Research Article

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Blow-up results of the positive solution for a class of degenerate parabolic equations

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Abstract: This paper is devoted to discussing the blow-up problem of the positive solution of the following degenerate parabolic equations:

$$\begin{cases} (r(u))_t = \operatorname{div}(|\nabla u|^p \nabla u) + f(x, t, u, |\nabla u|^2), & (x, t) \in D \times (0, T^*), \\ \frac{\partial u}{\partial \nu} + \sigma u = 0, & (x, t) \in \partial D \times (0, T^*), \\ u(x, 0) = u_0(x), & x \in \bar{D}. \end{cases}$$

Here $p > 0$, the spatial region $D \subset \mathbb{R}^n$ ($n \geq 2$) is bounded, and its boundary ∂D is smooth. We give the conditions that cause the positive solution of this degenerate parabolic problem to blow up. At the same time, for the positive blow-up solution of this problem, we also obtain an upper bound of the blow-up time and an upper estimate of the blow-up rate. We mainly carry out our research by means of maximum principles and first-order differential inequality technique.

Keywords: blow-up solution, degenerate parabolic equation, blow-up time

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1 Introduction

Over the past decade, the blow-up problem of the degenerate parabolic equations has attracted the attention and research of many scholars (see, for example [1–9]). We have also noted that in recent years, there have been many papers discussing and studying the blow-up solutions of parabolic equations with non-linear gradient source terms, and many meaningful results have been obtained (see, for example [2, 10–15]). The purpose of this paper is to study the blow-up positive solutions of the following degenerate parabolic problems:

$$\begin{cases} (r(u))_t = \operatorname{div}(|\nabla u|^p \nabla u) + f(x, t, u, |\nabla u|^2), & (x, t) \in D \times (0, T^*), \\ \frac{\partial u}{\partial \nu} + \sigma u = 0, & (x, t) \in \partial D \times (0, T^*), \\ u(x, 0) = u_0(x), & x \in \bar{D}. \end{cases} \quad (1)$$

In problem (1), $p > 0$, the spatial region $D \subset \mathbb{R}^n$ ($n \geq 2$) is bounded, and its boundary ∂D is smooth, T^* represents the blow-up time of the solution, $\frac{\partial}{\partial \nu}$ represents the external normal derivative, the function

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$r \in C^2(\mathbb{R}_+)$ satisfies $r'(s) > 0$, $s \in \mathbb{R}_+$, the function $f \in C^1(D \times \mathbb{R}_+ \times \mathbb{R}_+ \times \overline{\mathbb{R}_+})$ is positive, the initial value function $u_0 \in C^2(\overline{D})$ is positive and satisfies $\frac{\partial u_0(x)}{\partial \nu} + \sigma u_0 = 0$, $x \in \partial D$, and σ is a positive constant.

There are many papers on the blow-up of the parabolic equation with Robin boundary conditions, and people can refer to the literature [13,16–21]. The research work on problem (1) in this paper is mainly inspired by the papers [13,16]. Ding studied the blow-up problem of the following nondegenerate parabolic equations in the paper [13]:

$$\begin{cases} (r(u))_t = \operatorname{div}(b(u)\nabla u) + f(x, t, u, |\nabla u|^2), & (x, t) \in D \times (0, T^*), \\ \frac{\partial u}{\partial \nu} + \sigma u = 0, & (x, t) \in \partial D \times (0, T^*), \\ u(x, 0) = u_0(x), & x \in \overline{D}. \end{cases} \quad (2)$$

In problem (2), the spatial region $D \subset \mathbb{R}^n$ ($n \geq 2$) is bounded, and its boundary ∂D is smooth. With the aid of maximum principles and first-order differential inequality technique, he gave the conditions for the blow-up of the positive solution of problem (2). At the same time, for the positive blow-up solution, the upper bound of the blow-up time and the upper estimate of the blow-up rate are also obtained. Tian and Zhang studied the blow-up problem of the following nondegenerate parabolic equations in the paper [16]:

$$\begin{cases} (r(u))_t = \operatorname{div}(b(|\nabla u|^p)\nabla u) + h(t)g(x)f(u), & (x, t) \in D \times (0, t^*), \\ \frac{\partial u}{\partial \nu} + \sigma u = 0, & (x, t) \in \partial D \times (0, t^*), \\ u(x, 0) = u_0(x), & x \in \overline{D}. \end{cases} \quad (3)$$

In problem (3), $p > 0$, the spatial region $D \subset \mathbb{R}^n$ ($n \geq 2$) is bounded, and its boundary ∂D is smooth. They used first-order differential inequality technique to give the conditions that make the positive solution of problem (3) blow up. They also derived the upper and lower bounds of the blow-up time for the positive blow-up solution of this problem.

Since there is a nonlinear gradient source term in the first equation of problem (1), and there is no nonlinear gradient source term in the first equation of problem (3), the research method in the paper [16] is not suitable for studying problem (1). In this paper, we used the research method in paper [13] to study problem (1). In other words, we rely on maximum principles and first-order differential inequality techniques for research. In using this research method to study problem (1), the biggest difficulty is that some suitable auxiliary functions need to be established. Since the main part of the first equation in problem (1) is different from the main part of the first equation in problem (2), the auxiliary functions that have been established in the paper [13] cannot be used to study problem (1). Therefore, in order to complete our research, we need to establish some new auxiliary functions suitable for problem (1), which is also the key point of this paper. We give the conditions that cause the positive solution of this degenerate parabolic problem to blow up. At the same time, for the positive blow-up solution of this problem, we also obtain an upper bound of the blow-up time and an upper estimate of the blow-up rate.

For convenience, throughout this paper, partial derivative is represented by a comma, and summation convention is used, for example,

$$\sum_{l,k=1}^n u_{x_l} u_{x_k} u_{x_l x_k} = u_{,l} u_{,k} u_{,lk}.$$

2 The main result and its proof

In this section, two constants are defined as follows:

$$\gamma = \min_{x \in \overline{D}} \frac{\operatorname{div}(|\nabla u_0|^p \nabla u_0) + f(x, 0, u_0, |\nabla u_0|^2)}{u_0 e^{u_0} r'(u_0)}, \quad (4)$$

$$\eta = \inf_{(x,t,h) \in D \times \mathbb{R}_+ \times \mathbb{R}_+} \frac{f(x,t,h,0)}{he^h r'(h)}. \quad (5)$$

On this basis, for research needs, two auxiliary functions are established as follows:

$$H(x,t) = -\frac{1}{u}u_t + \gamma e^u, \quad (x,t) \in \bar{D} \times [0, T^*), \quad (6)$$

$$G(h) = \int_h^{+\infty} \frac{1}{se^s} ds, \quad h \in \mathbb{R}_+. \quad (7)$$

Theorem 2.1 is the blow-up result of problem (1).

Theorem 2.1. Let $u \in C^3(D \times (0, T^*)) \cap C^2(\bar{D} \times [0, T^*))$ be a positive solution of problem (1). Suppose the following:

(i) The two constants γ and η defined by (4) and (5), respectively, satisfy

$$0 < \gamma \leq \eta. \quad (8)$$

(ii) The functions f and r satisfy that for $(x,t,h,w) \in D \times \mathbb{R}_+ \times \mathbb{R} \times \overline{\mathbb{R}_+}$,

$$\begin{aligned} f_w(x,t,h,w) &\geq 0, \quad f_t(x,t,h,w) \geq 0, \\ \frac{f_h(x,t,h,w)}{f(x,t,h,w)} - (p-3)\left(1 + \frac{1}{h}\right) &\geq 0, \quad p\left(1 + \frac{1}{h}\right) - \frac{r''(h)}{r'(h)} \geq 0. \end{aligned} \quad (9)$$

Then, u blows up at some finite time T^* , T^* is bounded from above by

$$T^* \leq \frac{1}{\gamma} \int_M^{+\infty} \frac{1}{se^s} ds, \quad M = \max_{x \in \bar{D}} u_0(x),$$

and $u(x,t)$ has the following upper estimate:

$$u(x,t) \leq G^{-1}(\gamma(T^* - t)), \quad (x,t) \in \bar{D} \times [0, T^*),$$

where G^{-1} is the inverse function of function G defined by (7).

Proof. By taking the partial derivative of $H(x,t)$ established in (6), we get

$$H_{,l} = \frac{1}{u^2}u_t u_{,l} - \frac{1}{u}u_{t,l} + \gamma e^u u_{,l} \quad (10)$$

and

$$H_{,lk} = \frac{1}{u^2}u_t u_{,lk} - \frac{2}{u^3}u_t(u_{,l} u_{,k}) + \frac{1}{u^2}(u_{t,k} u_{,l}) + \frac{1}{u^2}(u_{t,l} u_{,k}) - \frac{1}{u}u_{t,lk} + \gamma e^u u_{,lk} + \gamma e^u(u_{,l} u_{,k}). \quad (11)$$

It follows from (11) that

$$\Delta H = H_{,ll} = \frac{1}{u^2}\Delta u u_t - \frac{2}{u^3}|\nabla u|^2 u_t + \frac{2}{u^2}(\nabla u \cdot \nabla u_t) - \frac{1}{u}\Delta u_t + \gamma e^u \Delta u + \gamma e^u |\nabla u|^2. \quad (12)$$

With the help of the first equation of problem (1) and defining $q = |\nabla u|^2$, we obtain

$$\begin{aligned} H_t &= \frac{1}{u^2}(u_t)^2 - \frac{1}{u}(u_t)_t + \gamma e^u u_t \\ &= \frac{1}{u^2}(u_t)^2 + \gamma e^u u_t - \frac{1}{u} \left(\frac{1}{r'(u)} \operatorname{div}(|\nabla u|^p \nabla u) + \frac{f(x,t,u,q)}{r'(u)} \right)_t \end{aligned} \quad (13)$$

$$\begin{aligned}
&= \frac{1}{u^2}(u_t)^2 + \gamma e^u u_t - \frac{1}{u} \left(\frac{1}{r'(u)} |\nabla u|^p \Delta u + p \frac{1}{r'(u)} |\nabla u|^{p-2} (u_{,l} u_{,k} u_{,lk}) + \frac{f(x, t, u, q)}{r'(u)} \right)_t \\
&= \frac{1}{u^2}(u_t)^2 + \gamma e^u u_t + \frac{r''}{u(r')^2} |\nabla u|^p \Delta u u_t - p \frac{1}{ur'} |\nabla u|^{p-2} \Delta u (\nabla u \cdot \nabla u_t) - \frac{1}{ur'} |\nabla u|^p \Delta u_t \\
&\quad + p \frac{r''}{u(r')^2} |\nabla u|^{p-2} (u_{,l} u_{,k} u_{,lk}) u_t - p(p-2) \frac{1}{ur'} |\nabla u|^{p-4} (u_{,l} u_{,k} u_{,lk}) (\nabla u \cdot \nabla u_t) \\
&\quad - 2p \frac{1}{ur'} |\nabla u|^{p-2} (u_{t,l} u_{,k} u_{,lk}) - p \frac{1}{ur'} |\nabla u|^{p-2} (u_{,l} u_{,k} u_{t,lk}) + \frac{f r''}{u(r')^2} u_t - \frac{f_u}{ur'} u_t - 2 \frac{f_q}{ur'} (\nabla u \cdot \nabla u_t) - \frac{f_t}{ur'}.
\end{aligned}$$

Using (11)–(13), we deduce

$$\begin{aligned}
&\frac{1}{r'} |\nabla u|^p \Delta H + p \frac{1}{r'} |\nabla u|^{p-2} (u_{,l} u_{,k} H_{,lk}) - H_t \\
&= \left(\frac{1}{u^2 r'} - \frac{r''}{u(r')^2} \right) |\nabla u|^p \Delta u u_t - 2(p+1) \frac{1}{u^3 r'} |\nabla u|^{p+2} u_t + 2(p+1) \frac{1}{u^2 r'} |\nabla u|^p (\nabla u \cdot \nabla u_t) \\
&\quad + \gamma \frac{e^u}{r'} |\nabla u|^p \Delta u + \gamma(p+1) \frac{e^u}{r'} |\nabla u|^{p+2} + p \left(\frac{1}{u^2 r'} - \frac{r''}{u(r')^2} \right) |\nabla u|^{p-2} (u_{,l} u_{,k} u_{,lk}) u_t \\
&\quad + \gamma p \frac{e^u}{r'} |\nabla u|^{p-2} (u_{,l} u_{,k} u_{,lk}) - \frac{1}{u^2} (u_t)^2 + \left(\frac{f_u}{ur'} - \frac{f r''}{u(r')^2} - \gamma e^u \right) u_t \\
&\quad + p \frac{1}{ur'} |\nabla u|^{p-2} \Delta u (\nabla u \cdot \nabla u_t) + p(p-2) \frac{1}{ur'} |\nabla u|^{p-4} (u_{,l} u_{,k} u_{,lk}) (\nabla u \cdot \nabla u_t) \\
&\quad + 2p \frac{1}{ur'} |\nabla u|^{p-2} (u_{t,l} u_{,k} u_{,lk}) + 2 \frac{f_q}{ur'} (\nabla u \cdot \nabla u_t) + \frac{f_t}{ur'}.
\end{aligned} \tag{14}$$

By (10), we derive

$$u_{t,l} = -u H_{,l} + \frac{1}{u} u_t u_{,l} + \gamma u e^u u_{,l} \tag{15}$$

and

$$\nabla u_t = -u \nabla H + \frac{1}{u} u_t \nabla u + \gamma u e^u \nabla u. \tag{16}$$

We substitute (15) and (16) into (14) to obtain

$$\begin{aligned}
&\frac{1}{r'} |\nabla u|^p \Delta H + p \frac{1}{r'} |\nabla u|^{p-2} (u_{,l} u_{,k} H_{,lk}) + \frac{1}{r'} \left[2(p+1) \frac{1}{u} |\nabla u|^p \right. \\
&\quad \left. + p |\nabla u|^{p-2} \Delta u + p(p-2) |\nabla u|^{p-4} (u_{,l} u_{,k} u_{,lk}) + 2f_q \right] (\nabla u \cdot \nabla H) + 2p \frac{1}{r'} |\nabla u|^{p-2} (u_{,k} u_{,lk} H_{,l}) - H_t \\
&= \left((p+1) \frac{1}{u^2 r'} - \frac{r''}{u(r')^2} \right) |\nabla u|^p \Delta u u_t + \gamma(p+1) \frac{e^u}{r'} \left(1 + \frac{2}{u} \right) |\nabla u|^{p+2} \\
&\quad + \gamma(p+1) \frac{e^u}{r'} |\nabla u|^p \Delta u + p \left((p+1) \frac{1}{u^2 r'} - \frac{r''}{u(r')^2} \right) |\nabla u|^{p-2} (u_{,l} u_{,k} u_{,lk}) u_t \\
&\quad + \gamma p(p+1) \frac{e^u}{r'} |\nabla u|^{p-2} (u_{,l} u_{,k} u_{,lk}) - \frac{1}{u^2} (u_t)^2 + \left(\frac{f_u}{ur'} - \frac{f r''}{u(r')^2} - \gamma e^u \right) u_t \\
&\quad + 2 \frac{f_q}{u^2 r'} |\nabla u|^2 u_t + 2 \gamma \frac{f_q e^u}{r'} |\nabla u|^2 + \frac{f_t}{ur'}.
\end{aligned} \tag{17}$$

From the first equation of problem (1), we infer

$$|\nabla u|^p \Delta u = r' u_t - p |\nabla u|^{p-2} (u_{,l} u_{,k} u_{,lk}) - f. \tag{18}$$

We substitute (18) into (17) to get

$$\begin{aligned} & \frac{1}{r'} |\nabla u|^p \Delta H + p \frac{1}{r'} |\nabla u|^{p-2} (u_{,l} u_{,k} H_{,lk}) + \frac{1}{r'} \left[2(p+1) \frac{1}{u} |\nabla u|^p + p |\nabla u|^{p-2} \Delta u \right. \\ & \quad \left. + p(p-2) |\nabla u|^{p-4} (u_{,l} u_{,k} u_{,lk}) + 2f_q \right] (\nabla u \cdot \nabla H) + 2p \frac{1}{r'} |\nabla u|^{p-2} (u_{,k} u_{,lk} H_{,l}) - H_t \\ & = \left(p \frac{1}{r'^2} - \frac{r''}{ur'} \right) (u_t)^2 + \left(\frac{f_u}{ur'} - (p+1) \frac{f}{u^2 r'} + \gamma p e^u \right) u_t \\ & \quad + \gamma (p+1) \frac{e^u}{r'} \left(1 + \frac{2}{u} \right) |\nabla u|^{p+2} - \gamma (p+1) \frac{f e^u}{r'} + 2 \frac{f_q}{u^2 r'} |\nabla u|^2 u_t + 2\gamma \frac{f_q e^u}{r'} |\nabla u|^2 + \frac{f_t}{ur'}. \end{aligned} \quad (19)$$

With (6), we have

$$u_t = -uH + \gamma u e^u. \quad (20)$$

We substitute (20) into (21) to deduce

$$\begin{aligned} & \frac{1}{r'} |\nabla u|^p \Delta H + p \frac{1}{r'} |\nabla u|^{p-2} (u_{,l} u_{,k} H_{,lk}) + \frac{1}{r'} \left[2(p+1) \frac{1}{u} |\nabla u|^p + p |\nabla u|^{p-2} \Delta u \right. \\ & \quad \left. + p(p-2) |\nabla u|^{p-4} (u_{,l} u_{,k} u_{,lk}) + 2f_q \right] (\nabla u \cdot \nabla H) + 2p \frac{1}{r'} |\nabla u|^{p-2} (u_{,k} u_{,lk} H_{,l}) \\ & \quad + \left[\left(\frac{ur''}{r'} - p \right) (H - 2\gamma e^u) + \frac{f_u}{r'} - (p+1) \frac{f}{ur'} + \gamma p u e^u + 2 \frac{f_q}{ur'} |\nabla u|^2 \right] H - H_t \\ & = \gamma (p+1) \frac{e^u}{r'} \left(1 + \frac{2}{u} \right) |\nabla u|^{p+2} + \gamma^2 u e^{2u} \left[p \left(1 + \frac{1}{u} \right) - \frac{r''}{r'} \right] \\ & \quad + \gamma \frac{f e^u}{r'} \left[\frac{f_u}{f} - (p+1) \left(1 + \frac{1}{u} \right) \right] + 2\gamma \frac{f_q e^u}{r} \left(1 + \frac{1}{u} \right) |\nabla u|^2 + \frac{f_t}{ur'}. \end{aligned} \quad (21)$$

Assumption (9) guarantees that the right end of (21) is nonnegative. In other words, we obtain that for $(x, t) \in D \times (0, T^*)$,

$$\begin{aligned} & \frac{1}{r'} |\nabla u|^p \Delta H + p \frac{1}{r'} |\nabla u|^{p-2} (u_{,l} u_{,k} H_{,lk}) + \frac{1}{r'} \left[2(p+1) \frac{1}{u} |\nabla u|^p + p |\nabla u|^{p-2} \Delta u \right. \\ & \quad \left. + p(p-2) |\nabla u|^{p-4} (u_{,l} u_{,k} u_{,lk}) + 2f_q \right] (\nabla u \cdot \nabla H) + 2p \frac{1}{r'} |\nabla u|^{p-2} (u_{,k} u_{,lk} H_{,l}) \\ & \quad + \left[\left(\frac{ur''}{r'} - p \right) (H - 2\gamma e^u) + \frac{f_u}{r'} - (p+1) \frac{f}{ur'} + \gamma p u e^u + 2 \frac{f_q}{ur'} |\nabla u|^2 \right] H - H_t \geq 0. \end{aligned} \quad (22)$$

Combining the regularity assumptions of functions r and f in Section 1 with maximum principles [22], it can be known from (22) that the function H can take its nonnegative maximum value on $\bar{D} \times [0, T^*)$ under the following three possible situations:

(i) in $\bar{D} \times \{0\}$; (ii) on $\partial D \times (0, T^*)$; (iii) at a point (x^*, t^*) in $D \times (0, T^*)$ where we have $|\nabla u(x^*, t^*)| = 0$.

Now first situation (i) is considered. According to the definition of constant γ in (4), we get that for $x \in \bar{D}$,

$$\begin{aligned} H(x, 0) &= -\frac{1}{u_0} \left(\frac{1}{r'(u_0)} [\operatorname{div}(|\nabla u_0|^p \nabla u_0) + f(x, 0, u_0, |\nabla u_0|^2)] \right) + \gamma e^{u_0} \\ &= e^{u_0} \left(\gamma - \frac{\operatorname{div}(|\nabla u_0|^p \nabla u_0) + f(x, 0, u_0, |\nabla u_0|^2)}{u_0 e^{u_0} r'(u_0)} \right) \leq 0. \end{aligned} \quad (23)$$

Then situation (ii) is considered. By means of the boundary condition of problem (1) and Assumption (8), we deduce that for $(x, t) \in \partial D \times (0, T^*)$,

$$\begin{aligned} \frac{\partial H}{\partial v} &= \frac{1}{u^2} u_t \frac{\partial u}{\partial v} - \frac{1}{u} \frac{\partial u_t}{\partial v} + \gamma e^u \frac{\partial u}{\partial v} = -\sigma \frac{1}{u} u_t - \frac{1}{u} \left(\frac{\partial u}{\partial v} \right)_t - \gamma \sigma u e^u \\ &= -\sigma \frac{1}{u} u_t - \frac{1}{u} (-\sigma u)_t - \gamma \sigma u e^u = -\gamma \sigma u e^u < 0. \end{aligned} \quad (24)$$

Finally, situation (iii) is considered. It follows from (5) and (8) that

$$\begin{aligned}
 H(x^*, t^*) &= \left(-\frac{1}{u} u_t + \gamma e^u \right) \Big|_{(x^*, t^*)} \\
 &= \left(-\frac{\operatorname{div}(|\nabla u|^p \nabla u) + f(x, t, u, |\nabla u|^2)}{ur'(u)} + \gamma e^u \right) \Big|_{(x^*, t^*)} \\
 &= \left(\frac{|\nabla u|^p (-\Delta u) + p|\nabla u|^{p-2} (-u_{,l} u_{,k} u_{,lk}) - f(x, t, u, |\nabla u|^2)}{ur'(u)} + \gamma e^u \right) \Big|_{(x^*, t^*)} \\
 &\leq \left(\frac{|\nabla u|^p |\Delta u| + p|\nabla u|^{p-2} |u_{,l}| |u_{,k}| |u_{,lk}| - f(x, t, u, |\nabla u|^2)}{ur'(u)} + \gamma e^u \right) \Big|_{(x^*, t^*)} \\
 &\leq \left(\frac{|\nabla u|^p |\Delta u| + p|\nabla u|^{p-2} |\nabla u| |\nabla u| |u_{,lk}| - f(x, t, u, |\nabla u|^2)}{ur'(u)} + \gamma e^u \right) \Big|_{(x^*, t^*)} \\
 &= \left(\frac{|\nabla u|^p (|\Delta u| + p|u_{,lk}|) - f(x, t, u, |\nabla u|^2)}{ur'(u)} + \gamma e^u \right) \Big|_{(x^*, t^*)} \\
 &= e^{u(x^*, t^*)} \left(\gamma - \frac{f(x^*, t^*, u(x^*, t^*), 0)}{u(x^*, t^*) e^{u(x^*, t^*)} r'(u(x^*, t^*))} \right) \\
 &\leq e^{u(x^*, t^*)} (\gamma - \eta) \leq 0.
 \end{aligned} \tag{25}$$

The maximum principle and (23)–(25) mean the maximum value of function H in $\bar{D} \times [0, T^*)$ must be zero. Therefore, we get

$$H(x, t) \leq 0, \quad (x, t) \in \bar{D} \times [0, T^*).$$

In other words, the following first-order differential inequality is obtained:

$$\frac{1}{\gamma u e^u} u_t \geq 1, \quad (x, t) \in \bar{D} \times [0, T^*]. \tag{26}$$

We now assume that $\hat{x}$ is the maximum point of u_0 in $\bar{D}$, that is, $u_0(\hat{x}) = \max_{x \in \bar{D}} u_0(x) = M$. By integrating (26) over $[0, t]$ at the point $\hat{x}$, we derive

$$\frac{1}{\gamma} \int_0^t \frac{1}{u e^u} u_t dt = \frac{1}{\gamma} \int_M^{u(\hat{x}, t)} \frac{1}{s e^s} ds \geq t, \tag{27}$$

which guarantees that u blows up at some finite time T^* . Actually, assuming that u remains global, then we know

$$0 < u(x, t) < +\infty, \quad (x, t) \in \bar{D} \times \mathbb{R}_+. \tag{28}$$

It follows from (27) and (28) that

$$\frac{1}{\gamma} \int_M^{+\infty} \frac{1}{s e^s} ds > \frac{1}{\gamma} \int_M^{u(\hat{x}, t)} \frac{1}{s e^s} ds \geq t, \quad t \in \mathbb{R}_+. \tag{29}$$

In (29), we take the limit $t \rightarrow +\infty$ and draw the following conclusion:

$$\frac{1}{\gamma} \int_M^{+\infty} \frac{1}{s e^s} ds = +\infty,$$

which contradicts the following conclusion:

$$\frac{1}{\gamma} \int_M^{+\infty} \frac{1}{s e^s} ds < +\infty.$$

This contradiction shows that the solution u blows up at some finite time T^* . Hence, in (27) we take the limit $t \rightarrow T^*$ to get

$$T^* \leq \frac{1}{\gamma} \int_M^{+\infty} \frac{1}{se^s} ds.$$

In the interval $[0, \tilde{t}]$, we integrate (26) to infer

$$G(u(x, t)) \geq G(u(x, t)) - G(u(x, \tilde{t})) = \int_{u(x, t)}^{u(x, \tilde{t})} \frac{1}{se^s} ds \geq \gamma(\tilde{t} - t), \quad (30)$$

where x is a fixed point in $\bar{D}$ and $\tilde{t}$ satisfies $0 < t < \tilde{t} < T^*$. In (30), we get the following conclusions by taking the limit $\tilde{t} \rightarrow T^*$:

$$G(u(x, t)) \geq \gamma(T^* - t),$$

which implies

$$u(x, t) \leq G^{-1}(\gamma(T^* - t)), \quad (x, t) \in \bar{D} \times [0, T^*].$$

At this point, we have completed the proof of Theorem 2.1. $\square$

Since all conclusions of Theorem 2.1 are correct when $r(u) \equiv u$ and $f(x, t, u, |\nabla u|^2) \equiv f(u)$, we have the following results on blow-up solution.

Corollary 2.1. *Let $u \in C^3(D \times (0, T^*)) \cap C^2(\bar{D} \times [0, T^*))$ be a positive solution of the following problems:*

$$\begin{cases} u_t = \operatorname{div}(|\nabla u|^p \nabla u) + f(u), & (x, t) \in D \times (0, T^*), \\ \frac{\partial u}{\partial \nu} + \sigma u = 0, & (x, t) \in \partial D \times (0, T^*), \\ u(x, 0) = u_0(x). & x \in \bar{D}. \end{cases}$$

Here $p > 0$, the spatial region $D \subset \mathbb{R}^n$ ($n \geq 2$) is bounded, and its boundary ∂D is smooth. Suppose the following:

(i)

$$0 < \gamma^* \leq \eta^*,$$

where

$$\gamma^* = \min_{x \in \bar{D}} \frac{\operatorname{div}(|\nabla u_0|^p \nabla u_0) + f(u_0)}{u_0 e^{u_0}}, \quad \eta^* = \inf_{h \in \mathbb{R}_+} \frac{f(h)}{h e^h}.$$

(ii) For $h \in \mathbb{R}_+$,

$$\frac{f'(h)}{f(h)} - (p+1) \left(1 + \frac{1}{h}\right) \geq 0.$$

Then, u blows up at some finite time T^* , T^* is bounded from above by

$$T^* \leq \frac{1}{\gamma^*} \int_M^{+\infty} \frac{1}{se^s} ds, \quad M = \max_{x \in \bar{D}} u_0(x),$$

and $u(x, t)$ has the following upper estimate:

$$u(x, t) \leq G^{-1}(\gamma^*(T^* - t)), \quad (x, t) \in \bar{D} \times [0, T^*].$$

3 Application

Now, we give the following example to illustrate the conclusions of Theorem 2.1 in Section 2.

Example 3.1. Let $u \in C^3(D \times (0, T^*)) \cap C^2(\bar{D} \times [0, T^*))$ be a positive solution of the following problem:

$$\begin{cases} ((u-1)e^u)_t = \operatorname{div}(|\nabla u| \nabla u) + (1+t|x|^2|\nabla u|^2)u^2e^u, & (x, t) \in D \times (0, T^*), \\ \frac{\partial u}{\partial \nu} + 2u = 0, & (x, t) \in \partial D \times (0, T^*), \\ u(x, 0) = 2 - |x|^2, & x \in \bar{D}. \end{cases}$$

Here $x = (x_1, x_2, x_3)$, $|x| = \left(\sum_{i=1}^3 x_i^2\right)^{\frac{1}{2}}$ and spatial region $D = \{x | |x| < 1\}$. We note

$$r(u) = (u-1)e^u, \quad f(x, t, u, |\nabla u|^2) = (1+t|x|^2|\nabla u|^2)u^2e^u, \quad u_0(x) = 2 - |x|^2, \quad p = 1, \quad \sigma = 2.$$

From (4) and (5), we have

$$\gamma = \min_{x \in \bar{D}} \frac{\operatorname{div}(|\nabla u_0|^p \nabla u_0) + f(x, 0, u_0, |\nabla u_0|^2)}{u_0 e^{u_0} r'(u_0)} = \min_{x \in \bar{D}} \frac{\operatorname{div}(|\nabla u_0|^p \nabla u_0) + 3u_0^2 e^{2u_0}}{u_0^2 e^{2u_0}} = \min_{u_0 \in [1, 2]} \left(3 - 16 \frac{\sqrt{2-u_0}}{u_0^2 - e^{2u_0}} \right) = 3 - \frac{16}{e^2}$$

and

$$\eta = \inf_{(x, t, h) \in D \times \mathbb{R}_+ \times \mathbb{R}_+} \frac{f(x, t, h, 0)}{h e^{hr}(h)} = \inf_{(x, t, h) \in D \times \mathbb{R}_+ \times \mathbb{R}_+} 3 = 3,$$

which means that Assumption (8) is true. We can easily verify that Assumption (9) is also true. Hence, it follows from Theorem 2.1 that u blows up at some finite time T^* and

$$\begin{aligned} T^* &\leq \frac{1}{\gamma} \int_M^{\infty} \frac{1}{se^s} ds = \frac{1}{3 - \frac{16}{e^2}} \int_2^{\infty} \frac{1}{se^s} ds = 0.0586, \\ u(x, t) &\leq G^{-1}(\gamma(T^* - t)) = G^{-1}\left(\left(3 - \frac{16}{e^2}\right)(T^* - t)\right), \end{aligned}$$

where G^{-1} is the inverse function of function G defined by (7).

4 Conclusions

This paper is devoted to discussing the blow-up problem of the positive solution of problem (1). We mainly use the research methods in paper [13]. In other words, we rely on maximum principles and first-order differential inequality techniques for research. In using this research method to study problem (1), the biggest difficulty is that some suitable auxiliary functions need to be established. Since the main part of the first equation in problem (1) is different from the main part of the first equation in problem (2), the auxiliary functions that have been established in the paper [13] cannot be used to study problem (1). There, the key and difficult point of our research is to establish auxiliary functions (6) and (7). With the help of auxiliary functions (6) and (7), we give the conditions that cause the positive solution of problem (1) to blow up. At the same time, for the positive blow-up solution of problem (1), we also obtain an upper bound of the blow-up time and an upper estimate of the blow-up rate.

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