Research Article

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so-metrizable spaces and images of metric spaces

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Abstract: *so*-metrizable spaces are a class of important generalized metric spaces between metric spaces and *sn*-metrizable spaces where a space is called an *so*-metrizable space if it has a σ -locally finite *so*-network. As the further work that attaches to the celebrated Alexandrov conjecture, it is interesting to characterize *so*-metrizable spaces by images of metric spaces. This paper gives such characterizations for *so*-metrizable spaces. More precisely, this paper introduces *so*-open mappings and uses the “Ponomarev’s method” to prove that a space X is an *so*-metrizable space if and only if it is an *so*-open, compact-covering, σ -image of a metric space, if and only if it is an *so*-open, σ -image of a metric space. In addition, it is shown that *so*-open mapping is a simplified form of *sn*-open mapping (resp. 2-sequence-covering mapping if the domain is metrizable). Results of this paper give some new characterizations of *so*-metrizable spaces and establish some equivalent relations among *so*-open mapping, *sn*-open mapping and 2-sequence-covering mapping, which further enrich and deepen generalized metric space theory.

Keywords: *so*-network, *so*-metrizable space, *so*-open mapping, compact-covering mapping, σ -mapping

MSC 2020: 54E35, 54E40, 54E45, 54E50

1 Introduction

As a class of important generalized metric spaces between *sn*-metrizable spaces and metric spaces, *so*-metrizable spaces were introduced by Ge in [1]. Here, a space is an *so*-metrizable space [1] if it has a σ -locally finite *so*-network. In Lin’s paper [2], an *so*-metrizable space was defined as a space with a σ -locally finite sequentially open network. Since then, people have been interested in this topic and many interesting results were obtained (e.g., see [1–6]). However, all these results that do not deal with images of metric spaces only demonstrate internal characterizations of *so*-metrizable spaces. This leads us to consider the following question, which comes from the celebrated Alexandrov conjecture [7] (also see [4]) and is a central question in generalized metric space theory.

Question 1.1. How characterize generalized metric spaces by images of metric spaces?

Note that it is difficult to discuss Question 1.1 by using known mapping and general method. Lin and Yan introduced σ -mapping and used “Ponomarev’s method” to characterize *g*-metrizable spaces by compact-covering, quotient, compact, σ -images of metric spaces [8]. Here, “Ponomarev’s method” was established by Ponomarev in order to characterize first countable spaces by open images of metric spaces [9].

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The key of this method is to construct a metric space M and a mapping $f : M \rightarrow X$ such that $f(M) = X$, where X is a given generalized metric space. In the past few years, many answers to Question 1.1 around some generalized metric spaces were obtained (see, e.g., [4,10–14]). This stimulates the following question (i.e., Question 1.1 with respect to *so*-metrizable spaces), which has no answers are readily available.

Question 1.2. How characterize *so*-metrizable spaces by images of metric spaces?

Our discussion spreads around Question 1.2. Having gained some enlightenments from previous work of Ponomarev [9] and Lin and Yan [8], we introduce *so*-open mappings and use the “Pomomarev’s method” to prove that a space X is an *so*-metrizable space if and only if X is an *so*-open, compact-covering, σ -image of a metric space, if and only if X is an *so*-open, σ -image of a metric space. In addition, we discuss some relations among *so*-open mapping, *sn*-open mapping and 2-sequence-covering mapping to show that *so*-open mapping is a simplified form of *sn*-open mapping (resp. 2-sequence-covering mapping if the domain is metrizable). That is, we obtain that a mapping f is an *so*-open mapping if and only if f is an *sn*-open mapping (resp. 2-sequence-covering mapping if the domain is metrizable). This makes it possible to replace *sn*-open mappings (resp. 2-sequence-covering mappings if the domain is metrizable) that appeared in some relevant papers with *so*-open mappings. Results of this paper give some new characterizations of *so*-metrizable spaces and establish some equivalent relations among *so*-open mapping, *sn*-open mapping and 2-sequence-covering mapping, which further enrich and deepen generalized metric space theory.

Throughout this paper, all spaces are regular and T_1 , all mappings are continuous and onto. $\mathbb{N}$ denotes the set of all natural numbers. Let P be a subset of a space X and $\{x_n\}$ be a sequence in X converging to $x \in X$. $\{x_n\}$ is eventually in P if $\{x_n : n > k\} \cup \{x\} \subset P$ for some $k \in \mathbb{N}$; it is frequently in P if $\{x_{n_k}\}$ is eventually in P for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Let $\mathcal{P}$ be a family of subsets of a space X , $x \in X$ and f be a mapping from X . $\bigcup \mathcal{P}$ and $\bigcap \mathcal{P}$ denote the union $\bigcup \{P : P \in \mathcal{P}\}$ and the intersection $\bigcap \{P : P \in \mathcal{P}\}$, respectively. $(\mathcal{P})_x = \{P \in \mathcal{P} : x \in P\}$ and $st(x, \mathcal{P}) = \bigcup (\mathcal{P})_x$. $f(\mathcal{P})$ denotes $\{f(P) : P \in \mathcal{P}\}$. A point $b = (\beta_n)_{n \in \mathbb{N}}$ of a Tychonoff-product space is abbreviated to (β_n) , where β_n is the n th coordinate of b .

2 Definitions and known lemmas

Definition 2.1. Let X be a space.

- (1) Let $x \in P \subset X$. P is called a sequential neighborhood of x in X [15] if whenever $\{x_n\}$ is a sequence in X converging to x , then $\{x_n\}$ is eventually in P .
- (2) Let $P \subset X$. P is called a sequentially-open subset of X [16] if P is a sequential neighborhood of x in X for each $x \in P$.

Remark 2.2. [17,18] The following are known.

- (1) P is a sequential neighborhood of x in X iff each sequence $\{x_n\}$ in X converging to x is frequently in P .
- (2) The intersection of finitely many sequential neighborhoods of x in X is a sequential neighborhood of x in X .
- (3) The intersection of finitely many sequentially-open subsets of X is a sequentially-open subset of X .

Definition 2.3. Let $\mathcal{P} = \bigcup \{\mathcal{P}_x : x \in X\}$ be a family of subsets of X , such that each $A \in \mathcal{P}_x$, $x \in A$.

- (1) $\mathcal{P}$ is called a network for X [19], if whenever $x \in U$ with U open in X there is $P \in \mathcal{P}_x$ such that $x \in P \subset U$. Here, the $\mathcal{P}_x$ is called a network at x in X .
- (2) $\mathcal{P}$ is called an *so*-network for X [1], if each element of $\mathcal{P}$ is a sequentially-open subset of X , and the following (a) and (b) hold for each $x \in X$.
 - (a) $\mathcal{P}_x$ is a network at x in X .
 - (b) If $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.
 Here, the $\mathcal{P}_x$ is called an *so*-network at x in X .

Definition 2.4. [1] A space X is called an so-metrizable space if X has a σ -locally finite so-network.

Remark 2.5. If X has a σ -locally finite network $\mathcal{P}$ consisting of sequentially-open subsets of X , then X is an so-metrizable space. In fact, put $\mathcal{F} = \{\bigcap \mathcal{P}' : \mathcal{P}' \text{ is a finite subfamily of } \mathcal{P}\}$, then $\mathcal{F}$ is a σ -locally finite so-network for X .

Definition 2.6. Let $\mathcal{P}$ be a family of subsets of a space X .

- (1) Let K be a subset of X . $\mathcal{P}$ is called a *cfp*-cover of K [8], if $\mathcal{P}$ is a finite cover of K in X such that it can be precisely refined by some finite cover of K consisting of closed subsets of K .
- (2) Let K be a subset of X . $\mathcal{P}$ is called to have property *cc* for K [20], if whenever H is a compact subset of K , and $H \subset U$ with U open in X , there is a subfamily $\mathcal{P}_H$ of $\mathcal{P}$ such that $\mathcal{P}_H$ is a *cfp*-cover of H and $\bigcup \mathcal{P}_H \subset U$.
- (3) $\mathcal{P}$ is called a *cfp*-network for X [21], if whenever K is a compact subset of X and $K \subset U$ with U open in X , there is a finite subfamily $\mathcal{P}_K$ of $\mathcal{P}$ such that $\mathcal{P}_K$ is a *cfp*-cover of K and $\bigcup \mathcal{P}_K \subset U$.
- (4) $\mathcal{P}$ is called a strong k -network for X [20], if whenever K is a compact subset of X , there is a countable subfamily $\mathcal{P}_K$ of $\mathcal{P}$ such that $\mathcal{P}_K$ has property *cc* for K .
- (5) $\mathcal{P}$ is called a k -network for X [22], if whenever $K \subset U$ with K compact in X and U open in X , there is a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ such that $K \subset \bigcup \mathcal{F} \subset U$.

Definition 2.7. Let $f : X \rightarrow Y$ be a mapping.

- (1) f is called an so-open mapping if whenever U is an open subset of X , $f(U)$ is a sequentially-open subset of Y .
- (2) f is called a compact-covering mapping [23] if whenever K is a compact subset of Y , there is a compact subset C in X such that $f(C) = K$.
- (3) f is called a σ -mapping [8] if there is a base $\mathcal{B}$ for X such that $f(\mathcal{B})$ is σ -locally-finite in Y .

Remark 2.8.

- (1) Let $f : X \rightarrow Y$ be a mapping and $\mathcal{B}$ be a base for X . If $f(B)$ is a sequentially-open subset of Y for each $B \in \mathcal{B}$, then f is an so-open mapping.
- (2) If $f : X \rightarrow Y$ is a σ -mapping, then X is a metric space [24, Remark 1].

The following two lemmas came from [25, Theorem 3.3] and [26, Lemma 3.8], respectively.

Lemma 2.9. [25] Let X be a compact space with a point-countable k -network. Then X is metrizable.

Lemma 2.10. [26] Let $\mathcal{P}$ be a point-countable cover of a space X . Then $\mathcal{P}$ is a *cfp*-network for X iff $\mathcal{P}$ is a strong k -network for X .

The following lemma is a particular case of [27, Proposition 1.2(2)].

Lemma 2.11. Let $\mathcal{P}$ be a σ -locally finite so-network for a space X . Then $\mathcal{P}$ is a k -network for X .

3 The main results

The following lemma shows that Lemma 2.11 can be improved by strengthening “ k -network” to “strong k -network”.

Lemma 3.1. Let $\mathcal{P}$ be a σ -locally finite so-network for a space X . Then $\mathcal{P}$ is a strong k -network for X .

Proof. By Lemma 2.10, we only need to prove that $\mathcal{P}$ is a *cfp*-network for X . Let K be a compact subset of X and $K \subset U$ with U open in X . Then it suffices to prove that there is a finite subfamily $\mathcal{P}_K$ of $\mathcal{P}$ such that $\mathcal{P}_K$ is a *cfp*-cover of K and $\bigcup \mathcal{P}_K \subset U$. By Lemma 2.11, $\mathcal{P}$ is a point-countable k -network for X . It follows that $\{P \cap K : P \in \mathcal{P}\}$ is a point-countable k -network for K . By Lemma 2.9, the subspace K is metrizable. Furthermore, for each $x \in K$, it is known that there is $P_x \in \mathcal{P}_x$ such that $P_x \subset U$. In addition, U_x is constructed as follows.

Let $\{V_n\}$ be a decreasing neighborhood base at x in K . For every $P \in \mathcal{P}$, $P \cap K$ is open in K because $P \cap K$ is sequentially-open in K and K is metrizable. Put $F_x = P_x \cap K$. Then there is $m \in \mathbb{N}$ such that $V_m \subset F_x \subset P_x \subset U$. Thus, F_x is a neighborhood of x in K . So $x \in \text{int}_K(F_x) \subset P_x \subset U$, where $\text{int}_K(F_x)$ is the interior of F_x in K . By regularity of K , there is an open (in K) subset U_x of K such that $x \in U_x \subset \text{cl}_K(U_x) = \text{cl}(U_x) \subset \text{int}_K(F_x)$, where $\text{cl}_K(U_x)$ and $\text{cl}(U_x)$ are closures of U_x in K and X , respectively. Consequently, U_x is constructed.

Since $\{U_x : x \in K\}$ is an open (in K) cover of K , there is a finite subset K' of K such that $\{U_x : x \in K'\}$ covers K . Put $\mathcal{P}_K = \{P_x : x \in K'\}$. Then $\mathcal{P}_K$ is a finite subfamily of $\mathcal{P}$ and $\bigcup \mathcal{P}_K \subset U$. Moreover, $K = \bigcup \{\text{cl}(U_x) : x \in K'\}$ and $\text{cl}(U_x) \subset \text{int}_K(F_x) \subset P_x$ for each $x \in K'$. This shows that $\mathcal{P}_K$ is a *cfp*-cover of K . $\square$

The following theorem is the main theorem in this paper, where “Pomomarev’s method” is used in its proof. That is, in the proof of this theorem, we are based on a given *so*-metrizable space X to construct a metric space M and an *so*-open, compact-covering, σ -mapping $f : M \rightarrow X$ such that $f(M) = X$.

Theorem 3.2. *The following are equivalent for a space X .*

- (1) X is an *so*-metrizable space.
- (2) X is an *so*-open, compact-covering, σ -image of a metric space.
- (3) X is an *so*-open, σ -image of a metric space.

Proof. (1) $\Rightarrow$ (2): Let X be an *so*-metrizable space, and let $\mathcal{P} = \{P_\beta : \beta \in \Lambda\}$ be a σ -locally finite *so*-network for X . For each $n \in \mathbb{N}$, put $\Lambda_n = \Lambda$ and endow Λ_n a discrete topology. Put

$$M = \{b = (\beta_n) \in \prod_{n \in \mathbb{N}} \Lambda_n : \{P_{\beta_n}\} \text{ is a network at some point } x_b \text{ in } X\},$$

then M is a metric space and x_b is unique for each $b \in M$. Define $f : M \rightarrow X$ by $f(b) = x_b$, then f is continuous and onto, so f is a mapping. We only need to prove the following three facts.

Fact 1. f is a σ -mapping.

Put $\mathcal{P}' = \{\bigcap \mathcal{F} : \mathcal{F} \text{ is a finite subfamily of } \mathcal{P}\}$, then $\mathcal{P}'$ is σ -locally finite because $\mathcal{P}$ is σ -locally finite. For each $k \in \mathbb{N}$, put $\mathcal{B}_k = \{((\prod_{n \leq k} \{\beta_n\}) \times (\prod_{n > k} \Lambda_n)) \cap M : \beta_n \in \Lambda_n \text{ for each } n \leq k\}$ and put $\mathcal{B} = \bigcup_{k \in \mathbb{N}} \mathcal{B}_k$. Then $\mathcal{B}$ is a base for M . We only need to prove that $f(\mathcal{B}) \subset \mathcal{P}'$. Let $B = ((\prod_{n \leq k} \{\beta_n\}) \times (\prod_{n > k} \Lambda_n)) \cap M \in \mathcal{B}$. It suffices to show that $f(B) = \bigcap_{n \leq k} P_{\beta_n}$. Let $x = f(c)$, where $c = (\gamma_n) \in B$. Then $\gamma_n = \beta_n$ for each $n \leq k$, and hence $x = f(c) \in P_{\gamma_n} = P_{\beta_n}$ for each $n \leq k$. So $f(B) \subset \bigcap_{n \leq k} P_{\beta_n}$. Conversely, let $x \in \bigcap_{n \leq k} P_{\beta_n}$. Then there is $a = (\alpha_n) \in M$ such that $f(a) = x$, that is, $\{P_{\alpha_n}\}$ is a network at x in X . Put $\gamma_n = \beta_n$ for each $n \leq k$ and put $\gamma_n = \alpha_{n-k}$ for each $n > k$. It is clear that $\{P_{\gamma_n}\}$ is a network at x in X . Put $c = (\gamma_n)$, then $x = f(c)$ and $c \in B$. So $x \in f(B)$, this proves that $\bigcap_{n \leq k} P_{\beta_n} \subset f(B)$. Thus, $f(B) = \bigcap_{n \leq k} P_{\beta_n}$.

Fact 2. f is a compact-covering mapping.

Let K be a non-empty compact subset of X . Since $\mathcal{P}$ is a strong k -network for X (by Lemma 3.1), there is a countable subfamily $\mathcal{P}_K$ of $\mathcal{P}$ such that $\mathcal{P}_K$ has property *cc* for K , and the family of *cfp*-covers of K consisting of elements of $\mathcal{P}_K$ is countable. Let $\{\mathcal{P}_n : n \in \mathbb{N}\}$ be the family. For each $n \in \mathbb{N}$, put $\mathcal{P}_n = \{P_\beta : \beta \in \Gamma_n\}$, where Γ_n is a finite subset of Λ . Note that $\mathcal{P}_n$ is a *cfp*-cover of K , there is a cover $\mathcal{F}_n = \{F_\beta : \beta \in \Gamma_n\}$ of K consisting of compact subsets of K such that $F_\beta \subset P_\beta$ for each $\beta \in \Gamma_n$. Put $C = \{(\beta_n) \in \prod_{n \in \mathbb{N}} \Gamma_n : \bigcap_{n \in \mathbb{N}} F_{\beta_n} \neq \emptyset\}$. It suffices to prove the following (a), (b) and (c).

(a) $C \subset M$ and $f(C) \subset K$.

Let $b = (\beta_n) \in C$, then $\bigcap_{n \in \mathbb{N}} F_{\beta_n} \neq \emptyset$. Choose $x \in \bigcap_{n \in \mathbb{N}} F_{\beta_n}$. We only need to prove that $\{P_{\beta_n} : n \in \mathbb{N}\}$ is a network at x in X , then $b \in M$ and $f(b) = x \in K$, thus $C \subset M$ and $f(C) \subset K$. Let V be a neighborhood of x in X . By regularity of K , there is an open neighborhood W of x in the subspace K such that $W \subset E \subset V$, where E is the closure of W in the subspace K . Since E is a compact subset of K and $\mathcal{P}_K$ has cc-property on K , there is a finite family $\mathcal{F}'$ of $\mathcal{P}_K$ such that $\mathcal{F}'$ is a cfp-cover of E and $\bigcup \mathcal{F}' \subset V$. Similarly, since $K - W$ is a compact subset of K and $K - W \subset X - \{x\}$, there is a finite family $\mathcal{F}''$ of $\mathcal{P}_K$ such that $\mathcal{F}''$ is a cfp-cover of $K - W$ and $\bigcup \mathcal{F}'' \subset X - \{x\}$. Put $\mathcal{F} = \mathcal{F}' \cup \mathcal{F}''$, then $\mathcal{F}$ is a cfp-cover of K , so $\mathcal{F} = \mathcal{P}_n$ for some $n \in \mathbb{N}$. Note that $\bigcup \mathcal{F}'' \subset X - \{x\}$ and $x \in F_{\beta_n} \subset P_{\beta_n} \in \mathcal{P}_n = \mathcal{F} = \mathcal{F}' \cup \mathcal{F}''$. So $P_{\beta_n} \notin \mathcal{F}''$. It follows that $P_{\beta_n} \in \mathcal{F}'$, and hence $P_{\beta_n} \subset V$. Thus, we prove that $\{P_{\beta_n} : n \in \mathbb{N}\}$ is a network at x in X .

(b) $K \subset f(C)$.

Let $x \in K$. For each $n \in \mathbb{N}$, choose $\beta_n \in \Gamma_n$ such that $x \in F_{\beta_n}$. Put $b = (\beta_n)$, then $b \in C$. By the same method as in the proof of (a), it is easy to prove that $f(b) = x$. So $K \subset f(C)$.

(c) C is a compact subset of M .

Note that $\prod_{n \in \mathbb{N}} \Gamma_n$ is a compact subset of $\prod_{n \in \mathbb{N}} \Lambda_n$. It suffices to prove that C is a closed subset of $\prod_{n \in \mathbb{N}} \Gamma_n$. Obviously, $C \subset \prod_{n \in \mathbb{N}} \Gamma_n$. Put $b = (\beta_n) \in \prod_{n \in \mathbb{N}} \Gamma_n - C$, then $\bigcap_{n \in \mathbb{N}} F_{\beta_n} = \emptyset$. By the compactness of K , there is $n_0 \in \mathbb{N}$ such that $\bigcap_{n \leq n_0} F_{\beta_n} = \emptyset$. Let $W = (\prod_{n \leq n_0} \{\beta_n\}) \times (\prod_{n > n_0} \Gamma_n)$, then W is an open subset of $\prod_{n \in \mathbb{N}} \Gamma_n$ such that $b \in W$ and $W \cap C = \emptyset$. Therefore, C is a closed subset of $\prod_{n \in \mathbb{N}} \Gamma_n$.

Fact 3. f is an so-open mapping.

Put $\mathcal{B}_k = \{(\prod_{n \leq k} \{\beta_n\}) \times (\prod_{n > k} \Lambda_n) \cap M : \beta_n \in \Lambda_n \text{ for each } n \leq k\}$ and put $\mathcal{B} = \bigcup_{k \in \mathbb{N}} \mathcal{B}_k$. Then $\mathcal{B}$ is a base for M . Whenever $B \in \mathcal{B}$, then $B = (\prod_{n \leq k} \{\beta_n\}) \times (\prod_{n > k} \Lambda_n) \cap M$ for some $k \in \mathbb{N}$ and $\beta_n \in \Lambda_n$ ($n \leq k$). In the proof of Fact 1, we obtain that $f(B) = \bigcap_{n \leq k} P_{\beta_n}$. Note that P_{β_n} is a sequentially-open subset of Y for each $n \leq k$. By Remark 2.2(3), $f(B)$ is a sequentially-open subset of Y . It follows that f is an so-open mapping from Remark 2.8(1).

(2) $\Rightarrow$ (3): It is clear.

(3) $\Rightarrow$ (1): Let $f : M \longrightarrow X$ be an so-open, σ -mapping, where M is a metric space. Since f is a σ -mapping, there is a base $\mathcal{B}$ of M such that $f(\mathcal{B})$ is σ -locally-finite in X . By Remark 2.5, it suffices to prove the following two facts.

Fact 1. $f(\mathcal{B})$ is a network for X .

If $x \in X$ and U is a neighborhood of x , then $f^{-1}(x) \subset f^{-1}(U)$. Choose $b \in f^{-1}(x) \subset f^{-1}(U)$. $\mathcal{B}$ is a base for M , so there is $B \in \mathcal{B}$ such that $b \in B \subset f^{-1}(U)$. Thus, $x = f(b) \in f(B) \subset ff^{-1}(U) = U$ and $f(B) \in f(\mathcal{B})$. This proves that $f(\mathcal{B})$ is a network for X .

Fact 2. Each element of $f(\mathcal{B})$ is a sequentially-open subset of X .

Let $B \in \mathcal{B}$, then B is an open subset of M . f is an so-open mapping, so $f(B)$ is a sequentially-open subset of X . This proves that each element of $f(\mathcal{B})$ is a sequentially-open subset of X . $\square$

In [28], Ge introduced sn -open mappings, which is lengthy and complicated. We point out that so-open mappings are a simple form of sn -open mappings.

Definition 3.3. A network $\mathcal{P} = \bigcup \{\mathcal{P}_x : x \in X\}$ is called an sn -network for X [29] if the following conditions hold for each $x \in X$.

- (a) $\mathcal{P}_x$ is a network at x in X .
- (b) If $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.
- (c) Each element of $\mathcal{P}_x$ is a sequential neighborhood of x in X .

Here, the $\mathcal{P}_x$ is called an sn -network at x in X .

Definition 3.4. A mapping $f : X \longrightarrow Y$ is called an *sn-open mapping* [28] if for each $y \in Y$, there is an *sn-network* $\mathcal{P}_y$ at y in Y such that for each $x \in f^{-1}(y)$, whenever U is a neighborhood of x in X , then $P \subset f(U)$ for some $P \in \mathcal{P}_y$.

Proposition 3.5. Let $f : X \longrightarrow Y$ be a mapping. Then the following are equivalent.

- (1) f is an *so-open mapping*.
- (2) f is an *sn-open mapping*.

Proof. (1) $\Rightarrow$ (2): Let f be an *so-open mapping*. For each $x \in X$, let $\mathcal{B}_x$ be an open neighborhood base at x in X . For each $y \in Y$, put $\mathcal{P}'_y = \bigcup \{f(B) : B \in \mathcal{B}_x : x \in f^{-1}(y)\}$ and $\mathcal{P}_y = \{\bigcap \mathcal{P}' : \mathcal{P}' \text{ is a finite subfamily of } \mathcal{P}'_y\}$. Now we prove that $\mathcal{P}_y$ is an *sn-network* at y in Y . First, let $y \in G$ with G open in Y . Choose $x \in f^{-1}(y) \subset f^{-1}(G)$. Then there is $B \in \mathcal{B}_x$ such that $x \in B \subset f^{-1}(G)$. Thus, $y = f(x) \in f(B) \subset ff^{-1}(G) = G$ and $f(B) \in \mathcal{P}'_y$. So $\mathcal{P}_y$ is a network at y in Y . Second, it is clear that $\mathcal{P}_y$ is closed under finite intersections. Third, we need to prove that each element of $\mathcal{P}_y$ is a sequential neighborhood of y in Y . By Remark 2.2(2), it suffices to prove that each element of $\mathcal{P}'_y$ is a sequential neighborhood of y in Y . Let $P \in \mathcal{P}'_y$, then there is $B \in \mathcal{B}_x$ such that $P = f(B)$ for some $x \in f^{-1}(y)$. Since f is an *so-open mapping* and B is an open subset of X , $P = f(B)$ is a sequentially-open subset of Y . It follows that P is a sequential neighborhood of y in Y . By the above, $\mathcal{P}_y$ is an *sn-network* at y in Y . If $x \in f^{-1}(y)$ and W is a neighborhood of x in X , then there is $B \in \mathcal{B}_x$ such that $x \in B \subset W$. Put $P = f(B)$, then $P \subset f(W)$ for some $P \in \mathcal{P}_y$. Thus, f is an *sn-open mapping*.

(2) $\Rightarrow$ (1): Let f be an *sn-open mapping*. If $y \in Y$, then there is an *sn-network* $\mathcal{P}_y$ at y in Y such that for each $x \in f^{-1}(y)$, whenever U is a neighborhood of x in X , then $P \subset f(U)$ for some $P \in \mathcal{P}_y$. Let W be an open subset of X and $y \in f(W)$. It suffices to prove that $f(W)$ is a sequential neighborhood of y in Y . In fact, choose $x \in W \cap f^{-1}(y)$. Since W is a neighborhood of x in X , there is $P \in \mathcal{P}_y$ such that $P \subset f(W)$. Note that P is a sequential neighborhood of y in Y . It follows that $f(W)$ is a sequential neighborhood of y in Y . $\square$

Proposition 3.5 shows that “*so-open*” in Theorem 3.2 can be replaced by “*sn-open*”. Recall that a mapping $f : X \longrightarrow Y$ is called a 2-sequence-covering mapping [2] if for each $y \in Y$ and each $x \in f^{-1}(y)$, then whenever $\{y_n\}$ is a sequence in Y converging to y , there is a sequence $\{x_n\}$ in X converging to x with each $x_n \in f^{-1}(y_n)$. Ge proved that a mapping from a metric space is a 2-sequence-covering mapping if and only if it is an *sn-open mapping* [28]. So “*so-open*” in Theorem 3.2 can also be replaced by “2-sequence-covering”. In fact, the following remark shows that Proposition 3.5 is more useful to simplify the description of “*sn-open mapping*” or “2-sequence-covering mapping”.

Remark 3.6. *sn-open mapping* (resp. 2-sequence-covering mapping if the domain is metrizable) that appeared in some relevant papers can be replaced with *so-open mapping*.

4 Conclusions

In this paper, we characterize *so-metrizable spaces* by images of metric spaces. More precisely, we introduce *so-open mapping* and use the “Pomomarev’s method” to obtain the main result of this paper as follows.

Conclusion 4.1. A space X is an *so-metrizable space* if and only if X is an *so-open*, compact-covering, σ -image of a metric space, if and only if X is an *so-open*, σ -image of a metric space.

In addition, note that *sn-open mapping* that was introduced by Ge [28] is lengthy and complicated. We prove that *so-open mapping* is a simplified form of *sn-open mapping*.

Conclusion 4.2. Let $f : X \longrightarrow Y$ be a mapping. Then f is an *so*-open mapping if and only if f is an *sn*-open mapping.

Thus, by Conclusion 4.2, *so*-open mapping is also a simplified form of 2-sequence-covering mapping if the domain is a metric space.

Conclusion 4.3. Let $f : X \longrightarrow Y$ be a mapping where X is a metric space. Then f is an *so*-open mapping if and only if f is a 2-sequence-covering mapping.

By Conclusions 4.2 and 4.3, *so*-open in Conclusion 4.1 can be replaced by *sn*-open or 2-sequence-covering. However, the following is more useful to simplify the description of *sn*-open mapping or 2-sequence-covering mapping.

Conclusion 4.4. *sn*-open mapping (resp. 2-sequence-covering mapping if the domain is metrizable) that appeared in some relevant papers can be replaced with *so*-open mapping.

These results give some new characterizations of *so*-metrizable spaces and establish some equivalent relations among *so*-open mappings, *sn*-open mappings and 2-sequence-covering mappings, which further enrich and deepen generalized metric space theory.

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