Research Article

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Symmetric pairs and pseudosymmetry of Θ -Yetter-Drinfeld categories for Hom-Hopf algebras

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Abstract: In this paper, we investigate a more general category of Θ -Yetter-Drinfeld modules ($\Theta \in \text{Aut } H(H)$) over a Hom-Hopf algebra, which unifies two different definitions of Hom-Yetter-Drinfeld category introduced by Makhlouf and Panaite, Li and Ma, respectively. We show that the category of Θ -Yetter-Drinfeld modules with a bijective antipode S is a braided tensor category and some solutions of the Hom-Yang-Baxter equation and the Yang-Baxter equation can be constructed by this category. Also by the method of symmetric pairs, we prove that if a Θ -Yetter-Drinfeld category over a Hom-Hopf algebra H is symmetric, then H is trivial. Finally, we find a sufficient and necessary condition for a Θ -Yetter-Drinfeld category to be pseudosymmetric.

Keywords: Hom-Hopf algebra, Θ -Yetter-Drinfeld module, symmetric pair

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1 Introduction

The genesis of Hom-structures may be found in the physics literature from the 1990, concerning quantum deformations of algebras of vector fields, especially Witt and Virasoro algebras. These classes of examples led to the development of Hom-Lie algebras; see [1–4]. Recently, various Hom-Lie structures have been studied further by many scholars; see [5–12]. The idea of tailoring associativity-like conditions by linear maps was migrated to other algebraic structures. The concepts of Hom-algebras, Hom-coalgebras, Hom-Hopf algebras and quasi-triangular Hom-bialgebras were introduced and further developed; see [13–22].

Yetter-Drinfeld modules are known to be at the origin of a very vast family of solutions to the Yang-Baxter equation. Therefore, it is very natural to extend the notion of Yetter-Drinfeld module into the Hom-setting. The main purpose to study Yetter-Drinfeld module in the Hom-setting is to prove that the category of Yetter-Drinfeld modules over a Hom-Hopf algebra with a bijective antipode is also a braided tensor category. Furthermore, we can construct some solutions of the classical Yang-Baxter equation and the Hom-Yang-Baxter equation introduced by Yau [12]. Two directions of the study were developed, one considering the category of Yetter-Drinfeld modules under monoidal Hom-Hopf algebras introduced by Caenepeel and Goyvaerts [23]; see also [24–26] and others, discussing the category of Yetter-Drinfeld modules under Hom-Hopf algebras in [14, 27]. For a monoidal Hom-Hopf algebra, Caenepeel and Goyvaerts explained it from a categorical point of view. Therefore, we can write the corresponding conditions by the categorical method. Different from the monoidal Hom-Hopf algebra, the conditions in Hom-Hopf algebra

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are obtained more unpredictable than the monoidal version. For a Hom-Hopf algebra, there exists two definitions of Yetter-Drinfeld modules in the studies of Makhlouf and Panaite [14] and Li and Ma [27], respectively. It is obvious that these two definitions of Yetter-Drinfeld modules are not equivalent. Surprisingly, such two categories of Yetter-Drinfeld modules are both braided tensor categories and provide solutions of the Yang-Baxter equation and the Hom-Yang-Baxter equation. The motivation of the definition of Yetter-Drinfeld modules by Makhlouf and Panaite relies on the main tool called “twisting principle,” which was introduced by Yau [11]. But for the definition by Li and Ma, they gave it by massive calculations to satisfy the corresponding conditions. Naturally inspired by the above two definitions, the authors in [28] gave the definition of ν -Yetter-Drinfeld modules, $\nu \in \mathbb{Z}$; see (2.15). And they showed that every category of ν -Yetter-Drinfeld modules is also a braided tensor category and provides some solutions of the Yang-Baxter equation and the Hom-Yang-Baxter equation. Therefore, the ν -Yetter-Drinfeld module unifies the above two definitions of Yetter-Drinfeld modules. Observing the compatibility of ν -Yetter-Drinfeld modules, we find that the condition (2.16) is equivalent to the condition (2.15). In other words, we have proven that if $p = 2\nu - 2$ is even, the definition of ν -Yetter-Drinfeld is favorable. One natural question to ask is whether the definition of ν is favorable or not if p is odd. Furthermore, there exist more general favorable definitions of Yetter-Drinfeld module over a Hom-Hopf algebra. In this paper, we give the definition of Θ -Yetter-Drinfeld modules, $\Theta \in \text{Aut } H(H)$ where we denote the group of all Hom-Hopf algebra automorphisms of H by $\text{Aut } H(H)$ and prove that even if p is odd, the definition of Yetter-Drinfeld is in the same way favorable. Meanwhile, all the definitions of Yetter-Drinfeld modules over a Hom-Hopf algebra that appeared in [14, 27, 28] can be regarded as special cases of the Θ -Yetter-Drinfeld module.

This paper is organized as follows. In Section 1, we recall some definitions and results which will be used later. In Section 2, after introducing the concept of Θ -Yetter-Drinfeld category ${}^H_H\mathcal{HYD}^\Theta$, we prove that every Θ -Yetter Drinfeld category over a Hom-Hopf algebra with a bijective antipode is a braided tensor category and that every Θ -Yetter Drinfeld category over a Hom-Hopf algebra provides a solution of the Hom-Yang-Baxter equation and the Yang-Baxter equation. In Section 3, by the method of symmetric pairs, we show that symmetric Θ -Yetter-Drinfeld categories ${}^H_H\mathcal{HYD}^\Theta$ over a Hom-Hopf algebra are all trivial. The results obtained in this section generalize the corresponding results in [28–31]. In Section 4, we find a necessary and sufficient condition for a Θ -Yetter-Drinfeld category ${}^H_H\mathcal{HYD}^\Theta$ over a Hom-Hopf algebra H to be pseudosymmetric.

2 Preliminaries

We work over a base field k . All algebras, linear spaces etc. will be over k , and unadorned $\otimes$ means $\otimes_k$. For a comultiplication $\Delta : C \rightarrow C \otimes C$ on a vector space C , we use a Sweedler-type notation $\Delta(c) = c_1 \otimes c_2$ for all $c \in C$. In what follows, we assume that k -linear maps $\alpha_\cdot$ are bijective, although some notions are not supposed to be bijective.

Definition 2.1. [14, 32] A *Hom-associative algebra* is a quadruple $(A, \mu, 1_A, \alpha)$ (abbr. (A, α)) where A is a k -linear space, $\mu : A \otimes A \rightarrow A$ and $\alpha : A \rightarrow A$ are k -linear maps such that

$$\alpha(aa') = \alpha(a)\alpha(a'), \quad \alpha(1_A) = 1_A, \quad (2.1)$$

$$\alpha(a)(a'a'') = (aa')\alpha(a''), \quad a1_A = 1_Aa = \alpha(a) \quad (2.2)$$

are satisfied for $a, a', a'' \in A$. Here we use the notation $\mu(a \otimes a') = aa'$.

A morphism $f : (A, \mu_A, 1_A, \alpha_A) \rightarrow (B, \mu_B, 1_B, \alpha_B)$ of Hom-associative algebra is a linear map $f : A \rightarrow B$ such that $\alpha_B \circ f = f \circ \alpha_A$, $f(1_A) = 1_B$ and $f \circ \mu_A = \mu_B \circ (f \otimes f)$.

Let (A, α) and (B, β) be two Hom-algebras. Then $(A \otimes B, \alpha \otimes \beta)$ is a Hom-associative algebra (called a *tensor product Hom-associative algebra*) with the multiplication $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ and its unit $1_A \otimes 1_B$.

Definition 2.2. [14,18,33] A *Hom-coassociative coalgebra* is a quadruple $(C, \Delta, \varepsilon_C, \beta)$ (abbr. (C, β)), where C is a k -linear space, $\Delta : C \rightarrow C \otimes C$, $\varepsilon_C : C \rightarrow k$ and $\beta : C \rightarrow C$ are k -linear maps such that

$$\beta(c)_1 \otimes \beta(c)_2 = \beta(c_1) \otimes \beta(c_2), \quad \varepsilon_C \circ \beta = \varepsilon_C, \quad (2.3)$$

$$\beta(c_1) \otimes c_{21} \otimes c_{22} = c_{11} \otimes c_{12} \otimes \beta(c_2), \quad \varepsilon_C(c_1)c_2 = c_1\varepsilon_C(c_2) = \beta(c) \quad (2.4)$$

are satisfied for $c \in C$.

A morphism $g : (C, \Delta_C, \alpha_C) \rightarrow (D, \Delta_D, \alpha_D)$ of Hom-coassociative coalgebras is a linear map $g : C \rightarrow D$ such that $\alpha_D \circ g = g \circ \alpha_C$, $\varepsilon_D = \varepsilon_C \circ g$ and $(g \otimes g) \circ \Delta_C = \Delta_D \circ g$.

Let (C, α) and (D, β) be two Hom-coassociative coalgebras. Then $(C \otimes D, \alpha \otimes \beta)$ is a Hom-coassociative coalgebra (called a *tensor product Hom-coassociative coalgebra*) with the comultiplication $\Delta(c \otimes d) = c_1 \otimes d_1 \otimes c_2 \otimes d_2$ and its counit $\varepsilon_C \otimes \varepsilon_D$.

Definition 2.3. [18,19] A *Hom-bialgebra* is a sextuple $(H, \mu, 1_H, \Delta, \varepsilon, \alpha)$ (abbr. (H, α)), where $(H, \mu, 1_H, \alpha)$ is a Hom-associative algebra and $(H, \Delta, \varepsilon, \alpha)$ is a Hom-coassociative coalgebra such that Δ and ε are morphisms of Hom-algebras, i.e.,

$$\Delta(hh') = \Delta(h)\Delta(h'), \quad \Delta(1_H) = 1_H \otimes 1_H, \quad (2.5)$$

$$\varepsilon(hh') = \varepsilon(h)\varepsilon(h'), \quad \varepsilon(1_H) = 1. \quad (2.6)$$

Furthermore, if there exists a linear map $S : H \rightarrow H$ such that

$$S(h_1)h_2 = h_1S(h_2) = \varepsilon(h)1_H, \quad S(\alpha(h)) = \alpha(S(h)), \quad (2.7)$$

then we call $(H, \mu, 1_H, \Delta, \varepsilon, \alpha, S)$ (abbr. (H, α, S)) a *Hom-Hopf algebra*.

Let (H, α, S) and (H', α', S') be two Hom-Hopf algebras. The linear map $f : H \rightarrow H'$ is called a *Hom-Hopf algebra map* if $f \circ \alpha = \alpha' \circ f$, $f \circ S = S' \circ f$ and f is a Hom-algebra map and a Hom-coalgebra map.

Definition 2.4. [18,32] Let $(A, \mu, 1_A, \alpha_A)$ be a Hom-associative algebra, M a linear space and $\alpha_M : M \rightarrow M$ a linear map. A *left A-Hom-module* (M, α_M) consists of a linear map $A \otimes M \rightarrow M$, $a \otimes m \mapsto a \cdot m$ satisfying the following conditions:

$$\alpha_M(a \cdot m) = \alpha_A(a) \cdot \alpha_M(m); \quad (2.8)$$

$$\alpha_A(a) \cdot (a' \cdot m) = (aa') \cdot \alpha_M(m), \quad 1_H \cdot m = \alpha_M(m) \quad (2.9)$$

for all $a, a' \in A$ and $m \in M$. If (M, α_M) and (N, α_N) are left A -Hom-modules (both are A -Hom-actions, denoted by $\cdot$), a morphism of left A -Hom-modules $f : M \rightarrow N$ is a linear map satisfying the following conditions:

$$\alpha_N \circ f = f \circ \alpha_M \quad \text{and} \quad f(a \cdot m) = a \cdot f(m), \quad a \in A, \quad m \in M.$$

Definition 2.5. [18,32] Let $(C, \Delta_C, \varepsilon, \alpha_C)$ be a Hom-coassociative coalgebra, M a linear space and $\alpha_M : M \rightarrow M$ a linear map. A *left C-Hom-comodule* (M, α_M) consists of a linear map $\lambda : M \rightarrow C \otimes M$ (usually denoted by $\lambda(m) = m_{(-1)} \otimes m_{(0)}$) satisfying the following conditions:

$$(\alpha_C \otimes \alpha_M) \circ \lambda = \lambda \circ \alpha_M; \quad (2.10)$$

$$(\Delta_C \otimes \alpha_M) \circ \lambda = (\alpha_C \otimes \lambda) \circ \lambda, \quad (\varepsilon \otimes id) \circ \lambda = \alpha_M. \quad (2.11)$$

If (M, α_M) and (N, α_N) are left C -Hom-comodules, with Hom-coactions $\lambda_M : M \rightarrow C \otimes M$ and $\lambda_N : N \rightarrow C \otimes N$, a morphism of left C -Hom-comodules $g : M \rightarrow N$ is a linear map satisfying the conditions $\alpha_N \circ g = g \circ \alpha_M$ and $(id_C \otimes g) \circ \lambda_M = \lambda_N \circ g$.

Definition 2.6. [14] Let $(H, \mu_H, 1_H, \Delta_H, \varepsilon_H, \alpha_H)$ be a Hom-bialgebra, M a linear space and $\alpha_M : M \rightarrow M$ a linear map such that (M, α_M) is a left H -Hom-module with Hom-action $H \otimes M \rightarrow M$, $h \otimes m \mapsto h \cdot m$ and

a left H -Hom-comodule with Hom-coaction $M \rightarrow H \otimes M$, $m \mapsto m_{(-1)} \otimes m_{(0)}$. Then (M, α_M) is called a *left-left Yetter-Drinfel module* over H if, for all $h \in H$ and $m \in M$, there holds the following identity:

$$(h_1 \cdot m)_{(-1)} \alpha_H^2(h_2) \otimes (h_1 \cdot m)_{(0)} = \alpha_H^2(h_1) \alpha_H(m_{(-1)}) \otimes \alpha_H(h_2) \cdot m_{(0)}. \quad (2.12)$$

Notice that the choice of the compatibility condition (2.12) due to [14] was motivated by the twisting principle, but the compatibility condition of Yetter-Drinfeld module over a Hom-bialgebra is not the only choice as in (2.12). The following definition of Yetter-Drinfeld module was due to Li and Ma [27].

Definition 2.7. [27] Let $(H, \mu_H, 1_H, \Delta_H, \varepsilon_H, \alpha_H)$ be a Hom-bialgebra, (M, α_M) a left H -Hom-module with Hom-action $H \otimes M \rightarrow M$ denoted by $h \otimes m \mapsto h \cdot m$ and (M, α_M) a left H -Hom-comodule with Hom-coaction $M \rightarrow H \otimes M$ denoted by $m \mapsto m_{(-1)} \otimes m_{(0)}$. Then we call (M, α_M) a *left-left Yetter-Drinfeld module* over a Hom-bialgebra (H, α_H) if, for all $h \in H$ and $m \in M$, there holds the following condition:

$$h_1 \alpha_H(m_{(-1)}) \otimes \alpha_H^3(h_2) \cdot m_{(0)} = (\alpha_H^2(h_1) \cdot m)_{(-1)} h_2 \otimes (\alpha_H^2(h_1) \cdot m)_{(0)}. \quad (2.13)$$

Remark 2.8. Obviously, the compatibility condition (2.12) is different from the condition (2.13). It is not hard to show that the definition of Yetter-Drinfeld module in Definition 2.7 does not satisfy the twisting principle. Indeed, by a direct computation, we have

$$(\alpha_H^2(h_{(1)}) \triangleright m)_{(-1)} * h_{(2)} \otimes (\alpha_H^2(h_{(1)}) \triangleright m)_{(0)} \neq h_{(1)} * \alpha_H(m_{(-1)}) \otimes \alpha_H^3(h_{(2)}) * m_{(0)},$$

where the multiplication and comultiplication of H_{α_H} were denoted by $h * h' = \alpha_H(hh')$ and $h_{(1)} \otimes h_{(2)} = \alpha_H(h_1) \otimes \alpha_H(h_2)$, respectively.

Remark 2.9. Since α is bijective, it is not hard to obtain that the condition (2.13) is equivalent to

$$(h_1 \cdot m)_{(-1)} \alpha_H^{-2}(h_2) \otimes (h_1 \cdot m)_{(0)} = \alpha_H^{-2}(h_1) \alpha_H(m_{(-1)}) \otimes \alpha_H(h_2) \cdot m_{(0)}. \quad (2.14)$$

In [28], the authors gave a more general compatibility condition including the definitions in [14,27] as the special cases.

Definition 2.10. [28] Let (H, α_H) be a Hom-bialgebra, (M, α_M) a left H -Hom-module with Hom-action $H \otimes M \rightarrow M$, $h \otimes m \mapsto h \cdot m$ and (M, α_M) a left H -Hom-comodule with Hom-coaction $M \rightarrow H \otimes M$, $m \mapsto m_{(-1)} \otimes m_{(0)}$. Then we call (M, α_M) a *left-left ν -Yetter-Drinfeld module* over a Hom-bialgebra (H, α_H) if, for all $h \in H$, $m \in M$ and $\nu \in \mathbb{Z}$, there holds the following condition:

$$(\alpha_H^{2-\nu}(h_1) \cdot m)_{(-1)} \alpha_H^\nu(h_2) \otimes (\alpha_H^{2-\nu}(h_1) \cdot m)_{(0)} = \alpha_H^\nu(h_1) \alpha_H(m_{(-1)}) \otimes \alpha_H^{3-\nu}(h_2) \cdot m_{(0)}. \quad (2.15)$$

Remark 2.11. Being similar to Remark 2.9, we readily see that the condition (2.15) is equivalent to

$$(h_1 \cdot m)_{(-1)} \alpha_H^{2\nu-2}(h_2) \otimes (h_1 \cdot m)_{(0)} = \alpha_H^{2\nu-2}(h_1) \alpha_H(m_{(-1)}) \otimes \alpha_H(h_2) \cdot m_{(0)}, \quad \nu \in \mathbb{Z}. \quad (2.16)$$

3 Θ -Yetter-Drinfeld category

In this section, we introduce a more general Θ -Yetter-Drinfeld category ($\Theta \in \text{Aut } H(H)$), which unifies two different definitions of Hom-Yetter-Drinfeld category of Makhlouf and Panaite [14] and Li and Ma [27], respectively. We show that the category of Θ -Yetter-Drinfeld modules with a bijective antipode is a braided tensor category and some solutions of the Hom-Yang-Baxter equation and the Yang-Baxter equation can be constructed by this category.

In case that $(H, \mu_H, \Delta_H, \alpha_H, S)$ is a Hom-Hopf algebra, we denote the group of all Hom-Hopf algebra automorphisms of H by $\text{Aut } H(H)$. Inspired by the above three definitions of Hom-Yetter-Drinfeld modules, we can give a more general definition of Hom-Yetter-Drinfeld modules as follows.

Definition 3.1. Let (H, α_H) be a Hom-bialgebra and $\Theta \in \text{Aut } H(H)$, (M, α_M) a left H -Hom-module with Hom-action $H \otimes M \rightarrow M$, $h \otimes m \mapsto h \cdot m$ and (M, α_M) a left H -Hom-comodule with Hom-coaction $M \rightarrow H \otimes M$, $m \mapsto m_{(-1)} \otimes m_{(0)}$. Then we call (M, α_M) a *left-left Θ -Yetter-Drinfeld module* over a Hom-bialgebra (H, α_H) if, for all $h \in H$ and $m \in M$, there holds the following condition:

$$(h_1 \cdot m)_{(-1)} \Theta(h_2) \otimes (h_1 \cdot m)_{(0)} = \Theta(h_1) \alpha_H(m_{(-1)}) \otimes \alpha_H(h_2) \cdot m_{(0)}. \quad (3.1)$$

Remark 3.2. We find an interesting fact that every definition that appeared in references can be regarded as the special cases of Θ -Yetter-Drinfeld module.

- (1) If $\Theta = \alpha^{2\nu-2}$, $\nu \in \mathbb{Z}$, the definition of Θ -Yetter-Drinfeld module is the same as Definition 2.1 in [28].
- (2) If $\Theta = \alpha^{-2}$, the definition of Θ -Yetter-Drinfeld module is the same as Definition 2.7;
- (3) If $\Theta = \alpha^2$, the definition of Θ -Yetter-Drinfeld module is just Definition 2.6;
- (4) If $\alpha_H = \text{id}$, $\alpha_M = \text{id}$ and $\Theta = \text{id}$, the definition of Θ -Yetter-Drinfeld module is exactly the usual Yetter-Drinfeld module;
- (5) We should note that it is impossible to get the definition only by the twisting principle.

Example 3.3. Let (H, S, α_H) be a Hom-Hopf algebra and $\Theta \in \text{Aut } H(H)$. Then (H, α_H) is a Θ -Yetter-Drinfeld module in Definition 3.1 with a left H -Hom-action

$$h \triangleright g = (\Theta \alpha_H^{-4}(h_1) \alpha_H^{-1}(g)) S(\Theta \alpha_H^{-3}(h_2))$$

and a left H -Hom-coaction by the Hom-comultiplication Δ . Denote it by $H_1 = (H, \triangleright, \Delta, \alpha_H)$. Dually, (H, α) is a Θ -Yetter-Drinfeld module in Definition 3.1 with a left H -Hom-action by the Hom-multiplication μ and a left H -Hom-coaction

$$\lambda(h) = \Theta \alpha_H^{-4}(h_{11}) \Theta \alpha_H^{-3}(h_2) \otimes \alpha_H^{-1}(h_{12});$$

it will be denoted by $H_2 = (H, \mu, \lambda, \alpha_H)$.

Proof. We only establish the first part of the above example here. First, it is easy to see that (H_1, Δ, α) is a left (H, α) -comodule. Second, we show that $(H_1, \triangleright, \alpha)$ is a left (H, α) -module. In fact, for any $h, g, l \in H_1$, we have

$$\alpha(h) \triangleright \alpha(g) = (\Theta \alpha^{-3}(h_1) g) S(\Theta \alpha^{-2}(h_2)) = \alpha(\Theta \alpha^{-4}(h_1) \alpha^{-1}(g)) S(\Theta \alpha^{-3}(h_2)) = \alpha(h \triangleright g),$$

which proves the equality (2.8). For proving (2.9), we observe that $1_H \triangleright g = (\Theta \alpha^{-4}(1) \alpha^{-1}(g)) S(\Theta \alpha^{-3}(1)) = (1 \cdot g) \cdot 1 = \alpha(g)$ and

$$\begin{aligned} \alpha(h) \triangleright (g \triangleright l) &= \alpha(h) \triangleright ((\Theta \alpha^{-4}(g_1) \alpha^{-1}(l)) S(\Theta \alpha^{-3}(g_2))) \\ &= (\Theta \alpha^{-3}(h_1) \alpha^{-1}((\Theta \alpha^{-4}(g_1) \alpha^{-1}(l)) S(\Theta \alpha^{-3}(g_2)))) S(\Theta \alpha^{-2}(h_2)) \\ &\stackrel{(2.2)}{=} (\Theta \alpha^{-3}(h_1) (\Theta \alpha^{-4}(g_1) \alpha^{-1}(l))) (S(\Theta \alpha^{-3}(g_2)) S(\Theta \alpha^{-3}(h_2))) \\ &= ((\Theta \alpha^{-4}(h_1) \Theta \alpha^{-4}(g_1)) l_1) (S(\Theta \alpha^{-3}(g_2)) S(\Theta \alpha^{-3}(h_2))) \\ &= (\Theta \alpha^{-4}(h_1 g_1) l) S(\Theta \alpha^{-3}(h_2 g_2)) = (hg) \triangleright \alpha(l). \end{aligned}$$

Finally, we show the compatible condition (3.1) of Θ -Yetter-Drinfeld module. For this end, the left-hand side of (3.1) can be expressed as

$$\Theta(h_1) \alpha(l_1) \otimes \alpha(h_2) \cdot l_2 \stackrel{(2.3)}{=} \Theta(h_1) \alpha(l_1) \otimes (\Theta \alpha^{-3}(h_{21}) \alpha^{-1}(l_2)) S \Theta \alpha^{-2}(h_{22}),$$

and also its right-hand side can be given by

$$\begin{aligned}
 & (h_1 \cdot l)_{(-1)} \Theta(h_2) \otimes (h_1 \cdot l)_{(0)} \\
 & \stackrel{(2.5), (2.3)}{=} ((\Theta \alpha^{-4}(h_{11}) \alpha^{-1}(l_1)) S(\Theta \alpha^{-3}(h_{122}))) \Theta(h_2) \otimes (\Theta \alpha^{-4}(h_{112}) \alpha^{-1}(l_2)) S(\Theta \alpha^{-3}(h_{121})) \\
 & \stackrel{(2.4), (2.2)}{=} ((\Theta \alpha^{-2}(h_{11}) l_1) S(\Theta \alpha^{-3}(h_{221})) \Theta \alpha^{-3}(h_{222})) \otimes (\Theta \alpha^{-3}(h_{12}) \alpha^{-1}(l_2)) S(\Theta \alpha^{-2}(h_{21})) \\
 & \stackrel{(2.7), (2.2), (2.4)}{=} \Theta \alpha^{-1}(h_{11}) \alpha(l_1) \otimes (\Theta \alpha^{-3}(h_{12}) \alpha^{-1}(l_2)) S(\Theta \alpha^{\nu-1}(h_2)) \\
 & \stackrel{(2.4)}{=} \Theta(h_1) \alpha(l_1) \otimes (\Theta \alpha^{-3}(h_{21}) \alpha^{-1}(l_2)) S \Theta \alpha^{-2}(h_{22}),
 \end{aligned}$$

yielding the compatible condition (3.1) of Θ -Yetter-Drinfeld module and the proof is complete.

Remark 3.4. If we take $\Theta = \alpha^{2\nu-2}$, $\nu \in \mathbb{Z}$ in Example 3.3, we obtain Example 2.4 in [28]. Also, if we take $\Theta = \alpha^{-2}$, Example 3.3 is just Lemmas 3.3 and 3.4 in [31].

Definition 3.5. Let (H, α_H) be a Hom-bialgebra and $\Theta \in \text{Aut } H(H)$. We denote by ${}^H_H \mathcal{HYD}^\Theta$ the category whose objects are Θ -Yetter-Drinfeld modules (M, α_M) over H , and the morphisms in the category are morphisms of left H -Hom-modules and left H -Hom-comodules.

Proposition 3.6. Let (H, S, α_H) be a Hom-Hopf algebra and $\Theta \in \text{Aut } H(H)$. Let (M, α_M) be both a left H -Hom-module and a left H -Hom-comodule with notations as in Definition 3.1. Then (3.1) in Definition 3.1 is equivalent to

$$(\Theta \alpha_H^{-4}(h_{11}) \alpha_H^{-1}(m_{(-1)})) \Theta \alpha_H^{-2} S(h_2) \otimes \alpha_H^{-1}(h_{12}) m_{(0)} = (h \cdot m)_{(-1)} \otimes (h \cdot m)_{(0)} \quad (3.2)$$

for all $h \in H$ and $m \in M$.

Proof. (3.1) $\Rightarrow$ (3.2). For all $h \in H$ and $m \in M$, we compute

$$\begin{aligned}
 & (\Theta \alpha_H^{-4}(h_{11}) \alpha_H^{-1}(m_{(-1)})) \Theta \alpha_H^{-2} S(h_2) \otimes \alpha_H^{-1}(h_{12}) m_{(0)} \\
 & \stackrel{(2.1), (2.8), (2.10)}{=} \alpha_H^{-4}(\Theta(h_{11}) \alpha_H((\alpha_M^2(m))_{(-1)})) \alpha_H^{-2} S(h_2) \otimes \alpha_M^{-2}(\alpha(h_{12}) \cdot (\alpha_M^2(m))_{(0)}) \\
 & \stackrel{(3.1)}{=} \alpha_H^{-4}((h_{11} \cdot \alpha_M^2(m))_{(-1)} \Theta(h_{12})) \Theta \alpha_H^{-2} S(h_2) \otimes \alpha_M^{-2}((h_{11} \cdot \alpha_M^2(m))_{(0)}) \\
 & \stackrel{(2.1), (2.2), (2.4)}{=} \alpha_H^{-3}((\alpha_H(h_1) \cdot \alpha_M^2(m))_{(-1)} (\Theta \alpha_H^{-4}(h_{21}) \Theta \alpha_H^{-4} S(h_{22}))) \otimes \alpha_M^{-2}((\alpha_H(h_1) \cdot \alpha_M^2(m))_{(-0)}) \\
 & \stackrel{(2.7), (2.3), (2.4)}{=} (h \cdot m)_{(-1)} \otimes (h \cdot m)_{(0)}.
 \end{aligned}$$

(3.2) $\Rightarrow$ (3.1). For all $h \in H$ and $m \in M$, we have

$$\begin{aligned}
 & (h_1 \cdot m)_{(-1)} \Theta(h_2) \otimes (h_1 \cdot m)_{(0)} \\
 & \stackrel{(3.2)}{=} ((\Theta \alpha_H^{-4}(h_{11}) \alpha_H^{-1}(m_{(-1)})) \Theta \alpha_H^{-2} S(h_{12})) \Theta(h_2) \otimes \alpha_H^{-1}(h_{12}) \cdot m_{(0)} \\
 & \stackrel{(2.2), (2.4)}{=} (\Theta \alpha_H^{-2}(h_{11}) m_{(-1)}) (\Theta \alpha_H^{-2} S(h_{21}) \alpha_H^2 \Theta(h_{22})) \otimes h_{21} \cdot m_{(0)} \\
 & \stackrel{(2.7), (2.1), (2.2)}{=} \Theta(h_1) \alpha_H(m_{-1}) \otimes \alpha_H(h_2) \cdot m_{(0)}.
 \end{aligned}$$

This completes the proof. $\square$

In the following, we give solutions of the Hom-Yang-Baxter introduced and studied by Yau [12].

Proposition 3.7. Let (H, α_H) be a Hom-bialgebra, $\Theta \in \text{Aut } H(H)$ and $(M, \alpha_M), (N, \alpha_N) \in {}^H_H \mathcal{HYD}^\Theta$. Define the linear map

$$\tau_{M,N} : M \otimes N \rightarrow N \otimes M, \quad m \otimes n \mapsto \Theta^{-1} \alpha_H(m_{(-1)}) \cdot n \otimes m_{(0)}, \quad (3.3)$$

where $m \in M$ and $n \in N$. Then,

- (i) $\tau_{M,N} \circ (\alpha_M \otimes \alpha_N) = (\alpha_N \otimes \alpha_M) \circ \tau_{M,N}$, and
(ii) if $(P, \alpha_P) \in {}^H_H\mathcal{HYD}^\Theta$, the maps $\tau_{-, -}$ satisfy the Hom-Yang-Baxter equation:

$$(\alpha_P \otimes \tau_{M,N}) \circ (\tau_{M,P} \otimes \alpha_N) \circ (\alpha_M \otimes \tau_{N,P}) = (\tau_{N,P} \otimes \alpha_M) \circ (\alpha_N \otimes \tau_{M,P}) \circ (\tau_{M,N} \otimes \alpha_P).$$

Proof. The property (i) can be easily proven. Now we claim the property (ii): given $m \in M, n \in N$ and $p \in P$, using the conditions (2.10), (2.1) and (2.2), we obtain

$$\begin{aligned} & (\alpha_P \otimes \tau_{M,N}) \circ (\tau_{M,P} \otimes \alpha_N) \circ (\alpha_M \otimes \tau_{N,P})(m \otimes n \otimes p) \\ &= \Theta^{-1} \alpha_H^2(m_{(-1)} n_{(-1)}) \cdot \alpha_P^2(p) \otimes \Theta^{-1} \alpha_H^2(m_{(0)(-1)}) \cdot \alpha_N(n_{(0)}) \\ & \quad \otimes \alpha_M(m_{(0)(0)}) (\tau_{N,P} \otimes \alpha_M) \circ (\alpha_N \otimes \tau_{M,P}) \circ (\tau_{M,N} \otimes \alpha_P)(m \otimes n \otimes p) \\ &= \Theta^{-1} \alpha_H((\alpha_H^2 \Theta^{-1}(m_{(-1)}) \cdot \alpha_N(n))_{(-1)}) \cdot (\Theta^{-1} \alpha_H(m_{(0)(-1)}) \cdot \alpha_P(p)) \\ & \quad \otimes (\alpha_H^2 \Theta^{-1}(m_{(-1)}) \cdot \alpha_N(n))_{(0)} \otimes \alpha_M(m_{(0)(0)}) \\ & \stackrel{(2.9), (2.11)}{=} (\Theta^{-1}((\Theta^{-1} \alpha_H(m_{(-1)1}) \cdot \alpha_N(n))_{(-1)}) \Theta^{-1} \alpha_H(m_{(-1)2})) \cdot \alpha_P^2(p) \\ & \quad \otimes (\Theta^{-1} \alpha_H(m_{(-1)1}) \cdot \alpha_N(n))_{(0)} \otimes \alpha_M^2(m_{(0)}) \\ & \stackrel{(2.1), (2.3)}{=} \Theta^{-1}((\Theta^{-1} \alpha_H(m_{(-1)1}) \cdot \alpha_N(n))_{(-1)} \Theta(\Theta^{-1} \alpha_H(m_{(-1)2}))) \cdot \alpha_P^2(p) \\ & \quad \otimes (\Theta^{-1} \alpha_H(m_{(-1)1}) \cdot \alpha_N(n))_{(0)} \otimes \alpha_M^2(m_{(0)}) \\ & \stackrel{(3.1)}{=} \Theta^{-1}(\Theta(\Theta^{-1} \alpha(m_{(-1)1})) \alpha_H(\alpha_N(n)_{(-1)})) \cdot \alpha_P^2(p) \\ & \quad \otimes \alpha_H(\Theta^{-1} \alpha_H(m_{(-1)2})) \cdot \alpha_N(n)_{(0)} \otimes \alpha_M^2(m_{(0)}) \\ &= \Theta^{-1} \alpha_H^2(m_{(-1)} n_{(-1)}) \cdot \alpha_P^2(p) \otimes \Theta^{-1} \alpha_H^2(m_{(0)(-1)}) \cdot \alpha_N(n_{(0)}) \\ & \quad \otimes \alpha_M(m_{(0)(0)}) \quad \text{by (2.3), (2.1) and (2.11),} \end{aligned}$$

and the proof is complete. $\square$

Now we will state the main result in this section.

Theorem 3.8. Let (H, α_H) be a Hom-Hopf algebra with a bijective antipode S and a bijective $\alpha_H, \Theta \in \text{Aut } H(H)$. Then the Θ -Yetter-Drinfeld category ${}^H_H\mathcal{HYD}^\Theta$ is a braided tensor category, with tensor product $\hat{\otimes}$, associativity constraints $a_{-, -, -}$, braiding $c_{-, -}$ and the unit defined as follows: for all $(M, \alpha_M), (N, \alpha_N), (P, \alpha_P) \in {}^H_H\mathcal{HYD}^\Theta$,

- (i) The tensor product $\hat{\otimes} = \otimes$, and the left H -Hom-module $H \otimes (M \otimes N) \rightarrow M \otimes N$ and the left H -Hom-comodule $M \otimes N \rightarrow H \otimes (M \otimes N)$ are defined, respectively, by

$$\begin{aligned} h \otimes (m \otimes n) &\mapsto h_1 \cdot m \otimes h_2 \cdot n, \\ \rho(m \otimes n) &\mapsto \alpha_H^{-2}(m_{(-1)} n_{(-1)}) \otimes m_{(0)} \otimes n_{(0)}; \end{aligned}$$

- (ii) $a_{M,N,P} : (M \hat{\otimes} N) \hat{\otimes} P \rightarrow M \hat{\otimes} (N \hat{\otimes} P)$ is given by

$$(m \otimes n) \otimes p \mapsto \alpha_M^{-1}(m) \otimes (n \otimes \alpha_P(p));$$

- (iii) $c_{M,N} : M \hat{\otimes} N \rightarrow N \hat{\otimes} M$ is defined by

$$m \otimes n \mapsto \Theta^{-1}(m_{(-1)}) \cdot \alpha_N^{-1}(n) \otimes \alpha_M^{-1}(m_{(0)});$$

- (iv) $c_{M,N}^{-1} : N \hat{\otimes} M \rightarrow M \hat{\otimes} N$ is given by

$$n \otimes m \mapsto \alpha_M^{-1}(m_{(0)}) \otimes S^{-1}(\Theta^{-1}(m_{(-1)})) \cdot \alpha_N^{-1}(n);$$

- (v) the unit is defined by $I = (k, id_k)$.

Proof. (Sketch of the proof) By the definition of a braided tensor category, we need to investigate many conditions by a large number of computations. In fact, we have to prove that $M \hat{\otimes} N$ is also a θ -Yetter-Drinfeld module, $a_{M,N,P}$ is an isomorphism of left H -Hom-modules and left H -Hom-comodules, $c_{M,N}$ is

a morphism of left H -Hom-modules and left H -Hom-comodules, $c_{M,N}^{-1} \circ c_{M,N} = id_{M \otimes N}$, the pentagon axiom and hexagonal relation are satisfied, and so on. We think that it is a good exercise to a reader. For example, now we will prove the Θ -Yetter-Drinfeld compatibility condition (3.1) for $M \hat{\otimes} N$. Indeed, given $m \in M$ and $n \in N$, simple computations yield

$$\begin{aligned}
 & (h_1 \cdot (m \otimes n))_{(-1)} \Theta(h_2) \otimes (h_1 \cdot (m \otimes n))_{(0)} \\
 &= \alpha_H^{-2}((h_{11} \cdot m)_{(-1)}(h_{12} \cdot n)_{(-1)}) \Theta(h_2) \otimes (h_{11} \cdot m)_{(0)} \otimes (h_{12} \cdot n)_{(0)} \\
 &= \alpha_H^{-1}((\alpha_H(h_1) \cdot m)_{(-1)}) \alpha_H^{-2}((h_{21} \cdot n)_{(-1)}) \Theta(h_{22}) \\
 &\quad \otimes (\alpha_H(h_1) \cdot m)_{(0)} \otimes (h_{21} \cdot n)_{(0)} \quad \text{by (2.1), (2.2) and (2.4)} \\
 &\stackrel{(3.1)}{=} \alpha_H^{-1}((\alpha_H(h_1) \cdot m)_{(-1)}) \alpha_H^{-2}(\Theta(h_{21}) \alpha_H(n_{(-1)})) \otimes (\alpha_H(h_1) \cdot m)_{(0)} \otimes \alpha_H(h_{22}) \cdot n_{(0)} \\
 &= \alpha_H^{-2}((h_{11} \cdot m)_{(-1)} \Theta(h_{12})) n_{(-1)} \otimes (h_{11} \cdot m)_{(0)} \otimes \alpha_H^2(h_2) \cdot n_{(0)} \quad \text{by (2.1), (2.2) and (2.4)} \\
 &\stackrel{(3.1)}{=} \alpha_H^{-2}(\Theta(h_{11}) \alpha_H(m_{(-1)})) n_{(-1)} \otimes \alpha_H(h_{12}) \cdot m_{(0)} \otimes \alpha_H^2(h_2) \cdot n_{(0)} \\
 &= \Theta(h_1) (\alpha_H(m_{(-1)}) \alpha_H(n_{(-1)})) \otimes \alpha_H(h_{21}) \cdot m_{(0)} \otimes \alpha_H(h_{22}) \cdot n_{(0)} \quad \text{by (2.1), (2.2) and (2.4)} \\
 &= \Theta(h_1) \alpha_H(m_{(-1)} n_{(-1)}) \otimes \alpha_H(h_2) \cdot (m_{(0)} \otimes n_{(0)}).
 \end{aligned}$$

Let us choose another part of the proof: the H -linearity of $c_{M,N}$ can be obtained by

$$\begin{aligned}
 c_{M,N}(h \cdot (m \otimes n)) &= \Theta^{-1}((h_1 \cdot m)_{(-1)}) \cdot \alpha_N^{-1}(h_2 \cdot n) \otimes \alpha_M^{-1}((h_1 \cdot m)_{(0)}) \\
 &\stackrel{(2.8),(2.9)}{=} (\Theta^{-1} \alpha^{-1}((h_1 \cdot m)_{(-1)}) \alpha_H^{-1}(h_2)) \cdot n \otimes \alpha_M^{-1}((h_1 \cdot m)_{(0)}) \\
 &\stackrel{(2.1)}{=} \Theta^{-1} \alpha_H^{-1}((h_1 \cdot m)_{(-1)} \Theta(h_2)) \cdot n \otimes \alpha_M^{-1}((h_1 \cdot m)_{(0)}) \\
 &\stackrel{(3.1)}{=} \Theta^{-1} \alpha_H^{-1}(\Theta(h_1) \alpha_H(m_{(-1)})) \cdot n \otimes \alpha_M^{-1}(\alpha_H(h_2) \cdot m_{(0)}) \\
 &\stackrel{(2.1),(2.2)}{=} h_1 \cdot (\Theta^{-1}(m_{(-1)}) \cdot \alpha_N^{-1}(n)) \otimes (h_2 \cdot \alpha_M^{-1}(m_{(0)})) = h \cdot c_{M,N}(m \otimes n),
 \end{aligned}$$

for all $h \in H$, $m \in M$ and $n \in N$. □

Remark 3.9. Theorem 4.4 in [14] can be regarded as a special case of Theorem 3.8 if we take $\Theta = \alpha^2$. Also, if we take $\Theta = \alpha^{-2}$, then Theorem 4.7 in [27] is just a special case of this theorem. Furthermore, if we take $\Theta = \alpha^{2\nu-2}$, $\nu \in \mathbb{Z}$, Theorem 3.8 is the same as Theorem 2.12 in [28].

By Theorem 3.8, we know that the Θ -Yetter-Drinfeld category ${}^H_H\mathcal{HYD}^\Theta$ is a braided tensor category. Therefore, we can obtain the following corollary.

Corollary 3.10. *Let (H, α_H) be a Hom-Hopf algebra with a bijective antipode S . Assume (M, α_M) , (N, α_N) and $(P, \alpha_P) \in {}^H_H\mathcal{HYD}^\nu$. Then the braiding c satisfies the braid relation*

$$(id_P \otimes c_{M,N}) \circ (c_{M,P} \otimes id_N) \circ (id_M \otimes c_{N,P}) = (c_{N,P} \otimes id_M) \circ (id_N \otimes c_{M,P}) \circ (c_{M,N} \otimes id_P).$$

Remark 3.11. This implies that $c_{M,M}$ is a solution of the Yang-Baxter equation for any object M of a Θ -Yetter-Drinfeld category ${}^H_H\mathcal{HYD}^\Theta$. By Proposition 3.6 and Theorem 3.8, we find an interesting fact that every Θ -Yetter-Drinfeld category ${}^H_H\mathcal{HYD}^\Theta$ can give not only a solution of the Hom-Yang-Baxter equation but also a solution of the classical Yang-Baxter equation.

4 Symmetric pairs of Θ -Yetter-Drinfeld categories over a Hom-Hopf algebra

By [30], we know that the Yetter-Drinfeld category over a Hopf algebra H is symmetric if and only if $H = k$. In this section, we extend some interesting results to Θ -Yetter-Drinfeld categories ${}^H_H\mathcal{HYD}^\Theta$, e.g., symmetric

Θ -Yetter-Drinfeld categories ${}^H_H\mathcal{HYD}^\Theta$ over a Hom-Hopf algebra are all trivial. The results obtained in this section generalize the corresponding results in [28–31].

Definition 4.1. Let (M, α_M) and (N, α_N) be two objects in a Θ -Yetter-Drinfeld category ${}^H_H\mathcal{HYD}^\Theta$. The pair (M, N) is called a *symmetric pair* if $c_{N,M}c_{M,N} = id_{M \otimes N}$. Particularly, the category ${}^H_H\mathcal{HYD}^\Theta$ is called *symmetric* whenever any pair of objects $M, N \in {}^H_H\mathcal{HYD}^\Theta$ is a symmetric pair.

Theorem 4.2. Let (H, α) be a Hom-Hopf algebra, then (H_2, H_1) is a symmetric pair if and only if $H = k$.

Proof. By Example 3.3, $H_1 = (H, \triangleright, \Delta, \alpha_H)$ and $H_2 = (H, \mu, \lambda, \alpha_H)$ are two ν -Yetter Drinfeld modules. On setting $1_H \otimes h \in H_2 \otimes H_1$, we have

$$c_{H_2, H_1}(1_H \otimes h) = \Theta^{-1}(1_{H(-1)}) \triangleright \alpha_H^{-1}(h) \otimes \alpha_H^{-1}(1_{H(0)}) = 1_H \triangleright \alpha_H^{-1}(h) \otimes 1_H \stackrel{(2.9)}{=} h \otimes 1_H.$$

Furthermore, we have

$$\begin{aligned} c_{H_1, H_2}c_{H_2, H_1}(1_H \otimes h) &= c_{H_1, H_2}(h \otimes 1_H) \\ &= \Theta^{-1}(h_{(-1)}) \cdot \alpha_H^{-1}(1_H) \otimes \alpha_H^{-1}(h_{(0)}) \\ &= \Theta^{-1}(h_1) \cdot \alpha_H^{-1}(1) \otimes \alpha_H^{-1}(h_2) \\ &= \Theta^{-1}\alpha_H(h_1) \otimes \alpha_H^{-1}(h_2). \end{aligned}$$

Since (H_2, H_1) is a symmetric pair, i.e., $c_{H_1, H_2}c_{H_2, H_1} = id_{H_2 \otimes H_1}$, it follows that $1_H \otimes h = \Theta^{-1}\alpha_H(h_1) \otimes \alpha_H^{-1}(h_2)$. Applying $id \otimes \varepsilon$ to the above equality gives

$$1_H \otimes \varepsilon(h) = 1_H \varepsilon(h) = \Theta^{-1}\alpha_H(h_1) \otimes \varepsilon(\alpha_H^{-1}(h_2)) \stackrel{(2.3)}{=} \Theta^{-1}\alpha_H(h_1) \otimes \varepsilon(h_2) \stackrel{(2.4)}{=} \Theta^{-1}\alpha^2(h)$$

and it then follows that $h = \Theta\alpha^{-2}(\varepsilon(h)1_H) = \varepsilon(h)1_H$. This means $H = k$, as desired. The converse is straightforward. $\square$

The following corollary is easily obtained from the above theorem.

Corollary 4.3. Let (H, α) be a Hom-Hopf algebra such that the Θ -Yetter-Drinfeld category ${}^H_H\mathcal{HYD}^\Theta$ is symmetric. Then $H = k$.

Remark 4.4.

- (1) If $\alpha = id$ and $\Theta = id$, Theorem 4.2 is exactly the famous conclusion in [30], namely, the symmetric Yetter-Drinfeld category ${}^H_H\mathcal{YD}$ over a Hopf algebra is trivial.
- (2) If $\Theta = \alpha^{-2}$, Theorem 4.2 is just Theorem 3.5 in [31]. Also, if $\Theta = \alpha^{2\nu-2}$, $\nu \in \mathbb{Z}$, Theorem 4.2 is just Theorem 3.2 in [28].
- (3) The special case of Theorem 4.2 has been considered in the setting of monoidal Hom-Hopf algebras; see [26].

Theorem 4.5. Let (H, α) be a Hom-Hopf algebra. Then (H_1, H_2) is a symmetric pair if and only if $H = k$.

Proof. Let $h \otimes 1_H \in H_1 \otimes H_2$. Then we obtain

$$\begin{aligned} c_{H_1, H_2}(h \otimes 1_H) &= \Theta^{-1}(h_{(-1)}) \cdot \alpha_H^{-1}(1_H) \otimes \alpha_H^{-1}(h_{(0)}) \\ &= \Theta^{-1}(h_1) \cdot 1_H \otimes \alpha_H^{-1}(h_2) \\ &= \Theta^{-1}\alpha_H(h_1) \otimes \alpha_H^{-1}(h_2) \end{aligned}$$

and also

$$\begin{aligned}
c_{H_2, H_1} c_{H_1, H_2}(h \otimes 1_H) &= (\Theta^{-1} \alpha_H(h_1) \otimes \alpha_H^{-1}(h_2)) \\
&= \Theta^{-1}(\Theta^{-1} \alpha(h_1)_{(-1)}) \triangleright \alpha_H^{-1}(\alpha_H^{-1}(h_2)) \otimes \alpha_H^{-1}(\Theta^{-1} \alpha_H(h_1)_{(0)}) \\
&= \Theta^{-1}(\Theta \alpha_H^{-4}(\Theta^{-1} \alpha_H(h_1)_{11}) \Theta \alpha_H^{-3} S(\Theta^{-1} \alpha_H(h_1)_{22})) \triangleright \alpha_H^{-2}(h_2) \otimes \alpha_H^{-1}(\alpha_H^{-1}(\Theta^{-1} \alpha_H(h_1)_{12})) \\
&\stackrel{(2.3)}{=} (\Theta^{-1} \alpha_H^{-3}(h_{111}) S \Theta^{-1} \alpha_H^{-2}(h_{12})) \triangleright \alpha_H^{-2}(h_2) \otimes \Theta^{-1} \alpha_H^{-1}(h_{112}) \\
&= (\Theta \alpha_H^{-4}((\Theta^{-1} \alpha_H^{-3}(h_{111}) S \Theta^{-1} \alpha_H^{-2}(h_{12}))_1) \alpha_H^{-1}(\alpha_H^{-2}(h_2)) \\
&\quad \times S(\Theta \alpha_H^{-3}(\Theta^{-1} \alpha_H^{-3}(h_{111}) S \Theta^{-1} \alpha_H^{-2}(h_{12}))_2) \otimes \Theta^{-1} \alpha_H^{-1}(h_{112})) \\
&= (\Theta \alpha_H^{-4}(\Theta^{-1} \alpha_H^{-3}(h_{1111}) S \Theta^{-1} \alpha_H^{-2}(h_{122})) \alpha_H^{-1}(\alpha_H^{-2}(h_2))) \\
&\quad \times S(\Theta \alpha_H^{-3}(\Theta^{-1} \alpha_H^{-3}(h_{1112}) S \Theta^{-1} \alpha_H^{-2}(h_{121}))) \otimes \Theta^{-1} \alpha_H^{-1}(h_{112}) \\
&= ((\alpha_H^{-7}(h_{1111}) S \alpha_H^{-6}(h_{122})) \alpha_H^{-3}(h_2)) S(\alpha_H^{-6}(h_{1112}) S \alpha_H^{-5}(h_{121})) \otimes \Theta^{-1} \alpha_H^{-1}(h_{112}),
\end{aligned}$$

where the sixth equality uses (2.3) and the property of the antipode S . Since (H_1, H_2) is a symmetric pair, i.e., $c_{H_2, H_1} c_{H_1, H_2} = id_{H_1 \otimes H_2}$, it follows that $h \otimes 1_H$ is equals to

$$((\alpha_H^{-7}(h_{1111}) S \alpha_H^{-6}(h_{122})) \alpha_H^{-3}(h_2)) S(\alpha_H^{-6}(h_{1112}) S \alpha_H^{-5}(h_{121})) \otimes \Theta^{-1} \alpha_H^{-1}(h_{112}).$$

Applying $\varepsilon \otimes id$ to the above equality yields $\varepsilon(h)1_H = \Theta^{-1} \alpha_H^2(h)$. It then follows that $h = \Theta \alpha_H^{-2}(\varepsilon(h)1_H) = \varepsilon(h)1_H$. This means $H = k$, as required. The converse is obvious. $\square$

Remark 4.6. Note that Corollary 4.3 can also be obtained by Theorem 4.5.

5 Pseudosymmetry of Θ -Yetter-Drinfeld categories over a Hom-Hopf algebra

In this section, we will find a necessary and sufficient condition for a Θ -Yetter-Drinfeld category ${}^H_H \mathcal{HYD}^\Theta$ over a Hom-Hopf algebra (H, S, α) to be pseudosymmetric. The main theorem obtained in this section generalizes the main conclusions in [28, 30, 31].

Definition 5.1. [34, 35] Let $\mathcal{C}$ be a tensor category and c a braiding on $\mathcal{C}$. The braiding c is called a *pseudosymmetry* if, for all $M, N, P \in \mathcal{C}$, there holds the following condition:

$$(id_P \otimes c_{M, N})(c_{P, M}^{-1} \otimes id_N)(id_M \otimes c_{N, P}) = (c_{N, P} \otimes id_M)(id_N \otimes c_{P, M}^{-1})(c_{M, N} \otimes id_P). \quad (5.1)$$

In this case, $\mathcal{C}$ is called a *pseudosymmetric braided tensor category*.

Obviously, a symmetric braided tensor category must be a pseudosymmetric braided tensor category.

Lemma 5.2. Let (H, S, α) be a cocommutative Hom-Hopf algebra. Then, the canonical braiding of the Θ -Yetter-Drinfeld category c_{H_2, H_1} is the usual flip map.

Proof. For all $x \in H_2$ and $y \in H_1$, we have

$$\begin{aligned}
c_{H_2, H_1}(x \otimes y) &= \Theta(x_{(-1)}) \triangleright \alpha_H^{-1}(y) \otimes \alpha_H^{-1}(x_{(0)}) \\
&= \Theta^{-1}(\Theta \alpha_H^{-4}(x_{11}) \Theta \alpha_H^{-3} S(x_2)) \triangleright \alpha_H^{-1}(y) \otimes \alpha_H^{-2}(x_{12}) \\
&= (\alpha_H^{-4}(x_{11}) \alpha_H^{-3} S(x_2)) \cdot \alpha_H^{-1}(y) \otimes \alpha_H^{-2}(x_{12}) \\
&= (\alpha_H^{-4}(x_{12}) \alpha_H^{-3} S(x_2)) \cdot \alpha_H^{-1}(y) \otimes \alpha_H^{-2}(x_{11}) \\
&\stackrel{(2.4)}{=} (\alpha_H^{-4}(x_{21}) \alpha_H^{-4} S(x_{22})) \cdot \alpha_H^{-1}(y) \otimes \alpha_H^{-1}(x_1) \\
&\stackrel{(2.7), (2.4)}{=} 1 \cdot \alpha_H^{-1}(y) \otimes x = y \otimes x,
\end{aligned}$$

where the fourth equality uses cocommutativity. $\square$

Lemma 5.3. *Let (H, S, α) be a Hom-Hopf algebra. Then, the canonical braiding of Θ -Yetter-Drinfeld category satisfies $(\varepsilon \otimes id)c_{H_2, H_1}(x \otimes y) = \varepsilon(y)x$ for all $x \in H_2$ and $y \in H_1$.*

Proof. For all $x \in H_2, y \in H_1$, we calculate

$$\begin{aligned} (\varepsilon \otimes id)c_{H_2, H_1}(x \otimes y) &= (\varepsilon \otimes id)(\alpha_H^{-2v+2}(x_{(-1)}) \cdot \alpha_H^{-1}(y) \otimes \alpha_H^{-1}(x_{(0)})) \\ &= \Theta^{-1}(\Theta\alpha_H^{-4}(x_{11})\Theta\alpha_H^{-3}S(x_2)) \triangleright \alpha_H^{-1}(y) \otimes \alpha_H^{-2}(x_{12}) \\ &= (\varepsilon \otimes id)((\alpha_H^{-4}(x_{11})\alpha_H^{-3}S(x_2)) \cdot \alpha_H^{-1}(y) \otimes \alpha_H^{-2}(x_{12})) \\ &= (\varepsilon \otimes id)((\Theta\alpha_H^{-4}((\alpha_H^{-4}(x_{11})\alpha_H^{-3}S(x_2))_1)\alpha_H^{-2}(y)) \\ &\quad \times S(\Theta\alpha_H^{-3}((\alpha_H^{-4}(x_{11})\alpha_H^{-3}S(x_2))_2))) \otimes \alpha_H^{-2}(x_{12}) \\ &= \varepsilon(x_{11})\varepsilon(x_2)\varepsilon(y)\alpha_H^{-2}(x_{12}) \stackrel{(2.4)}{=} \varepsilon(y)x, \end{aligned}$$

completing the proof. $\square$

Theorem 5.4. *Let (H, α_H) be a Hom-Hopf algebra with a bijective antipode S . Then, the canonical braiding of the Θ -Yetter-Drinfeld category ${}^H_H\mathcal{HYD}^\Theta$ is pseudosymmetric if and only if (H, α_H) is commutative and cocommutative.*

Proof. ($\Leftarrow$) Assume that (H, α_H) is commutative and cocommutative. We first prove that the compatibility (3.2) becomes the equality

$$(h \cdot m)_{(-1)} \otimes (h \cdot m)_{(0)} = \alpha_H(m_{(-1)}) \otimes \alpha_H(h) \cdot m_{(0)} \quad (5.2)$$

for all $h \in H$ and $m \in M$. Indeed, routine manipulations give

$$\begin{aligned} (h \cdot m)_{(-1)} \otimes (h \cdot m)_{(0)} &= ((\Theta\alpha_H^{-4}(h_{11}))\alpha_H^{-1}(m_{(-1)}))\Theta\alpha_H^{-2}S(h_2) \otimes \alpha_H^{-1}(h_{12})m_{(0)} \\ &\stackrel{(2.4)}{=} ((\Theta\alpha_H^{-3}(h_1))\alpha_H^{-1}(m_{(-1)}))\Theta\alpha_H^{-3}S(h_{22}) \otimes \alpha_H^{-1}(h_{21})m_{(0)} \\ &= (\alpha_H^{-1}(m_{(-1)})(\Theta\alpha_H^{-3}(h_1))\Theta\alpha_H^{-3}S(h_{21}) \otimes \alpha_H^{-1}(h_{22})m_{(0)} \\ &\stackrel{(2.1)}{=} m_{(-1)}((\Theta\alpha_H^{-3}(h_1))\Theta\alpha_H^{-4}S(h_{21})) \otimes \alpha_H^{-1}(h_{22})m_{(0)} \\ &\stackrel{(2.4)}{=} m_{(-1)}((\Theta\alpha_H^{-4}(h_{11}))\Theta\alpha_H^{-4}S(h_{12})) \otimes h_2m_{(0)} \\ &\stackrel{(2.7)}{=} m_{(-1)} \cdot \Theta\alpha_H^{-4}(1_H)\varepsilon(h_1) \otimes h_2 \cdot m_{(0)} \\ &= \alpha_H(m_{(-1)}) \otimes \alpha_H(h) \cdot m_{(0)}, \end{aligned}$$

where we used commutativity and cocommutativity of H in the third equality.

Now we claim the equality (5.1). For this end, let $(M, \alpha_M), (N, \alpha_N)$ and $(P, \alpha_P) \in {}^H_H\mathcal{HYD}^\Theta$. Then, for all $m \in M, n \in N$ and $p \in P$, we obtain that

$$\begin{aligned} (id_P \otimes c_{M,N})(c_{P,M}^{-1} \otimes id_N)(id_M \otimes c_{N,P})(m \otimes n \otimes p) \\ &= (id_P \otimes c_{M,N})(c_{P,M}^{-1} \otimes id_N)(m \otimes \Theta^{-1}(n_{(-1)}) \cdot \alpha_P^{-1}(p) \otimes \alpha_N^{-1}(n_{(0)})) \\ &= (id_P \otimes c_{M,N})(\alpha_H^{-1}((\Theta^{-1}(n_{(-1)}) \cdot \alpha_H^{-1}(p))_{(0)}) \otimes S^{-1}\Theta((\Theta^{-1}(n_{(-1)}) \cdot \alpha_H^{-1}(p))_{(-1)}) \cdot \alpha_H^{-1}(m) \otimes \alpha_N^{-1}(n_{(0)})) \\ &\stackrel{(5.2)}{=} (id_P \otimes c_{M,N})(\alpha_H^{-1}(\alpha_H\Theta^{-1}(n_{(-1)}) \cdot \alpha_P^{-1}(p))_{(0)}) \otimes S^{-1}\Theta^{-1}(\alpha_H(\alpha_P^{-1}(p))_{(-1)}) \cdot \alpha_M^{-1}(m) \otimes \alpha_N^{-1}(n_{(0)})) \\ &= (id_P \otimes c_{M,N})(\Theta^{-1}(n_{(-1)}) \cdot \alpha_P^{-2}(p_{(0)}) \otimes S^{-1}\Theta^{-1}(p_{(-1)}) \cdot \alpha_M^{-1}(m) \otimes \alpha_N^{-1}(n_{(0)})) \quad \text{by (2.10) and (2.8)} \\ &= \Theta^{-1}(n_{(-1)}) \cdot \alpha_P^{-2}(p_{(0)}) \otimes \Theta^{-1}((S^{-1}\Theta^{-1}(p_{(-1)})\alpha_M^{-1}(m))_{(-1)}) \cdot \alpha_N^{-2}(n_{(0)}) \otimes \alpha_H^{-1}((S^{-1}\Theta^{-1}(p_{(-1)}) \cdot \alpha_M^{-1}(m))_{(0)}) \\ &\stackrel{(5.2)}{=} \Theta^{-1}(n_{(-1)}) \cdot \alpha_P^{-2}(p_{(0)}) \otimes \Theta^{-1}\alpha_H((\alpha_M^{-1}(m))_{(-1)}) \cdot \alpha_N^{-2}(n_{(0)}) \otimes \alpha_M^{-1}(\alpha_H S^{-1}\Theta^{-1}(p_{(-1)}) \cdot \alpha_M^{-1}(m))_{(0)} \\ &\stackrel{(2.8)}{=} \Theta^{-1}(n_{(-1)}) \cdot \alpha_P^{-2}(p_{(0)}) \otimes \Theta^{-1}\alpha_H(\alpha_M^{-1}(m))_{(-1)} \cdot \alpha_N^{-2}(n_{(0)}) \otimes \Theta^{-1}S^{-1}(p_{(-1)}) \cdot \alpha_M^{-2}(m)_{(0)} \\ &\stackrel{(2.10)}{=} \Theta^{-1}(n_{(-1)}) \cdot \alpha_P^{-2}(p_{(0)}) \otimes \Theta^{-1}(m_{(-1)}) \cdot \alpha_N^{-2}(n_{(0)}) \otimes \Theta^{-1}S^{-1}(p_{(-1)}) \cdot \alpha_M^{-2}(m_{(0)}), \end{aligned}$$

and also

$$\begin{aligned}
& (c_{N,P} \otimes id_M)(id_N \otimes c_{P,M}^{-1})(c_{M,N} \otimes id_P)(m \otimes n \otimes p) \\
&= (c_{N,P} \otimes id_M)(id_N \otimes c_{P,M}^{-1})(\Theta^{-1}(m_{(-1)}) \cdot \alpha_N^{-1}(n) \otimes \alpha_M^{-1}(m_{(0)}) \otimes p) \\
&= (c_{N,P} \otimes id_M)(\Theta^{-1}(m_{(-1)}) \cdot \alpha_N^{-1}(n) \otimes \alpha_P^{-1}(p_{(0)}) \otimes S^{-1}\Theta^{-1}(p_{(-1)}) \cdot \alpha_M^{-2}(m_{(0)})) \\
&= \Theta^{-1}((\Theta^{-1}(m_{(-1)}) \cdot \alpha_N^{-1}(n))_{(-1)}) \cdot \alpha_P^{-2}(p_{(0)}) \otimes \alpha_N^{-1}((\Theta^{-1}(m_{(-1)}) \cdot \alpha_N^{-1}(n))_{(0)}) \otimes S^{-1}\Theta^{-1}(p_{(-1)}) \cdot \alpha_M^{-2}(m_{(0)})) \\
&\stackrel{(5,2)}{=} \Theta^{-1}(\alpha_H((\alpha_N^{-1}(n))_{(-1)})) \cdot \alpha_P^{-2}(p_{(0)}) \otimes \alpha_N^{-1}(\alpha_H\Theta^{-1}(m_{(-1)}) \cdot \alpha_N^{-1}(n)_{(0)}) \otimes S^{-1}\Theta^{-1}(p_{(-1)}) \cdot \alpha_M^{-2}(m_{(0)}) \\
&\stackrel{(2,8)}{=} \Theta^{-1}(n_{(-1)}) \cdot \alpha_P^{-2}(p_{(0)}) \otimes \Theta^{-1}(m_{(-1)}) \cdot \alpha_N^{-2}(n_{(0)}) \otimes \Theta^{-1}S^{-1}(p_{(-1)}) \cdot \alpha_M^{-2}(m_{(0)}),
\end{aligned}$$

as required.

($\Rightarrow$) Assume that the canonical braiding c is pseudosymmetric. We will first show that (H, α) is cocommutative. For this end, let $1 \otimes \Theta\alpha_H^{-2}(x) \otimes 1 \in H_2 \otimes H_1 \otimes H_2$. Then we obtain

$$\begin{aligned}
& (id \otimes \varepsilon \otimes id)(c_{H_1,H_2} \otimes id_{H_2})(id_{H_1} \otimes c_{H_2,H_2}^{-1})(c_{H_2,H_1} \otimes id_{H_2})(1 \otimes \Theta\alpha_H^{-2}(x) \otimes 1) \\
&= (id \otimes \varepsilon \otimes id)(c_{H_1,H_2} \otimes id_{H_2})(id_{H_1} \otimes c_{H_2,H_2}^{-1})(\Theta\alpha_H^{-2}(x) \otimes 1 \otimes 1) \\
&= (id \otimes \varepsilon \otimes id)(c_{H_1,H_2} \otimes id_{H_2})(\Theta\alpha_H^{-2}(x) \otimes 1 \otimes 1) \\
&= (id \otimes \varepsilon \otimes id)(\Theta^{-1}(\Theta\alpha_H^{-2}(x)_{(-1)}) \cdot 1 \otimes \alpha_H^{-1}(\Theta\alpha_H^{-2}(x)_{(0)}) \otimes 1) \\
&= (id \otimes \varepsilon \otimes id)(\alpha_H^{-1}(x_1) \otimes \Theta\alpha_H^{-3}(x_2) \otimes 1) \stackrel{(2,4)}{=} x \otimes 1,
\end{aligned}$$

where the fourth equality uses (2.2) and (2.3). On the other hand, we have

$$\begin{aligned}
& (id \otimes \varepsilon \otimes id)(id_{H_2} \otimes c_{H_2,H_1})(c_{H_2,H_2}^{-1} \otimes id_{H_1})(id_{H_2} \otimes c_{H_1,H_2})(1 \otimes \Theta\alpha_H^{-2}(x) \otimes 1) \\
&= (id \otimes \varepsilon \otimes id)(id_{H_2} \otimes c_{H_2,H_1})(c_{H_2,H_2}^{-1} \otimes id_{H_1})(1 \otimes \alpha_H^{-1}(x_1) \otimes \Theta\alpha_H^{-3}(x_2)) \\
&= (id \otimes \varepsilon \otimes id)(id_{H_2} \otimes c_{H_2,H_1})(\alpha_H^{-1}(\alpha_H^{-1}(x_1)_{(0)}) \otimes S^{-1}\Theta^{-1}(\alpha_H^{-1}(x_1)_{(-1)}) \cdot \alpha_H^{-1}(1) \otimes \Theta\alpha_H^{-3}(x_2)) \\
&= (id \otimes \varepsilon \otimes id)(id_{H_2} \otimes c_{H_2,H_1})(\alpha_H^{-1}(\alpha_H^{-1}(\alpha_H^{-1}(x_1)_{12}))) \otimes S^{-1}\Theta^{-1} \\
&\quad \times (\Theta\alpha_H^{-4}(x)(\alpha_H^{-1}(x_1)_{11})\Theta\alpha_H^{-3}(x)(S\alpha_H^{-1}(x_1)_{22})) \cdot 1 \otimes \Theta\alpha_H^{-3}(x_2)) \\
&= (id \otimes \varepsilon \otimes id)(id_{H_2} \otimes c_{H_2,H_1})(\alpha_H^{-3}(x_{112}) \otimes S^{-1}(\alpha_H^{-4}(x_{111})S\alpha_H^{-3}(x_{12})) \otimes \Theta\alpha_H^{-3}(x_2)) \quad \text{by (2.2) and (2.3)} \\
&= \alpha_H^{-3}(x_{112}) \otimes \varepsilon(\Theta\alpha_H^{-3}(x)(x_2))S^{-1}(\alpha_H^{-4}(x_{111})S\alpha_H^{-3}(x_{12})) \\
&= \alpha_H^{-2}(x_{12}) \otimes S^{-1}(\alpha_H^{-3}(x_{11})S\alpha_H^{-2}(x_2)) \quad \text{by (2.4) and (2.3)} \\
&= \alpha_H^{-2}(x_{12}) \otimes \alpha_H^{-2}(x_2)S^{-1}(\alpha_H^{-3}(x_{11})),
\end{aligned}$$

where the fifth equality follows by Lemma 5.3. So, we get $x \otimes 1 = \alpha_H^{-2}(x_{12}) \otimes \alpha_H^{-2}(x_2)S^{-1}(\alpha_H^{-3}(x_{11}))$. Furthermore, we have

$$\begin{aligned}
x_2 \otimes x_1 &= x_2 \otimes \alpha_H^{-1}(1 \cdot x_1) \\
&= \alpha_H^{-2}(x_{212}) \otimes \alpha_H^{-1}((\alpha_H^{-2}(x_{22})S^{-1}(\alpha_H^{-3}(x_{211})))x_1) \\
&\stackrel{(2,9)}{=} \alpha_H^{-2}(x_{212}) \otimes \alpha_H^{-1}(\alpha_H^{-1}(x_{22})(S^{-1}(\alpha_H^{-3}(x_{211}))\alpha_H^{-1}(x_1))) \\
&\stackrel{(2,4)}{=} \alpha_H^{-1}(x_{12}) \otimes \alpha_H^{-1}(x_2(S^{-1}(\alpha_H^{-3}(x_{112}))\alpha_H^{-3}(x_{111}))) \\
&\stackrel{(2,7)}{=} \alpha_H^{-1}(x_{12}) \otimes \alpha_H^{-1}(x_2 \cdot 1)\varepsilon(x_{11}) \stackrel{(2,4),(2,2)}{=} x_1 \otimes x_2,
\end{aligned}$$

as required. Next we claim that (H, α_H) is commutative. Indeed, given $1 \otimes \Theta\alpha_H^{-2}(x)(x) \otimes \Theta\alpha_H^{-2}(y) \in H_2 \otimes H_1 \otimes H_1$, on one hand, we have

$$\begin{aligned}
& (c_{H_1,H_1} \otimes id_{H_2})(id_{H_1} \otimes c_{H_1,H_2}^{-1})(c_{H_2,H_1} \otimes id_{H_1})(1 \otimes \Theta\alpha_H^{-2}(x) \otimes \Theta\alpha_H^{-2}(y)) \\
&= (c_{H_1,H_1} \otimes id_{H_2})(id_{H_1} \otimes c_{H_1,H_2}^{-1})(\Theta\alpha_H^{-2}(x) \otimes 1 \otimes \Theta\alpha_H^{-2}(y)) \\
&= (c_{H_1,H_1} \otimes id_{H_2})(\Theta\alpha_H^{-2}(x) \otimes \Theta\alpha_H^{-3}(y_2) \otimes S^{-1}(\alpha_H^{-1}(y_1))) \\
&= \Theta^{-1}(\Theta\alpha_H^{-2}(x)_{(-1)}) \cdot \alpha_H^{-1}(\Theta\alpha_H^{-3}(y_2)) \otimes \alpha_H^{-1}(\Theta\alpha_H^{-2}(x)_{(0)}) \otimes S^{-1}(\alpha_H^{-1}(y_1))
\end{aligned}$$

$$\begin{aligned}
&= \alpha_H^{-2}(x_1) \cdot \Theta \alpha_H^{-4}(y_2) \otimes \Theta \alpha_H^{-3}(x_2) \otimes S^{-1}(\alpha_H^{-1}(y_1)) \\
&= (\Theta \alpha_H^{-4}(\alpha_H^{-2}(x_{11})) \alpha_H^{-1}(\Theta \alpha_H^{-4}(y_2))) S(\Theta \alpha_H^{-3}(\alpha_H^{-2}(x_{12}))) \otimes \Theta \alpha_H^{-3}(x_2) \otimes S^{-1}(\alpha_H^{-1}(y_1)) \\
&= (\Theta \alpha_H^{-6}(x_{11}) \Theta \alpha_H^{-5}(y_2)) S(\Theta \alpha_H^{-5}(x_{12})) \otimes \Theta \alpha_H^{-3}(x_2) \otimes S^{-1}(\alpha_H^{-1}(y_1)),
\end{aligned}$$

where the first equality is followed by Lemma 5.2. On the other hand, we compute

$$\begin{aligned}
&(id_{H_1} \otimes c_{H_2, H_1})(c_{H_1, H_2}^{-1} \otimes id_{H_1})(id_{H_2} \otimes c_{H_1, H_1})(1 \otimes \Theta \alpha_H^{-2}(x) \otimes \Theta \alpha_H^{-2}(y)) \\
&= (id_{H_1} \otimes c_{H_2, H_1})(c_{H_1, H_2}^{-1} \otimes id_{H_1})(1 \otimes \alpha_H^{-2}(x_1) \cdot \Theta \alpha_H^{-3}(y) \otimes \Theta \alpha_H^{-3}(x_2)) \\
&= (id_{H_1} \otimes c_{H_2, H_1})(c_{H_1, H_2}^{-1} \otimes id_{H_1})(1 \otimes (\Theta \alpha_H^{-6}(x_{11}) \Theta \alpha_H^{-4}(y)) S(\Theta \alpha_H^{-5}(x_{12})) \otimes \Theta \alpha_H^{-3}(x_2)) \\
&= (id_{H_1} \otimes c_{H_2, H_1})((\Theta \alpha_H^{-7}(x_{112}) \Theta \alpha_H^{-5}(y_2)) S(\Theta \alpha_H^{-6}(x_{121})) \otimes \alpha_H^{-4}(x_{122}) S^{-1}(\alpha_H^{-5}(x_{111}) \alpha_H^{-3}(y_1)) \otimes \Theta \alpha_H^{-3}(x_2)) \\
&= (\Theta \alpha_H^{-7}(x_{112}) \Theta \alpha_H^{-5}(y_2)) S(\Theta \alpha_H^{-6}(x_{121})) \otimes \Theta \alpha_H^{-3}(x_2) \otimes \alpha_H^{-4}(x_{122}) S^{-1}(\alpha_H^{-5}(x_{111}) \alpha_H^{-3}(y_1)),
\end{aligned}$$

where the last equality relies on Lemma 5.2. Since ${}^H_H \mathcal{HY} \mathcal{D}^\Theta$ is pseudosymmetric, it follows that

$$\begin{aligned}
&(\Theta \alpha_H^{-6}(x_{11}) \Theta \alpha_H^{-5}(y_2)) S(\Theta \alpha_H^{-5}(x_{12})) \otimes \Theta \alpha_H^{-3}(x_2) \otimes S^{-1}(\alpha_H^{-1}(y_1)) \\
&= (\Theta \alpha_H^{-7}(x_{112}) \Theta \alpha_H^{-5}(y_2)) S(\Theta \alpha_H^{-6}(x_{121})) \otimes \Theta \alpha_H^{-3}(x_2) \otimes \alpha_H^{-4}(x_{122}) S^{-1}(\alpha_H^{-5}(x_{111}) \alpha_H^{-3}(y_1)).
\end{aligned}$$

Now applying $\varepsilon \otimes \Theta^{-1} \alpha_H^2 \otimes S$ to the above equality yields

$$x \otimes y = \alpha_H^{-1}(x_2) \otimes (\alpha_H^{-4}(x_{11}) \alpha_H^{-2}(y)) S \alpha_H^{-3}(x_{12}).$$

and then we obtain

$$\begin{aligned}
yx &= ((\alpha_H^{-4}(x_{11}) \alpha_H^{-2}(y)) S \alpha_H^{-3}(x_{12})) \alpha_H^{-1}(x_2) \\
&\stackrel{(2.2)}{=} (\alpha_H^{-3}(x_{11}) \alpha_H^{-1}(y)) (S \alpha_H^{-3}(x_{12}) \alpha_H^{-2}(x_2)) \\
&\stackrel{(2.4)}{=} (\alpha_H^{-2}(x_1) \alpha_H^{-1}(y)) (S \alpha_H^{-3}(x_{21}) \alpha_H^{-3}(x_{22})) \\
&\stackrel{(2.7)}{=} (\alpha_H^{-2}(x_1) \alpha_H^{-1}(y)) \cdot 1 \varepsilon(x_2) \stackrel{(2.2), (2.4)}{=} xy.
\end{aligned}$$

Therefore H is commutative, as desired. The proof is complete. $\square$

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