bar{L} = L/N$ be subgroups of $\bar{G}$ and $\bar{K} \not\leq \bar{L}, \bar{L} \not\leq \bar{K}$. Then $K \not\leq L, L \not\leq K$. By the hypotheses, we see $\bar{K} \cap \bar{L} \trianglelefteq \langle \bar{K}, \bar{L} \rangle$. It follows that $\bar{K} \cap \bar{L} \trianglelefteq \langle \bar{K}, \bar{L} \rangle$. Thus, the lemma is true. $\square$

Lemma 3.2. *Let G be a Q - p -group. Then, for any $x \in G$, $\langle x^p \rangle \trianglelefteq G$.*

Proof. Without loss of generality, we may assume $x \notin Z(G)$ and $o(x) > p$. Take $y \in G$. If $x \in \langle y \rangle$ or $y \in \langle x \rangle$, then $[x, y] = 1$ and so $y \in C_G(x) \leq C_G(x^p)$. If $y \notin \langle x \rangle$, then $\langle y, x^p \rangle \not\leq \langle x \rangle$. If $x \notin \langle y \rangle$, then $\langle x \rangle \not\leq \langle y, x^p \rangle$. Otherwise, we see $x^p \in \Phi(\langle y, x^p \rangle)$ and so $\langle y, x^p \rangle = \langle y \rangle$, a contradiction. So $\langle y, x^p \rangle$ and $\langle x \rangle$ are two non-incident subgroups of G . Since G is a Q - p -group, we see $y \in N_G(x^p)$. So $\langle x^p \rangle \trianglelefteq G$. $\square$

Lemma 3.3. *Let G be a Q - p -group. Then, for any $a \in G \setminus \Phi(G)$, $b \in G \setminus \langle a, \Phi(G) \rangle$, we see $\langle a, b \rangle \trianglelefteq G$.*

Proof. Without loss of generality, we may assume $d(G) = 3$ and $c \in G \setminus \langle a, b \rangle$. Let $A = \langle a, b \rangle$, $B = \langle b, c \rangle$ and $C = \langle a, c \rangle$. Then $b \in A \cap B \trianglelefteq G$. By the hypotheses, $[b, a], [b, c] \in A \cap B$. Similarly, $[a, c] \in A \cap C \trianglelefteq G$. It follows that $G' \leq A$ and the lemma is true. $\square$

4 Q - p -groups of even order

Theorem 4.1. Suppose that G is a non-abelian metacyclic 2-group of order 2^n . Then G is a Q -2-group if and only if G is one of the following groups:

- (1) G is a minimal non-abelian 2-group;
- (2) 2-groups of maximal class;
- (3) $\langle a, b | a^8 = b^{2^{n-3}} = 1, [a, b] = b^2 \rangle, n \geq 6$;
- (4) $\langle a, b | a^8 = b^{2^{n-3}} = 1, [a, b] = b^{2^{n-4}+2} \rangle, n \geq 6$;
- (5) $\langle a, b | a^4 = b^{2^{n-2}} = 1, [a, b] = b^2 \rangle, n \geq 5$;
- (6) $\langle a, b | a^4 = b^{2^{n-2}} = 1, [a, b] = b^{2^{n-3}+2} \rangle, n \geq 5$;
- (7) $\langle a, b | a^8 = b^{2^{n-2}} = 1, a^4 = b^{2^{n-3}}, [a, b] = b^2 \rangle, n \geq 5$.

Proof. Since G is a non-abelian metacyclic 2-group, we see G is one of the groups listed in Lemma 2.6.

If G is a group listed in (1) in Lemma 2.6, then G is of maximal class or G is minimal non-abelian. So G is as given in parts (1) and (2).

If G is a group listed in (2) in Lemma 2.6, then $G = \langle a, b | a^{2^{r+s+u}} = 1, b^{2^{r+s+t}} = a^{2^{r+s}}, [a, b] = a^{2^r} \rangle$ with $r \geq 2$ and $u \leq r$. Since $[a, b^2] \in \langle a^{2^{r+1}} \rangle$, by Lemma 3.2, we see $a^{2^{r+1}} \in \langle b^2 \rangle$. Let $a_1 = ab^{-2^t}$. Then $\langle a_1 \rangle \cap \langle b \rangle = 1$. It is easy to see that $\langle [a_1^2, b] \rangle = \langle a^{2^{r+1}} \rangle$. Thus, $\langle a^{2^{r+1}} \rangle \leq \langle a_1 \rangle \cap \langle b \rangle = 1$. So $|G'| = 2$. By Lemma 2.5, G is a minimal non-abelian 2-group.

If G is of type (3) in Lemma 2.6, then $G = \langle a, b | a^{2^{r+s+v+t'+u}} = 1, b^{2^{r+s+t}} = a^{2^{r+s+v+t'}}, [a, b] = a^{-2+2^{r+v}} \rangle$ with $r \geq 2$ and $u \leq 1$. It follows from $[a, b^2] \in \langle b \rangle$ that $s + t' \leq 1$ and so $s + t' + u \leq 2$. Let $o(a) = 2^m$. If $s + t' + u = 2$, then $s + t' = u = 1$. Thus, $b^{2^{r+s+t}} = a^{2^{m-1}}$ and $[a, b] = a^{-2+2^{m-2}}$. Let $c = b^{2^{r+s+t}-1}a^{2^{m-2}}$. Then $c^2 = 1$. If $m > 4$ or $r + s + t > 2$, then $(b^{2^{r+s+t}-2}a^{2^{m-3}})^2 = c$. If $m = 4$ and $r + s + t = 2$, then $(ba)^2 = c$. By Lemma 3.2, $\langle c \rangle \trianglelefteq G$, which implies $c \in Z(G)$. However, it is impossible. So $s + t' + u \leq 1$. Thus, $[a, b] = a^{-2}$ or $a^{-2+2^{m-1}}$, $b^{2^{r+s+t}} = a^{2^{m-1}}$ or 1. It is easy to see that $(ab^2)^2 = a^2b^4$ and so $\langle a^2b^4 \rangle \trianglelefteq G$. Since $[a^2b^4, b] = a^{-4}$, we see $a^4 \in \langle a^2b^4 \rangle$. If $b^{2^{r+s+t}} = 1$, then $b^4 = 1$ or $b^8 = 1$. So G is as given in parts (3)–(6). If $b^{2^{r+s+t}} = a^{2^{m-1}}$, then $b^4 = a^{2^{m-1}}$ or $b^8 = a^{2^{m-1}}$. If $b^8 = a^{2^{m-1}}$, then $o(a^{2^{m-2}}b^4) = 2$. It follows from $(a^{2^{m-3}}b^2)^2 = a^{2^{m-2}}b^4$ that $\langle a^{2^{m-2}}b^4 \rangle \trianglelefteq G$, which implies $a^{2^{m-2}}b^4 \in Z(G)$. However, $[a^{2^{m-2}}b^4, b] \neq 1$, a contradiction. So $b^4 = a^{2^{m-1}}$ and G is given in part (7).

Conversely, every group listed in the theorem is a Q -2-group and they are pairwise non-isomorphic. $\square$

Corollary 4.2. Let G be a Q -2-group. Then $\Phi(G)$ is abelian.

Proof. For any $a, b \in G$, we may assume that $H = \langle a^2, b \rangle$ is not abelian and $o(a) = 2^n$. By the hypotheses, we see $\langle a^2 \rangle \trianglelefteq G$ and so H is metacyclic. It follows from Theorem 4.1 that $[a^2, b] = a^{2^{n-1}}, a^{-4}$ or $a^{-4+2^{n-1}}$. Then, it is easy to see that $[a^2, b^2] = 1$, which implies $\Phi(G)$ is abelian. $\square$

Lemma 4.3. Let G be a Q -2-group and $d(G) = 2$. If $\exp(G') = 2$, then G is one of the following groups:

- (1) G is a minimal non-abelian 2-group;
- (2) $\langle a, b, c | a^8 = c^2 = 1, a^4 = b^2, [a, b] = c, [c, a] = a^4, [c, b] = 1 \rangle$.

Proof. If $|G| \leq 2^4$, then the conclusion holds by checking the list of groups of order 2^3 and 2^4 . Assume $|G| \geq 2^5$. If $|G'| = 2$, then by Lemma 2.5, G is a minimal non-abelian 2-group. So we may assume $|G'| > 2$, G is non-metacyclic and $G = \langle a, b \rangle$ with $o(a) = 2^n, o(b) = 2^m, [a, b] = c, [c, a] = d$. Then $[a^2, b] = d$. By Lemma 3.2, we see $d \in \langle a^{2^{n-1}} \rangle \leq Z(G)$. Similarly, $[c, b] \in \langle b^{2^{m-1}} \rangle \leq Z(G)$. Let $A = \langle a, c \rangle$ and $B = \langle b, c \rangle$. Noting that G is a Q -2-group, we see $G' \leq A \cap B$. Since $\Omega_1(A) = \langle c \rangle \times \langle a^{2^{n-1}} \rangle, G' = \Omega_1(B) = \langle c \rangle \times \langle a^{2^{n-1}} \rangle$. So for any $x \in G \setminus \Phi(G), y \in G \setminus \langle \Phi(G), x \rangle$, we see $G' = \Omega_1(\langle c, x \rangle) = \Omega_1(\langle c, y \rangle)$. It follows that $\langle x \rangle \cap \langle y \rangle \neq 1$. Without loss of generality, we may assume $a^{2^r} = b^{2^t}, [c, a] = a^{2^{n-1}}$ and $[c, b] = 1$, where $r \geq 2$.

If $r \leq t$, then, by letting $a_1 = ab^{-2^{t-r}}$, we see $\langle a_1 \rangle \cap \langle b \rangle = 1$. If $r > t > 1$ or $r > t, r > 2$, then, by letting $b_1 = a^{-2^{r-t}}b$, we see $\langle a \rangle \cap \langle b_1 \rangle = 1$. So $r = 2, t = 1$, which implies $a^4 = b^2$. If $n \geq 4$, then, by letting $b_1 = a^{-2^{n-2}-1}b$, we also have $\langle a \rangle \cap \langle b_1 \rangle = 1$. If $n = 3$, then $o(a) = 8$ and G is given in part (2). $\square$

Lemma 4.4. *Let G be a non-metacyclic Q -2-group and $d(G) = 2$. If G' is cyclic with order 4, then G is one of the following groups:*

- (1) $\langle a, b, c | a^{2^{n-2}} = b^2 = 1, c^2 = a^{2^{n-3}}, [b, a] = c, [c, a] = 1, [c, b] = c^2 \rangle$;
- (2) $\langle a, b, c | a^8 = 1, a^4 = b^2 = c^2, [b, a] = c, [c, a] = 1, [c, b] = c^2 \rangle$.

Proof. We may assume $G = \langle a, b \rangle, [b, a] = c, o(a) = 2^m, o(b) = 2^n, o(c) = 4, [c, a] = 1, [c, b] = 1$ or c^2 . Then $[a^2, b] = c^2$. By Lemma 3.2, we see $c^2 = a^{2^{m-1}}$.

If $\langle a \rangle \cap \langle b \rangle \neq 1$, then we may assume $a^{2^r} = b^{2^t}, r \geq 2$. If $r < t$, then, by letting $a_1 = ab^{-2^{t-r}}$, we see $\langle a_1 \rangle \cap \langle b \rangle = 1$. If $r \geq t > 2$ or $r > t = 2$ or $r \geq 3, t = 1$, then, by letting $b_1 = a^{-2^{r-t}}b$, we see $\langle a \rangle \cap \langle b_1 \rangle = 1$.

If $r = 2, t = 1$, then, by letting $b_1 = a^{-2}b$, we see $b_1^2 = c^2 = a^{2^{m-1}}$. We may assume $m - 1 = 2$ and so $a^4 = b^2 = c^2$. If $[c, b] = 1$, then $(bc)^2 = 1$. However, $[(bc)^2, a] = c^2 \neq 1$, a contradiction. So $[c, b] = c^2$ and G is given in part (2).

If $r = t = 2$, then, by letting $b_1 = a^{-2^{r-t}}b$, we see $b_1^4 = c^2$. We may assume $a^4 = b_1^4 = c^2$. Let $A = \langle a^2c, b_1^2c \rangle, B = \langle b, a^2c \rangle$. Then $A \cap B = \langle a^2c \rangle$. However, $[a^2c, b] = c^2 \notin \langle a^2c \rangle$, a contradiction.

Now we may assume $\langle a \rangle \cap \langle b \rangle = 1$. If $[c, b] = 1$, then $[a, b^2] = c^2$ and so $c^2 \in \langle b \rangle \cap \langle a \rangle = 1$, a contradiction. So $[c, b] = c^2$. If $o(a) = 8$, then $a^4 = c^2$. Let $A = \langle a^2c, ba^2 \rangle$ and $B = \langle b, a^2 \rangle$. Then $A \not\leq B$ and $B \not\leq A$. However, $A \cap B = \langle ba^2 \rangle \not\leq B$, a contradiction. So $o(a) \geq 16$. Let $A = \langle a^{2^{n-2}}c, b^2 \rangle$ and $B = \langle a^{2^{n-2}}c, a^2b \rangle$. If $o(b) > 2$, then $A \not\leq B$ and $B \not\leq A$. However, $A \cap B \not\leq \langle A, B \rangle$, a contradiction. So $b^2 = 1$ and we get the group in part (1). $\square$

According to Lemma 4.4, we have the following result:

Corollary 4.5. *Let G be a Q -2-group and $d(G) = 2$. If $c(G) = 2$, then $\exp(G') = 2$.*

Lemma 4.6. *Let G be a non-metacyclic Q -2-group and $d(G) = 2$. Then $|G'| \leq 4$.*

Proof. Otherwise, $|G'| \geq 8$. Assume $|G| = 2^n$. Then $n \geq 6$. Let $N \leq G'$ such that $|N| = 2$ and $N \trianglelefteq G$. Then, by Lemmas 2.1 and 3.1, G/N is a non-metacyclic Q -2-group. By induction, $|G'/N| \leq 4$ and so $|G'| = 8$. Set $\bar{G} = G/N$ and $N = \langle x \rangle$. Then $\bar{G} \cong \langle \bar{a}, \bar{b}, \bar{c} | \bar{a}^{2^{n-3}} = \bar{b}^2 = \bar{1}, \bar{c}^2 = \bar{a}^{2^{n-4}}, [\bar{b}, \bar{a}] = \bar{c}, [\bar{c}, \bar{b}] = \bar{c}^2, [\bar{c}, \bar{a}] = \bar{1} \rangle$ by Lemma 4.4.

If G' is cyclic, then $c^4 = a^{2^{n-3}} = x$ and $\langle c^2 \rangle = \langle a^{2^{n-4}} \rangle$. It follows from $[b^2, a] = 1$ that $[c, b] = c^{-2}$ and therefore $[c^2, b] = c^4$, which implies $[a^{2^{n-4}}, b] = a^{2^{n-3}}$. On the other hand, we see $\langle [b, a^2] \rangle = \langle c^2 \rangle$ and so $[b, a^4] = c^4 = a^{2^{n-3}}$, which implies $n = 6$. Thus, $c^2 = a^4$ and $o(a) = 16$. Let $A = \langle a^2c, ba^2 \rangle$ and $B = \langle b, a^2 \rangle$. Then $A \not\leq B$ and $B \not\leq A$. However, $A \cap B = \langle ba^2 \rangle \not\leq B$, a contradiction.

If G' is not cyclic, then $G' = \langle c, x \rangle, c^4 = a^{2^{n-3}} = 1$. It follows from $[b^2, a] = c^2[c, b] = 1$ that $[c, b] = c^2$. Since $x \in G'$, we see $[c, a] = x$. Thus, $[(ba)^2, a] = [c, a] = x \in \langle (ba)^2 \rangle$. However, it is impossible. $\square$

According to Lemmas 4.3, 4.4 and 4.6, we have the following results:

Theorem 4.7. *Let G be a non-metacyclic Q -2-group, $|G| = 2^n$ and $d(G) = 2$. Then G is one of the following pairwise non-isomorphic groups:*

- (1) G is a minimal non-abelian 2-group;
- (2) $\langle a, b, c | a^8 = c^2 = 1, a^4 = b^2, [a, b] = c, [c, a] = a^4, [c, b] = 1 \rangle$;
- (3) $\langle a, b, c | a^8 = 1, a^4 = b^2 = c^2, [b, a] = c, [c, a] = 1, [c, b] = c^2 \rangle$;
- (4) $\langle a, b, c | a^{2^{n-2}} = b^2 = 1, c^2 = a^{2^{n-3}}, [b, a] = c, [c, a] = 1, [c, b] = c^2 \rangle$.

Lemma 4.8. *Let G be a Q -2-group and $d(G) = 3$. Then G has no subgroup A such that $A \cong \langle a, b, x | a^8 = x^2 = 1, a^4 = b^2, [a, b] = x, [x, a] = a^4, [x, b] = 1 \rangle$.*

Proof. Otherwise, we assume $A = \langle a, b, x | a^8 = x^2 = 1, a^4 = b^2, [a, b] = x, [x, a] = a^4, [x, b] = 1 \rangle$ and $A \leq G$. Then $[a, x] \neq 1$ and $[a^2, b] = a^4 \neq 1$. Since a^2 and $x \in \Phi(G)$, by Corollary 4.2, we see $a \notin \Phi(G)$ and $b \notin \Phi(G)$. Since $d(G) = 3$, we may assume that there exists $c \in G \setminus A$ such that $G = \langle a, b, c \rangle$. Let $B = \langle a, c \rangle$. Then $A \cap B \trianglelefteq G$. Since $a \in A \cap B$, we see $\langle a \rangle^G \leq A \cap B$, which implies $\langle a, x \rangle \leq A \cap B$. Noting that $|A| = 2^5$ and $|\langle a, x \rangle| = 2^4$, we see $A \cap B = \langle a, x \rangle$. Thus, $[c, a], [c, x] \in \langle a, x \rangle$. It follows that $[c, x] \in \langle a^4 \rangle$. Since $[a, x] \neq 1$, we see B is non-abelian. If $[a, c] = a^i x$, then $[a, bc] \in \langle a \rangle$. So, without loss of generality, we may assume $[a, c] \in \langle a \rangle$, which implies B is metacyclic. By Theorem 4.1, we see $C_B(a) < B$. It follows that $B = C_B(a)\langle x \rangle = \langle a, c^2, x \rangle = \langle a, x \rangle$. Thus, $c \in \langle a, x \rangle \leq A$, a contradiction. $\square$

Theorem 4.9. *Let G be a Q -2-group and $d(G) \geq 3$. Then $\exp(G') \leq 2$.*

Proof. Otherwise, $\exp(G') > 2$. Let $N \leq G'$ such that $|N| = 2$ and $N \trianglelefteq G$. By Lemma 3.1, $\exp(G'/N) \leq 2$ and so $\exp(G') = 4$. Then there exist $a, b \in G$ such that $o([a, b]) = 4$. We may assume $d(G) = 3$ and $G = \langle a, b, c \rangle$, $[a, b] = x$, $[b, c] = y$. By Corollary 4.5, $x \notin Z(G)$. We may assume $[b, x] \neq 1$. If $o(y) = 4$, then, by calculation, we see $o([b, ac]) \leq 2$. So we may assume $o(y) \leq 2$. Let $A = \langle a, b \rangle$ and $B = \langle b, c \rangle$. By Lemma 3.3, we see $B \trianglelefteq G$ and so $x \in B$, which implies that B is non-abelian. According to Lemmas 4.3 and 4.8, $\langle b, c \rangle$ is minimal non-abelian. It follows from $[b, x] \neq 1$ that $B = \langle b, x \rangle$. However, it is impossible. $\square$

Theorem 4.10. *Let G be a Q -2-group and $d(G) \geq 3$. Then $c(G) \leq 2$.*

Proof. For any $g_1, g_2 \in G$, by Lemmas 4.3, 4.8 and 4.9, we see $\langle g_1, g_2 \rangle$ is abelian or minimal non-abelian. We may assume $d(G) = 3$, $G = \langle a, b, c \rangle$ and $[a, b] = x$. Let $A = \langle a, b \rangle$ and $B = \langle b, c \rangle$. By Lemma 3.3, $B \trianglelefteq G$ and so $x \in \Phi(B)$. According to Lemma 2.5, $x \in Z(B)$. So $x \in Z(G)$ and the theorem is true. $\square$

Lemma 4.11. *Let G be a non-Dedekind 2-group with $d(G) \geq 4$ and $|G'| = 2$. Then G is a Q -2-group if and only if G is one of the following pairwise non-isomorphic groups:*

- (1) $D_8 * Q_8$;
- (2) $M_2(2, 1, 1) * Q_8$.

Proof. Since $|G'| = 2$, there exist $a, b \in G$ such that $\langle a, b \rangle$ is minimal non-abelian. By Lemma 2.2, $\langle a, b \rangle$ is isomorphic to Q_8 or $M_2(n, m)$, $n \geq 2$ or $M_2(n, m, 1)$ with $n \geq m \geq 1$ and $m + n \geq 3$. For any $x, y \in G$, since $|G'| = 2$, we have $[x, y^2] = 1$, which implies $\Phi(G) \leq Z(G)$. So $a \notin \Phi(G)$ and $b \notin \Phi(G)$. Assume $d(G) = t + 2$. Then there exist $x_1, x_2, \dots, x_t$ with $o(x_i) = 2^{e_i}$, where $1 \leq i \leq t$, such that $G = \langle a, b, x_1, x_2, \dots, x_t \rangle$. Without loss of generality, we may assume $e_1 \geq e_2 \geq \dots \geq e_t$ and $[x_i, a] = [x_i, b] = 1$.

If $\langle a, b \rangle \cong M_2(n, m)$, then we may assume $a^{2^n} = b^{2^m} = 1$, $[a, b] = a^{2^{n-1}}$, where $n \geq 2$. For any x_i , by Lemma 3.3, we see $G' \leq \langle b, x_i \rangle$, where $1 \leq i \leq t$. It follows that $a^{2^{n-1}} \in \langle x_i \rangle \cap \langle x_j \rangle \neq 1$, where $1 \leq i, j \leq t$ and $i \neq j$. We may assume $x_i^{2^s} = x_j^{2^k}$, where $s \geq k \geq 1$. If $s > t \geq 1$ or $s = t \geq 2$, then, by letting $x' = x_i^{-2^{s-t}} x_j$, we see $\langle x' \rangle \cap \langle x_i \rangle = 1$. So $s = k = 2$ and $x_1^2 = x_2^2 = (x_1 x_2)^2$. Then, for any $1 \leq i, j \leq t$ and $i \neq j$, we see $\langle x_i, x_j \rangle \neq Q_8$. It follows that $t = 2$. If $n \geq 3$, then $(x_1 a^{2^{n-2}})^2 = 1$, a contradiction. So $n = 2$. If $m > 1$, then, by letting $A = \langle b x_1, a \rangle$ and $B = \langle b x_1, a x_2 \rangle$, we see $A \cap B \not\trianglelefteq A$. So $m = 1$ and $G \cong D_8 * Q_8$.

If $\langle a, b \rangle \cong M_2(n, m, 1)$, then we may assume $a^{2^n} = b^{2^m} = c^2 = 1$, $[a, b] = c$, $[c, a] = [c, b] = 1$, where $n \geq m \geq 1$ and $m + n \geq 3$. For any x_i , by Lemma 3.3, we see $G' \leq \langle b, x_i \rangle$, where $1 \leq i \leq t$. By the above, we see $t = 2$ and $\langle x_1, x_2 \rangle \neq Q_8$. Since $n \geq m \geq 1$ and $m + n \geq 3$, we see $n \geq 2$. Let $A = \langle a^2 x_1, a^2 x_2 \rangle$ and $B = \langle a^2 x_1, b \rangle$. If $n \geq 3$, then $A \cap B = \langle a^2 x_1 \rangle \not\trianglelefteq A$. So $n = 2$. Let $C = \langle a x_1, b \rangle$ and $D = \langle a x_1, b x_2 \rangle$. If $m = 2$, then $C \cap D = \langle a x_1 \rangle \not\trianglelefteq C$. So $m = 1$ and $G \cong M_2(2, 1, 1) * Q_8$.

If $\langle a, b \rangle \cong Q_8$, then $a^4 = 1$, $a^2 = b^2$, $[a, b] = a^2$. For any $1 \leq i, j \leq t$ and $i \neq j$, by the above, we may assume $\langle x_i, x_j \rangle \cong Q_8$ or $\langle x_i, x_j \rangle$ is abelian. If $\langle x_i, x_j \rangle \cong Q_8$, then $x_i^2 = x_j^2 = a^2$. Let $A = \langle ax_1, b \rangle$ and $B = \langle ax_1, bx_2 \rangle$. Then $A \cap B = \langle ax_1 \rangle \not\subseteq A$. So $\langle x_i, x_j \rangle$ is abelian and we may assume $\langle x_i \rangle \cap \langle x_j \rangle = 1$. If $\langle a, b \rangle \cap \langle x_i \rangle = \langle a^2 \rangle$, then $\langle ax_i^{2^{e_i-2}}, bx_i^{2^{e_i-2}} \rangle \cong M_2(2, 1)$. Thus, we may assume $\langle a, b \rangle \cap \langle x_i \rangle = 1$, where $1 \leq i \leq t$. Let $C = \langle ax_1, b \rangle$ and $D = \langle ax_1, x_2 \rangle$. If $o(x_1) > 2$, then $C \cap D = \langle ax_1 \rangle \not\subseteq D$. Thus, $o(x_1) = 2$ and so $o(x_i) = 2$. So G is a Dedekind group, in contradiction to the hypothesis.

Conversely, every group listed in the lemma is a non-Dedekind Q -2-group and they are pairwise non-isomorphic. $\square$

Lemma 4.12. *Let G be a Q -2-group and $N \leq G' \cap Z(G)$ with $|N| = 2$. Then the quotient group G/N is not isomorphic to $M_2(2, 1, 1) * Q_8$.*

Proof. Otherwise, we may assume $N = \langle x \rangle$ and $\bar{G} = G/N = \langle \bar{a}, \bar{b}, \bar{c}, \bar{d} | \bar{a}^4 = \bar{b}^2 = \bar{c}^4 = \bar{d}^4 = \bar{1}, \bar{c}^2 = \bar{d}^2, [\bar{c}, \bar{d}] = \bar{c}^2, [\bar{a}, \bar{b}] = \bar{c}^2, [\bar{a}, \bar{c}] = [\bar{a}, \bar{d}] = [\bar{b}, \bar{c}] = [\bar{d}, \bar{d}] = \bar{1} \rangle$. By Lemma 4.9, we see $o(c) = o(d) = 4$ and $G' = \langle c, x \rangle \cong C_2 \times C_2$. According to Lemma 3.3, we see $G' \leq \langle c, b \rangle$. If $b^2 = 1$, then, since $x \in \langle c, b \rangle$, we see $[c, b] = x$. Similarly, $[d, b] = x$ and $[cd, b] = x$. However, it is impossible. So $b^2 = x$. Similarly, $a^4 = x$. By letting $b_1 = ba^2$, we see $b_1^2 = 1$. So we also have a contradiction. $\square$

Theorem 4.13. *Let G be a Q -2-group with $d(G) \geq 4$ and $|G| \geq 2^7$. Then G is Dedekind.*

Proof. Otherwise, we may assume $|G| = 2^7$ by Lemma 2.4. Let $N \leq G' \cap Z(G)$ such that $|N| = 2$. Set $\bar{G} = G/N$. Then, by Lemma 2.4 again, we may assume $\bar{G}$ is a non-Dedekind group. Thus, $\bar{G}$ is of the type (2) with order 2^6 in Lemma 4.11. However, it is impossible by Lemma 4.12. $\square$

5 Q - p -groups of odd order

Lemma 5.1. *Let G be a Q - p -group with $d(G) = 2$ and $p > 3$. If $\exp(G') = p$, then $c(G) \leq 3$.*

Proof. We may assume $G = \langle a, b \rangle$, $[a, b] = c$ and $[c, a] = d \neq 1$. Assume $A = \langle a, c \rangle$, $B = \langle b, c \rangle$, $C = \langle c, d \rangle$, $D = \langle a, d \rangle$ and $o(a) = p^n$. Then, by the hypotheses, we see $G' \leq A \cap B$ and $A' \leq C \cap D$. Now we prove $A' \leq \langle a^{p^{n-1}}, d \rangle$. Since $[a, d] \in \langle [c, d] \rangle$, it is easy to see that there exists an element $g \in A \setminus \langle c, \Phi(A) \rangle$ such that $[g, d] = 1$. Noting that $A' \leq \langle g, d \rangle$, we see $A' \leq \langle \cup_1(g), d \rangle$. Since $p > 3$ and $c(A) \leq 3$, by Lemma 2.3, we see $\cup_1(A) = \langle a^p \rangle$, which implies $A' \leq \langle a^{p^{n-1}}, d \rangle$. It follows that $G' \leq \langle a^{p^{n-1}}, c, d \rangle$. If $d \in \langle a \rangle$, then $G_3 \leq Z(G)$. So we may assume $d \notin \langle a \rangle$. Without loss of generality, we may assume $[c, b] \in \langle a^{p^{n-1}} \rangle$. It follows from $G' \leq \langle c, b \rangle$ that $d \in \Phi(\langle c, b \rangle) = \langle b^p \rangle \trianglelefteq G$. So $d \in Z(G)$, $G_3 \leq Z(G)$ and the lemma is proved. $\square$

Theorem 5.2. *Let G be a Q -3-group with $d(G) = 2$. Then $c(G) \leq 3$.*

Proof. We may assume $G = \langle a, b \rangle$, $o(a) = 3^n$, $[a, b] = c$ and $[c, a] = d \neq 1$. By Lemma 3.2, for any $x \in G$, we see $|\langle x \rangle / \langle x \rangle_G| \leq 3$. Thus, by Lemmas 2.7 and 2.8, we see $c(G) \leq 4$, $\exp(G') \leq 9$ and $\exp(G_3) \leq 3$. Then $[c, d] \in [G_2, G_3] \leq G_5 = 1$, $o(d) = 3$ and $c(\langle a, d \rangle) \leq 2$. Let $A = \langle c, d \rangle$, $B = \langle a, d \rangle$. Then $[a, d] \in A$ and so $[a, d] \in \langle c^3 \rangle$. Noting that $G' \leq \langle a, c \rangle$, we see $[c, b] \in \Omega_1(\langle a, c \rangle) = \langle a^{3^{n-1}}, c^3, d \rangle$. Without loss of generality, we may assume $[c, b] \in \langle a^{i3^{n-1}} c^{3j} \rangle$, which implies $c(\langle c, b \rangle) < 2$. It follows from $G' \leq \langle c, b \rangle$ that $d \in \Phi(\langle c, b \rangle) = \langle b^3, c^3 \rangle \trianglelefteq G$. Thus, $d \in Z(G)$ and $G_3 \leq Z(G)$. So $c(G) \leq 3$. $\square$

Theorem 5.3. *Let G be a Q - p -group and $p > 2$. Then $\exp(G') \leq p$.*

Proof. Suppose the result is not true and G is a counterexample of minimal order. Take $x, y \in G'$ with $o(x) \leq p$ and $o(y) \leq p$. Let $K = \langle x, y \rangle$. Then, by Lemma 2.3, $(xy)^p = x^p y^p z$, where $z \in \mathcal{U}_1(K')K_p$. If $p > 3$, then, by Lemma 5.1, $\mathcal{U}_1(K')K_p = 1$. If $p = 3$, then, by Lemma 2.7, we see $c(G) \leq 5$. Thus, $\mathcal{U}_1(K')K_3 = 1$. It follows that $(xy)^p = x^p y^p = 1$. So we may assume $d(G) = 2$, $G = \langle a, b \rangle$, $[a, b] = c$, $L = \langle a, c \rangle$ and $o(c) \geq p^2$. Then, according to Lemma 2.3, we see $[a^p, b] = c^p w$, where $w \in \mathcal{U}_1(L')L_p$. Since $L < G$, by Lemma 5.1 and Theorem 5.2, we see $\mathcal{U}_1(L')L_p = 1$. Thus, $w = 1$ and so $c^p \in \langle a \rangle$. Similarly, $c^p \in \langle b \rangle$. By induction, $o(c^p) = p$ and so $o(c) = p^2$.

Without loss of generality, we may assume $\langle a \rangle \cap \langle b \rangle = \langle a^{p^s} \rangle = \langle b^{p^t} \rangle$, $a^{p^s} = b^{p^t}$ and $s \geq t \geq 2$. If $s > t$, then, by letting $b_1 = a^{-p^{s-t}}b$, we see $[a, b_1^p] = c^p$ and $c^p \notin \langle b_1^p \rangle$. So $s = t$. Let $b_2 = a^{-1}b$. Then, by Lemma 2.3, we see $b_2^p = a^p b^{-p}g$, where $g \in \mathcal{U}_1(G')G_p$. Then $o(g) \leq p$ and $o(b_2) = p^s$. Noting that $[a, b_2^p] = c^p$, we see $c^p \in \langle b_2^p \rangle$. If $s = 2$, then $b_2^p \in Z(G)$, a contradiction. If $s > 2$, then $\langle c^p \rangle = \langle b_2^{p^{s-1}} \rangle = \langle a^{p^{s-1}}b^{p^{s-1}} \rangle$. It follows that $\langle a \rangle \cap \langle b \rangle = \langle a^{p^{s-1}} \rangle$, another contradiction. $\square$

According to Lemma 5.1 and Theorems 5.2 and 5.3, we have the following results:

Corollary 5.4. *Let G be a Q - p -group with $d(G) = 2$ and $p > 2$. Then $c(G) \leq 3$.*

Theorem 5.5. *Let G be a Q - p -group and $p > 2$. Then $c(G) \leq 3$.*

Proof. We may assume $d(G) = 3$, $G = \langle a, b, c \rangle$, $[a, b] = x$ and $[a, x] = y \neq 1$. We only need to prove that $[c, y] = 1$. Let $A = \langle a, y \rangle$, $B = \langle c, y \rangle$, $C = \langle x, y \rangle$. Then $[c, y] \in A \cap C$. It follows that $[c, y] = 1$, which implies $c(G) \leq 3$. $\square$

Theorem 5.6. *Let G be a Q - p -group with $d(G) \geq 4$ and $p > 2$. Then G is abelian.*

Proof. Otherwise, we may assume $d(G) = 4$ and $G' \neq 1$. Let $N \leq G'$ such that $|N| = p$ and $N \trianglelefteq G$. Then G/N is a Q - p -group by Lemma 3.1. By induction, G/N is abelian and so $|G'| = p$. Assume $A < G$ and A is minimal non-abelian with $A = \langle a, b \rangle$, $\langle a \rangle \cap \langle b \rangle = 1$ and $[a, b] \notin \langle b \rangle$. Then $C_G(b) < G$. Assume that there exists an element $g \in \Omega_1(C_G(b)) \setminus A$. Let $A = \langle a, b \rangle$, $B = \langle b, g \rangle$. Then, we see $A \cap B = \langle b \rangle$ and $\langle b \rangle \not\leq A$, a contradiction. So, $\Omega_1(C_G(b)) \leq A$. Since $|\Omega_1(A)| \leq p^3$, by Lemma 2.9, we see $\Omega_1(C_G(b)) = \Omega_1(A) = p^3$. Then $A = \langle a, b, c | a^{p^n} = b^{p^m} = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1 \rangle$ is non-metacyclic with $n \geq 2$. By Lemma 2.9, we see $|\Omega_1(G)| \geq p^4$. Then there exists an element $h \in \Omega_1(G) \setminus C_G(b)$. Let $A = \langle a, b \rangle$, $B = \langle b, ah \rangle$. Then, we see $A \cap B = \langle b, a^{p^{n-1}} \rangle \not\leq A$, another contradiction. Thus, the theorem is true. $\square$

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