Research Article

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Classification of f -biharmonic submanifolds in Lorentz space forms

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Abstract: In this paper, f -biharmonic submanifolds with parallel normalized mean curvature vector field in Lorentz space forms are discussed. When f is a constant, we prove that such submanifolds have parallel mean curvature vector field with the minimal polynomial of the shape operator of degree ≤ 2 . When f is a function, we completely classify such pseudo-umbilical submanifolds.

Keywords: Lorentz space forms, f -biharmonic submanifolds, parallel normal mean curvature vector field, the shape operator, pseudo-umbilical

MSC 2020: 53C40

1 Introduction

The theory of harmonic maps between manifolds is a very active and rich subject as an extension of important concepts of geodesics, minimal surfaces, and has been studied extensively by many mathematicians and physicists, we refer to [1–5] for details. As a generalization of harmonic maps, Eells and Lemaire in [1] suggested the ideas of k -harmonic maps. Later, Jiang in [6] defined the biharmonic maps or 2-harmonic maps (i.e., its bitension field vanishes identically) in the sense of Eells and Lemaire.

During the last decade, there have been tremendous developments in the study of biharmonic maps, which can be divided in two main research directions. On the one side is the analytic aspect from the point of view of partial differential equation (PDE): biharmonic maps are solutions of a fourth order strongly elliptic semilinear PDE, we refer to [7,8] for a review. On the other side, the differential geometric aspect has driven attention to biharmonic submanifolds as those submanifolds whose isometric immersions are biharmonic maps. The study of such submanifolds was initiated by Jiang as applications of biharmonic maps (cf. [6,9]), also independently by Chen (cf. [10]) in his study on finite type submanifolds, and had become a very dynamic research subject in modern differential geometry. We refer to [9,11–14] for a review, [15,16] with references therein for recent progress.

As a generalization of biharmonic submanifolds, η -biharmonic submanifolds with η being a constant (i.e., its bitension field and tension field satisfy a linear relation, see Section 2.3 for details) are concerned by many geometers, and important progress has been made. For example, Ferrández and Lucas in [17] classified completely η -biharmonic surfaces of Lorentz spaces $\mathbb{L}^3$. In [18], Ferrández and Lucas classified η -biharmonic hypersurfaces in high-dimensional Lorentz spaces $\mathbb{L}^{n+1}$ under the assumption that the minimal polynomial of the shape operator is at most of degree two. Most recently, Du (cf. [19]) obtained several full classification results of η -biharmonic surfaces in non-flat Lorentz 3-space forms and some examples of such nonminimal surfaces.

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In [20], Arvanitoyeorgos et al. proved that η -biharmonic hypersurfaces of pseudo-Euclidean spaces $\mathbb{E}_s^4$ with diagonalizable shape operator have constant mean curvature, and the same conclusion holds for Lorentz hypersurfaces in $\mathbb{L}^4$ (cf. [21]), hypersurfaces of index 2 in $\mathbb{E}_2^4$ (cf. [22]), and hypersurfaces in $\mathbb{E}_s^{n+1}$ with certain geometric properties (cf. [23] with references therein), respectively. More general, it was shown in [24] that the conclusion remains true for η -biharmonic hypersurfaces with certain geometric conditions in pseudo-Riemannian space forms. We need to point out that the conclusion also holds for the corresponding surfaces or hypersurfaces in [17–19].

In this paper, we attempt to study f -biharmonic submanifolds (instead of hypersurfaces) in Lorentz space forms, where f is a function. The outline of the paper is as follows. In Section 2, we recall some basic notions and formulas, and different forms of the shape operator of submanifolds in Lorentz space forms. Meanwhile, we give a more precise statement of f -biharmonic submanifolds in pseudo-Riemannian manifolds and the relationship between biharmonic submanifolds and f -biharmonic submanifolds. In Section 3, we study f -biharmonic submanifolds with parallel normalized mean curvature vector field from two aspects: When f is a constant, under the assumption that the minimal polynomial of the shape operator is at most of degree two, we prove that submanifolds have parallel mean curvature vector field, which means that the conclusion in [11] holds for such η -biharmonic submanifolds, and obtain an upper bound of that curvature. When f is a function, we classify completely pseudo-umbilical submanifolds.

2 Preliminaries

2.1 Notions and formulas

Let $N_1^{n+p}(c)$ be an $(n + p)$ -dimensional pseudo-Riemannian manifold of constant curvature c ($c \in \{-1, 0, 1\}$), of index 1, which we call a *Lorentz space form*. According to whether $c = 1$, $c = 0$ or $c = -1$, it is called a *de Sitter space* $\mathbb{S}_1^{n+p}(1)$, a *Lorentz space* $\mathbb{L}^{n+p}$ or an *anti-de Sitter space* $\mathbb{H}_1^{n+p}(-1)$. The curvature tensor $\tilde{R}$ of $N_1^{n+p}(c)$ is given by

$$\tilde{R}(X, Y)Z = c(\langle Y, Z \rangle X - \langle X, Z \rangle Y). \quad (2.1)$$

Let M_r^n be a submanifold with signature $(r, n - r)$ in a Lorentz space form $N_1^{n+p}(c)$. Let ∇ and $\tilde{\nabla}$ denote the Levi-Civita connections of M_r^n and $N_1^{n+p}(c)$, respectively. For any tangent vector fields X, Y , and a unit normal vector field ξ of M_r^n in $N_1^{n+p}(c)$, the Gauss and Weingarten formulas are given by (cf. [25] or [26])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \tilde{\nabla}_X \xi = -A_\xi X + D_X \xi,$$

where h is the second fundamental form of M_r^n , A_ξ is the shape operator with respect to ξ , and D is the normal connection of M_r^n . In general, the shape operator A_ξ is not diagonalizable, and A_ξ and h are related by

$$\langle A_\xi X, Y \rangle = \langle h(X, Y), \xi \rangle. \quad (2.2)$$

The mean curvature vector field $\vec{H}$ of M_r^n is defined by $\vec{H} = \frac{1}{n} \text{trace } h$. The mean curvature H of M_r^n in $N_1^{n+p}(c)$ is expressed as $H = \sqrt{|\langle \vec{H}, \vec{H} \rangle|}$.

For the second fundamental form h , the covariant derivative of h is defined by

$$(\tilde{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \quad (2.3)$$

The Codazzi equation is given by

$$(\tilde{\nabla}_X h)(Y, Z) = (\tilde{\nabla}_Y h)(X, Z).$$

Denote the curvature tensor R of M_r^n by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z. \quad (2.4)$$

The Gauss equation is given by

$$R(X, Y, Z, W) = c(\langle Y, Z \rangle \langle X, W \rangle - \langle X, Z \rangle \langle Y, W \rangle) - \langle h(Y, Z), h(X, W) \rangle + \langle h(X, Z), h(Y, W) \rangle.$$

If we denote the curvature tensor with the normal connection R^D of M_r^n by

$$R^D(X, Y)\xi = D_X D_Y \xi - D_Y D_X \xi - D_{[X, Y]}\xi, \quad (2.5)$$

then, for any normal vector fields ξ and η of M_r^n , the Ricci equation is given by

$$R(X, Y, \xi, \eta) = \langle R^D(X, Y)\xi, \eta \rangle = \langle [A_\xi, A_\eta]X, Y \rangle.$$

2.2 The shape operator of submanifolds

When a submanifold M_r^n is Riemannian, it is well known that the shape operator A_α of M_r^n is always diagonalizable, then we can choose a suitable pseudo-Riemannian orthonormal frame basis $\{e_1, e_2, \dots, e_n\}$, such that $A_\alpha = \text{diag}\{\lambda_1^\alpha, \lambda_2^\alpha, \dots, \lambda_n^\alpha\}$, $\alpha = n+1, n+2, \dots, n+p$.

When a submanifold M_r^n is Lorentzian, the shape operators A_α ($\alpha = n+1, n+2, \dots, n+p$) of M_r^n may not be diagonalizable. According to [26, p. 261] (or [25, pp. 33–34]), we know that A_α ($\alpha = n+1, n+2, \dots, n+p$) can be put into one of the following four forms, where D_k is $k \times k$ diagonal,

$$(a) = \begin{pmatrix} \lambda^\alpha & 0 \\ 1 & \lambda^\alpha \\ & D_{n-2} \end{pmatrix}; \quad (b) = \begin{pmatrix} a^\alpha & b^\alpha \\ -b^\alpha & a^\alpha \\ & D_{n-2} \end{pmatrix} \quad (b^\alpha \neq 0); \quad (c) = D_n; \quad (d) = \begin{pmatrix} \lambda^\alpha & 0 & 1 \\ 0 & \lambda^\alpha & 0 \\ 0 & 1 & \lambda^\alpha \\ & & D_{n-3} \end{pmatrix}.$$

Denote by, for $k, l = 1, 2, \dots, n$,

$$\delta_{kl} = \begin{cases} 1, & k = l; \\ 0, & k \neq l. \end{cases}$$

The forms (a) and (b) are represented with respect to a pseudo-orthonormal frame basis $\{v_1, v_2, \dots, v_n\}$ satisfying

$$\begin{aligned} \langle v_1, v_1 \rangle &= \langle v_2, v_2 \rangle = 0, & \langle v_1, v_i \rangle &= \langle v_2, v_i \rangle = 0, \\ \langle v_1, v_2 \rangle &= -1, & \langle v_i, v_j \rangle &= \delta_{ij}, \quad \text{for } i = 3, 4, \dots, n; j = 2, 3, \dots, n, \end{aligned}$$

while the forms (c) and (d) are represented with respect to a local pseudo-Riemannian orthonormal frame basis $\{e_1, e_2, \dots, e_n\}$ satisfying

$$\langle e_1, e_1 \rangle = -1, \quad \langle e_1, e_i \rangle = 0, \quad \langle e_i, e_j \rangle = \delta_{ij}, \quad \text{for } i, j = 2, 3, \dots, n.$$

When the minimal polynomial of the shape operator A_α is at most of degree two, combining with the four forms of A_α : (a), (b), (c), and (d), a straightforward algebraic computation can prove that A_α has one of the following four forms (cf. [18]):

$$(\tilde{a}) = \begin{pmatrix} \lambda^\alpha & 0 \\ 1 & \lambda^\alpha \\ & \lambda^\alpha I_{n-2} \end{pmatrix}; \quad (\tilde{b}) = \begin{pmatrix} a^\alpha & b^\alpha \\ -b^\alpha & a^\alpha \end{pmatrix} \quad (b^\alpha \neq 0); \quad (\tilde{c}) = \lambda^\alpha I_n; \quad (\tilde{d}) = \begin{pmatrix} \lambda^\alpha I_{n_1} & \\ & \mu^\alpha I_{n-n_1} \end{pmatrix},$$

where I_k is the unit matrix, and n_1 and $n - n_1$ are multiplicities of λ^α and μ^α in the form $(\tilde{d})$, respectively.

2.3 The f -biharmonic submanifolds in pseudo-Riemannian manifolds

Let M_r^n and N_q^{n+p} be pseudo-Riemannian manifolds of dimensions n and $(n + p)$, respectively, and $\phi : M_r^n \rightarrow N_q^{n+p}$ be an isometric immersion with the mean curvature vector field $\vec{H}$. Denote by ∇^ϕ the induced connection by ϕ on the bundle $\phi^{-1}TN_q^{n+p}$.

Harmonic isometric immersions $\phi : M_r^n \rightarrow N_q^{n+p}$ between two pseudo-Riemannian manifolds are defined as critical points of the *energy functional*

$$E(\phi) = \frac{1}{2} \int_{M_r^n} \langle d\phi, d\phi \rangle_{\nu_g}.$$

The corresponding Euler-Lagrange equation for $E(\phi)$ is given by the vanishing of the *tension field*

$$\tau(\phi) = \operatorname{tr} \nabla d\phi = n\vec{H}. \quad (2.6)$$

We say that ϕ is biharmonic if it is a critical point of the *bienergy functional*

$$E_2(\phi) = \frac{1}{2} \int_{M_r^n} \langle \tau(\phi), \tau(\phi) \rangle_{\nu_g}.$$

For the biharmonic isometric immersion ϕ , the *bitension field* $\tau_2(\phi)$ satisfies the associated Euler-Lagrange equation

$$\tau_2(\phi) = -n(\Delta^\phi \vec{H} + \operatorname{trace} \tilde{R}(d\phi, \vec{H})d\phi) = 0,$$

where $\Delta^\phi := -\operatorname{trace}(\nabla^\phi \nabla^\phi - \nabla_{\nabla^\phi}^\phi)$, which states the fact that M_r^n is a *biharmonic submanifold* if and only if its bitension field $\tau_2(\phi)$ vanishes identically (cf. [6,9,16] for details). This, together with the well-known fact that a submanifold is minimal if and only if the isometric immersion that defined the submanifold is harmonic, implies that a minimal submanifold is always a biharmonic one.

The energy functional $E_{21\eta}(\phi)$ of $\phi : M_r^n \rightarrow N_q^{n+p}$ between pseudo-Riemannian manifolds:

$$E_{21\eta}(\phi) = E_2(\phi) + \eta E(\phi),$$

is called the $(2, 1, \eta)$ -*energy functional* of ϕ (cf. [27] or [28] for $q = 0$, and [29] for $q \neq 0$), and we say that ϕ is η -biharmonic or $(2, 1, \eta)$ -harmonic if it is a critical point of the $(2, 1, \eta)$ -energy functional. The Euler-Lagrange equation of $E_{21\eta}(\phi)$ is

$$\tau_2(\phi) = \eta \tau(\phi).$$

In particular, when N_q^{n+p} is the pseudo-Euclidean space $\mathbb{E}_q^{n+p}$, $\tau_2(\phi) = \eta \tau(\phi)$ is equivalent to (cf. [20])

$$\Delta \vec{H} = \eta \vec{H},$$

with Δ be the Laplace operator of M_r^n .

If an isometric immersion $\phi : M_r^n \rightarrow N_q^{n+p}$ is η -biharmonic, then M_r^n is called an η -biharmonic submanifold in N_q^{n+p} , which is named by Chen submanifolds with proper mean curvature vector when $N_q^{n+p} = \mathbb{E}_q^{n+p}$.

More general, a submanifold is called a *f*-biharmonic submanifold if the isometric immersion ϕ satisfies $\tau_2(\phi) = f\tau(\phi)$, where f is a function, which is different from Ou's notion of *f*-biharmonic submanifolds (cf. [30]). It is obvious to see that minimal and biharmonic submanifolds (i.e., 0-biharmonic submanifolds) must be *f*-biharmonic.

Finally, we derive the equations for M_r^n of N_q^{n+p} to be *f*-biharmonic.

We choose a suitable local pseudo-Riemannian orthonormal frame field $\{e_1, e_2, \dots, e_{n+p}\}$ on N_q^{n+p} , and a long computation shows that the bitension field of M_r^n is given by (cf. [12,15,25])

$$\begin{aligned} \tau_2(\phi) = & -n\Delta^\phi \vec{H} - n \sum_{i=1}^n \varepsilon_i h(A_{\vec{H}}(e_i), e_i) - n \sum_{i=1}^n \varepsilon_i A_{D_{e_i} \vec{H}}(e_i) - n \sum_{i=1}^n \varepsilon_i (\nabla_{e_i} A_{\vec{H}})(e_i) \\ & - n \left[\sum_{i=1}^n \varepsilon_i \tilde{R}(e_i, \vec{H})e_i \right]^\perp - n \left[\sum_{i=1}^n \varepsilon_i \tilde{R}(e_i, \vec{H})e_i \right]^\top, \end{aligned} \quad (2.7)$$

where

$$\Delta^D = -\sum_{i=1}^n \varepsilon_i (D_{e_i} D_{e_i} - D_{\nabla_{e_i} e_i}). \quad (2.8)$$

Combining with (2.6) and (2.7), and using the elementally argument, we obtain that M_r^n is f -biharmonic if and only if the following two equations hold:

$$\begin{cases} -\Delta^D \vec{H} - \sum_{i=1}^n \varepsilon_i h(A_{\vec{H}}(e_i), e_i) - \left[\sum_{i=1}^n \varepsilon_i \tilde{R}(e_i, \vec{H}) e_i \right]^\perp = f \vec{H}, \\ n \nabla \langle \vec{H}, \vec{H} \rangle + 4 \sum_{i=1}^n \varepsilon_i A_{D_{e_i} \vec{H}}(e_i) + \left[\sum_{i=1}^n \varepsilon_i \tilde{R}(e_i, \vec{H}) e_i \right]^\top = 0. \end{cases} \quad (2.9)$$

3 Main results and their proofs

A submanifold in a pseudo-Riemannian space form $N_q^{n+p}(c)$ with constant sectional curvature c is said to have *parallel normalized mean curvature vector field* if it has nowhere zero mean curvature and the unit vector field in the direction of the mean curvature vector field is parallel in the normal bundle (cf. [10]), i.e., $D(\vec{H}/H) = 0$.

In this paper, we investigate f -biharmonic submanifolds M_r^n with parallel normalized mean curvature vector field in Lorentz space forms $N_1^{n+p}(c)$.

3.1 Characterization of f -biharmonic submanifolds with $f = \text{constant}$

In this section, we consider such f -biharmonic submanifolds with f being a constant. Denote by $\eta = f$. We choose a local orthonormal frame field $\{e_1, e_2, \dots, e_{n+p}\}$ on $N_1^{n+p}(c)$ such that $e_1, \dots, e_n$ are tangent to M_r^n and $e_{n+1} = \frac{\vec{H}}{H}$, $e_{n+2}, \dots, e_{n+p}$ are normal to M_r^n , then

$$\vec{H} = H e_{n+1}. \quad (3.1)$$

According to the definition of the mean curvature vector field and (2.2), we get

$$H = \frac{\varepsilon_{n+1}}{n} \text{trace} A_{n+1}, \quad (3.2)$$

and

$$\text{trace} A_\beta = 0, \quad \forall \beta > n+1. \quad (3.3)$$

Note that

$$D_{e_i} e_{n+1} = D_{e_i} (\vec{H}/H) = 0, \quad i = 1, 2, \dots, n, \quad (3.4)$$

then (2.5) implies that

$$R^D(e_i, e_j) e_{n+1} = 0, \quad i, j = 1, \dots, n,$$

together with the Ricci equation, we have

$$A_{n+1} A_a = A_a A_{n+1}, \quad a = n+2, \dots, n+p. \quad (3.5)$$

While, it follows from (3.1) that, for any $i = 1, 2, \dots, n$,

$$D_{e_i} \vec{H} = D_{e_i} (H e_{n+1}) = e_i(H) e_{n+1}. \quad (3.6)$$

In order to prove our main results, we give the following important lemma.

Lemma 3.1. *Let M_r^n be a submanifold with parallel normalized mean curvature vector field in a Lorentz space form $N_1^{n+p}(c)$. Then M_r^n is η -biharmonic if and only if*

$$\begin{cases} -\Delta H - \varepsilon_{n+1}H \sum_i^n \varepsilon_i \langle A_{n+1}(e_i), A_{n+1}(e_i) \rangle + ncH = \eta H, \\ \sum_i^n \varepsilon_i 2A_{n+1}(\nabla H) + n\varepsilon_{n+1}H\nabla H = 0, \\ \sum_{i=1}^n \langle A_{n+1}(e_i), A_\beta(e_i) \rangle = 0, \quad \forall \beta > n+1, \end{cases} \quad (3.7)$$

where

$$\nabla H = \sum_i^n \varepsilon_i e_i(H) e_i, \quad \Delta H = - \sum_{i=1}^n \varepsilon_i (e_i e_i - \nabla_{e_i} e_i) H.$$

Proof. Using (2.8), (3.1), (3.4), and (3.6), a straightforward calculation gives

$$\Delta^D \vec{H} = \Delta H e_{n+1},$$

and

$$\sum_{i=1}^n \varepsilon_i h(A_{\vec{H}}(e_i), e_i) = H \varepsilon_{n+1} \sum_i^n \varepsilon_i \langle A_{n+1}(e_i), A_{n+1}(e_i) \rangle e_{n+1} + H \sum_{\beta=n+2}^{n+p} \varepsilon_\beta \sum_i^n \varepsilon_i \langle A_{n+1}(e_i), A_\beta(e_i) \rangle e_\beta.$$

Moreover, it follows from (2.1) that

$$\left[\sum_{i=1}^n \varepsilon_i \tilde{R}(e_i, \vec{H}) e_i \right]^\perp = -ncH e_{n+1}, \quad \left[\sum_{i=1}^n \varepsilon_i \tilde{R}(e_i, \vec{H}) e_i \right]^\top = 0.$$

Putting those equations into the first equation of (2.9) yields

$$-\Delta H - \varepsilon_{n+1}H \sum_i^n \varepsilon_i \langle A_{n+1}(e_i), A_{n+1}(e_i) \rangle + ncH = \eta H,$$

and

$$\sum_i^n \varepsilon_i \langle A_{n+1}(e_i), A_\beta(e_i) \rangle = 0, \quad \forall \beta > n+1.$$

Then, it follows from (3.1) and (3.6) that

$$\nabla \langle \vec{H}, \vec{H} \rangle = 2\varepsilon_{n+1}H\nabla H, \quad \sum_{i=1}^n \varepsilon_i A_{D_{e_i}\vec{H}}(e_i) = A_{n+1}(\nabla H).$$

Taking into account the above two equations, the second equation of (2.9) becomes

$$2A_{n+1}(\nabla H) + n\varepsilon_{n+1}H\nabla H = 0,$$

and we complete the proof of Lemma 3.1. $\square$

As it is well known that non-minimal submanifolds with parallel mean curvature vector field (i.e., $D\vec{H} = 0$) must have parallel normalized mean curvature vector field, the converse is not true in general. The counterexamples are given in [10].

Using (3.6), we know that the submanifold with parallel normalized mean curvature vector field has parallel mean curvature vector field provided that its mean curvature is a constant. Next, we will prove that the mean curvature H of M_r^n is a constant when A_α has the form $(\tilde{a})$, $(\tilde{b})$, $(\tilde{c})$, or $(\tilde{d})$, respectively.

Suppose on the contrary that H is not a constant, then $\nabla H \neq 0$ on a certain open subset U . Using the second equation of (3.7), we obtain that one of the principal curvatures of A_{n+1} is $\left(-\frac{n\varepsilon_{n+1}H}{2}\right)$.

Note that the forms $(\tilde{a})$ and $(\tilde{b})$ are not diagonalizable, and $(\tilde{c})$ and $(\tilde{d})$ are diagonalizable, then we show that the assumption runs into a contradiction from the following two sides.

Case 1. The shape operator of M_r^n is not diagonalizable.

Proposition 3.1. *Let M_r^n be an η -biharmonic submanifold with parallel normalized mean curvature vector field in $N_1^{n+p}(c)$. Assume that the shape operator has the form $(\tilde{a})$ or $(\tilde{b})$, then M_r^n has parallel mean curvature vector field.*

Proof. When the shape operator has the form $(\tilde{a})$, we choose a pseudo-orthonormal frame basis $\{u_1, \dots, u_n\}$ with $\langle u_1, u_2 \rangle = \langle u_i, u_i \rangle = 1$ for $i = 2, \dots, n$, and the others be zero, such that (cf. [24]), the shape operator A_{n+1} of M_r^n takes the form

$$\begin{pmatrix} \lambda^{n+1} & 0 & & \\ 1 & \lambda^{n+1} & & \\ & & \ddots & \\ & & & \lambda^{n+1}I_{n-2} \end{pmatrix}.$$

Now, we construct a pseudo-Riemannian orthonormal basis $\{e_1, e_2, \dots, e_n\}$ from the pseudo-orthonormal basis $\{u_1, u_2, \dots, u_n\}$ such that

$$e_1 = \frac{u_1 - u_2}{\sqrt{2}}, \quad e_2 = \frac{u_1 + u_2}{\sqrt{2}}, \quad e_i = u_i, \quad \text{for } i = 3, 4, \dots, n,$$

where e_1 is time-like, and the others are space-like. Then A_{n+1} takes the following form with respect to this new basis,

$$\begin{pmatrix} \lambda^{n+1} + \frac{1}{2} & \frac{1}{2} & \dots & 0 \\ -\frac{1}{2} & \lambda^{n+1} - \frac{1}{2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda^{n+1} \end{pmatrix}.$$

Note that

$$\lambda^{n+1} = -\frac{1}{2}n\varepsilon_{n+1}H.$$

Together with (3.2), we obtain that $\frac{(n+2)H}{2} = 0$, which shows that $H = 0$. This contradicts the assumption that M_r^n has parallel normalized mean curvature vector field.

When the shape operator A_3 has the form $(\tilde{b})$, it is easy to compute that the shape operator of M_r^2 has two adjoint imaginary principal curvatures, which contradicts with the fact that one principal curvature of A_3 is $(-\varepsilon_3H)$. $\square$

Case 2. The shape operator of M_r^n is diagonalizable.

Proposition 3.2. *Let M_r^n be an η -biharmonic submanifold with parallel normalized mean curvature vector field in $N_1^{n+p}(c)$. Assume that the shape operator has the form $(\tilde{c})$ or $(\tilde{d})$, then M_r^n has parallel mean curvature vector field.*

Proof. When the shape operator has the form $(\tilde{c})$ or $(\tilde{d})$, then, correspondingly, A_{n+1} takes the form

$$(III) = \lambda^{n+1}I_n \quad \text{or} \quad (IV) = \begin{pmatrix} \lambda^{n+1}I_{n_1} & \\ & \mu^{n+1}I_{n-n_1} \end{pmatrix}.$$

Without loss of generality, we denote

$$\lambda^{n+1} = -\frac{1}{2}n\varepsilon_{n+1}H. \quad (3.8)$$

When the shape operator A_{n+1} has the form (III), $H = 0$ by (3.2) and (3.8), a contradiction.

When the shape operator A_{n+1} has the form (IV), we choose a pseudo-Riemannian orthonormal frame field $\{e_1, e_2, \dots, e_n\}$ on U so that e_1 is parallel to ∇H , $e_1, e_2, \dots, e_{n_1}$ are principal directions corresponding to λ^{n+1} ; $e_{n_1+1}, e_{n_1+2}, \dots, e_n$ are principal directions corresponding to μ^{n+1} , i.e.,

$$A_{n+1}(e_i) = \lambda_i^{n+1}e_i, \quad i = 1, 2, \dots, n, \quad (3.9)$$

where $\lambda_1^{n+1} = \lambda_2^{n+1} = \dots = \lambda_{n_1}^{n+1} = \lambda^{n+1}$, $\lambda_{n_1+1}^{n+1} = \lambda_{n_1+2}^{n+1} = \dots = \lambda_n^{n+1} = \mu^{n+1}$. Then it follows from (3.2) that

$$\varepsilon_{n+1}nH = n_1\lambda^{n+1} + (n - n_1)\mu^{n+1}. \quad (3.10)$$

Set

$$\nabla_{e_i}e_j = \sum_{k=1}^n \varepsilon_k \Gamma_{ij}^k e_k, \quad i, j = 1, 2, \dots, n. \quad (3.11)$$

Applying compatibility condition to calculate $e_i\langle e_j, e_k \rangle$, we conclude

$$\Gamma_{ij}^k = -\Gamma_{ik}^j. \quad (3.12)$$

We consider an arbitrary integral curve γ of e_1 and denote by H', H'' the first and the second derivatives of H along this curve.

Upon the hypothesis that H is not a constant, in order to simplify the proving process of Proposition 3.2, we prove the following four lemmas which are the four key steps in the remaining proof of Proposition 3.2. $\square$

Lemma 3.2. *We have, for distinct $i, j = 1, 2, \dots, n$,*

$$e_i(\lambda_j^{n+1})e_j + \sum_{l=1}^n \varepsilon_l(\lambda_j^{n+1} - \lambda_l^{n+1})\Gamma_{ij}^l e_l = e_j(\lambda_i^{n+1})e_i + \sum_{l=1}^n \varepsilon_l(\lambda_i^{n+1} - \lambda_l^{n+1})\Gamma_{ji}^l e_l. \quad (3.13)$$

Proof. Using (2.3) and the symmetry of the shape operator, we derive, for distinct $i, j, k = 1, 2, \dots, n$,

$$\begin{aligned} \langle (\tilde{\nabla}_{e_i}h)(e_j, e_k), e_{n+1} \rangle &= \langle D_{e_i}h(e_j, e_k) - h(\nabla_{e_i}e_j, e_k) - h(e_j, \nabla_{e_i}e_k), e_{n+1} \rangle \\ &= \langle D_{e_i}h(e_j, e_k), e_{n+1} \rangle - \langle A_{n+1}(\nabla_{e_i}e_j), e_k \rangle - \langle A_{n+1}(\nabla_{e_i}e_k), e_j \rangle, \end{aligned} \quad (3.14)$$

and

$$\langle (\tilde{\nabla}_{e_j}h)(e_i, e_k), e_{n+1} \rangle = \langle D_{e_j}h(e_i, e_k), e_{n+1} \rangle - \langle A_{n+1}(\nabla_{e_j}e_i), e_k \rangle - \langle A_{n+1}(\nabla_{e_j}e_k), e_i \rangle. \quad (3.15)$$

Using (2.2) and (3.6), a direct computation gives

$$\begin{cases} \langle D_{e_i}h(e_j, e_k), e_{n+1} \rangle = \nabla_{e_i}\langle h(e_j, e_k), e_{n+1} \rangle = \nabla_{e_i}\langle A_{n+1}(e_j), e_k \rangle, \\ \langle A_{n+1}(\nabla_{e_i}e_k), e_j \rangle = \nabla_{e_i}\langle A_{n+1}(e_j), e_k \rangle - \langle \nabla_{e_i}(A_{n+1}(e_j)), e_k \rangle, \end{cases} \quad (3.16)$$

and

$$\begin{cases} \langle D_{e_j}h(e_i, e_k), e_{n+1} \rangle = \nabla_{e_j}\langle h(e_i, e_k), e_{n+1} \rangle = \nabla_{e_j}\langle A_{n+1}(e_i), e_k \rangle, \\ \langle A_{n+1}(\nabla_{e_j}e_k), e_i \rangle = \nabla_{e_j}\langle A_{n+1}(e_i), e_k \rangle - \langle \nabla_{e_j}(A_{n+1}(e_i)), e_k \rangle. \end{cases} \quad (3.17)$$

Substituting (3.16) and (3.17) into (3.14) and (3.15), respectively, and according to the Codazzi equation $(\tilde{\nabla}_{e_i} h)(e_j, e_k) = (\tilde{\nabla}_{e_j} h)(e_i, e_k)$, we find

$$\nabla_{e_i}(A_{n+1}(e_j)) - A_{n+1}(\nabla_{e_i} e_j) = \nabla_{e_j}(A_{n+1}(e_i)) - A_{n+1}(\nabla_{e_j} e_i).$$

Together with (3.9), (3.11), and (3.12), we obtain (3.13) and complete the proof of Lemma 3.2. $\square$

Lemma 3.3. Assume that H is not a constant, then we have

$$3n\varepsilon_{n+1}H = 2(n-1)\mu^{n+1}. \quad (3.18)$$

Proof. We claim that $\lambda_\alpha^{n+1} \neq \lambda_1^{n+1}$, $\alpha = 2, \dots, n_1$. Suppose on the contrary $\alpha > 1$, then there exists a fix index α ($1 < \alpha \leq n_1$) such that $A_{n+1}(e_\alpha) = \lambda^{n+1}e_\alpha$. Taking $i = 1$ and $j = \alpha$ in (3.13), we have

$$e_1(\lambda_\alpha^{n+1})e_\alpha + \sum_{l=1}^n \varepsilon_l(\lambda_\alpha^{n+1} - \lambda_l^{n+1})\Gamma_{1\alpha}^l e_l = e_\alpha(\lambda_1^{n+1})e_1 + \sum_{l=1}^n \varepsilon_l(\lambda_1^{n+1} - \lambda_l^{n+1})\Gamma_{\alpha 1}^l e_l.$$

Taking the scalar product with e_α , we obtain that $e_1(\lambda^{n+1}) = 0$, that is,

$$H' = 0. \quad (3.19)$$

Note that $e_1 \parallel \nabla H$, then

$$e_i(H) = 0, \quad i = 2, 3, \dots, n, \quad (3.20)$$

thus

$$\nabla H = \sum_{i=1}^n \varepsilon_i e_i(H) e_i = \varepsilon_1 H' e_1, \quad (3.21)$$

which together with (3.19) shows that $\nabla H = 0$ on U , which is a contradiction.

With $n_1 = 1$, (3.8) and (3.10) lead to (3.18), which completes the proof of Lemma 3.3. $\square$

Lemma 3.4. We have, for $\alpha, \beta = 2, 3, \dots, n$,

$$(n+2)\Gamma_{\alpha 1}^\alpha H = -3\varepsilon_\alpha H'. \quad (3.22)$$

$$\Gamma_{\beta 1}^\alpha = 0, \quad \beta \neq \alpha. \quad (3.23)$$

$$\Gamma_{11}^\alpha = 0. \quad (3.24)$$

Proof. Putting $i = 1$ and $j = \alpha$ in (3.13), we arrive at

$$e_1(\lambda_\alpha^{n+1})e_\alpha + \sum_{l=1}^n \varepsilon_l(\lambda_\alpha^{n+1} - \lambda_l^{n+1})\Gamma_{1\alpha}^l e_l = e_\alpha(\lambda_1^{n+1})e_1 + \sum_{l=1}^n \varepsilon_l(\lambda_1^{n+1} - \lambda_l^{n+1})\Gamma_{\alpha 1}^l e_l, \quad (3.25)$$

and taking the scalar product with e_α , we obtain (3.22) from (3.18).

Putting $i = 1$ and $j = \alpha$ in (3.13) again, and taking the scalar product with e_β ($\beta \neq \alpha$) and e_1 , respectively, we have from (3.20) that (3.23) and (3.24) hold.

We complete the proof of Lemma 3.4. $\square$

Lemma 3.5. Assume that H is not a constant, then we obtain

$$C_1 HH'' + C_2 (H')^2 = C_3 H^2 + C_4 H^4, \quad (3.26)$$

$$\tilde{C}_1 HH'' + \tilde{C}_2 (H')^2 = \tilde{C}_3 H^2 + \tilde{C}_3 H^4, \quad (3.27)$$

where

$$\begin{aligned} C_1 &= 4(n-1)(n+2), & C_2 &= -12(n-1)^2, \\ C_3 &= -\varepsilon_1(n-1)(n+2)(nc-\eta), & C_4 &= \varepsilon_1 \varepsilon_{n+1}(n+2)(n+8)n^2, \end{aligned}$$

$$\begin{aligned}\tilde{C}_1 &= 12(n-1)(n+2), & \tilde{C}_2 &= -12(n-1)(n+5), \\ \tilde{C}_3 &= 4\varepsilon_1(n-1)(n+2)^2c, & \tilde{C}_4 &= -3\varepsilon_1\varepsilon_{n+1}(n+2)^2n^2.\end{aligned}$$

Proof. It follows from (3.8) and (3.18) that

$$4(n-1)\sum_{i=1}^n \varepsilon_i \langle A_{n+1}(e_i), A_{n+1}(e_i) \rangle = (n+8)n^2H^2. \quad (3.28)$$

By using (3.20), (3.22), and (3.24), it is straightforward to verify

$$\Delta H = -\varepsilon_1 H'' + \frac{3(n-1)\varepsilon_1}{(n+2)H} (H')^2. \quad (3.29)$$

Putting (3.28) and (3.29) into the first equation of (3.7) yields (3.26).

On the other hand, set $A_\beta(e_1) = \sum_{i=1}^n \lambda_i^\beta e_i$, $\beta = n+2, \dots, n+p$, where λ_i^β s are functions on U , then it follows from (3.9) that

$$A_{n+1}A_\beta(e_1) = \lambda^{n+1}e_1 + \mu^{n+1}\sum_{j=2}^n e_j, \quad A_\beta A_{n+1}(e_1) = \lambda^{n+1}\sum_{i=1}^n \lambda_i^\beta e_i,$$

together with (3.5), show that $A_\beta(e_1) = \lambda_1^\beta e_1$, $\beta = n+2, \dots, n+p$. Also, the third equation of (3.7) and (3.9) lead to

$$\lambda^{n+1}\varepsilon_1 \langle A_\beta(e_1), (e_1) \rangle + \mu^{n+1}\sum_{j=2}^n \varepsilon_j \langle A_\beta(e_j), (e_j) \rangle = 0.$$

By (3.3), we find from the above two equations that $\lambda_1^\beta = 0$. Those facts show that

$$A_\beta(e_1) = 0, \quad \beta = n+2, \dots, n+p. \quad (3.30)$$

Note that

$$h(e_i, e_j) = \sum_{\beta=n+1}^{n+p} \varepsilon_\beta \langle A_\beta(e_i), e_j \rangle e_\beta, \quad i, j = 1, 2, \dots, n.$$

Then, we infer from (3.30) that

$$h(e_1, e_\alpha) = 0, \quad h(e_1, e_1) = \varepsilon_1 \varepsilon_{n+1} \lambda^{n+1} e_{n+1}, \quad \langle h(e_1, e_1), h(e_\alpha, e_\alpha) \rangle = \varepsilon_1 \varepsilon_\alpha \varepsilon_{n+1} \lambda^{n+1} \mu^{n+1}, \quad \alpha = 2, \dots, n. \quad (3.31)$$

Making use of (3.11), (3.12), (3.23), and (3.24), it follows from (2.4) that

$$\langle R(e_1, e_\alpha)e_\alpha, e_1 \rangle = -e_1(\Gamma_{\alpha 1}^\alpha) - \varepsilon_\alpha(\Gamma_{\alpha 1}^\alpha)^2.$$

According to the Gauss equation, and combining with (3.22), (3.24), (3.31), we prove that (3.27) holds.

We complete the proof of Lemma 3.5. $\square$

Coming back to the proof of Proposition 3.2. Multiplying (3.27) by $3\varepsilon_1(n-1)$ and (3.26) by $(n+5)$, subtracting the results, we get

$$\begin{aligned}8(n-1)(n-4)H'' &= 4\varepsilon_1((n-1)(n+5)(nc-\eta) + (n-1)^2(n+2)c)H \\ &\quad - \varepsilon_1\varepsilon_{n+1}(3(n-1)(n+2) + (n+5)(n+8))n^2H^3.\end{aligned} \quad (3.32)$$

When $n = 4$ (since M_r^n has two distinct principal curvatures, $n > 1$), (3.32) can be rewritten as

$$H^2 = \varepsilon_{n+1} \left(\frac{1}{4}c - \frac{1}{24}\eta \right),$$

which means that H is a constant along integral curve γ , then $e_1(H) = 0$ on U , which together with (3.21) shows that $\nabla H = 0$, a contradiction.

When $n \neq 4$, then if $H' = 0$, then it follows from (3.32) that $e_1(H) = 0$. Together with (3.21) proves that $\Delta H = 0$ on U , a contradiction occurs. Otherwise, multiplying (3.32) by H' , and integrating the result along γ gives

$$48(n-1)(n-4)(H')^2 = 24\varepsilon_1(n-1)((n+5)(nc-\eta) + (n-1)(n+2)c)H^2 - 3\varepsilon_1\varepsilon_{n+1}(3(n-1)(n+2) + (n+5)(n+8))n^2H^4. \quad (3.33)$$

Also, multiplying (3.27) by $(-\varepsilon_1)$, and adding it to (3.26) leads to

$$48(n-1)(n-4)(H')^2 = 24\varepsilon_1(n-1)(n+2)(2(2n+1)c - 3\eta)H^2 - 12\varepsilon_1\varepsilon_{n+1}(n+2)(n+5)n^2H^4. \quad (3.34)$$

It is easy to check that $\varepsilon_1\varepsilon_{n+1}(3(n-1)(n+2) + (n+5)(n+8))n^2 \neq 4\varepsilon_1\varepsilon_{n+1}(n+2)(n+5)n^2$, thus, (3.33) and (3.34) show that H is a constant. It is impossible.

We complete the proof of Proposition 3.2. $\square$

Combining with Propositions 3.1 and 3.2, we have

Theorem 3.1. *Let M_r^n be a submanifold with parallel normalized mean curvature vector field and the minimal polynomial of the shape operator being at most of degree two in $N_1^{n+p}(c)$. If M_r^n is η -biharmonic, then it has parallel mean curvature vector field.*

Using Theorem 3.1, we obtain the following.

Theorem 3.2. *Let M_r^n be an η -biharmonic submanifold with parallel normalized mean curvature vector field and the minimal polynomial of the shape operator being at most of degree two in $N_1^{n+p}(c)$, then*

$$H \leq \max \left\{ \sqrt{\left| c - \frac{\eta}{n} \right|}, \sqrt{\left(c - \frac{\eta}{n} \right) + b^2} \right\}, \text{ where } b \text{ is a nonzero constant.}$$

Proof. According to Theorem 3.1, we know that H is a nonzero constant. Then it follows from the first equation in (3.7) that

$$\sum_i^n \varepsilon_i \langle A_{n+1}(e_i), A_{n+1}(e_i) \rangle = \varepsilon_{n+1}(nc - \eta). \quad (3.35)$$

Moreover, we note that A_{n+1} has the following four forms:

$$(I) = \begin{pmatrix} \lambda^{n+1} & 0 \\ 1 & \lambda^{n+1} \\ & & \lambda^{n+1}I_{n-2} \end{pmatrix}; \quad (II) = \begin{pmatrix} a^{n+1} & b^{n+1} \\ -b^{n+1} & a^{n+1} \end{pmatrix} \quad (b^{n+1} \neq 0); \quad (III) = \lambda^{n+1}I_n; \quad (IV) = \begin{pmatrix} \lambda^{n+1}I_{n_1} & \\ & \mu^{n+1}I_{n-n_1} \end{pmatrix}.$$

- For the form (I), we can easily obtain

$$\sum_i^n \varepsilon_i \langle A_{n+1}(e_i), A_{n+1}(e_i) \rangle = n(\lambda^{n+1})^2,$$

which together with (3.2) and (3.35), we find

$$H^2 = \varepsilon_{n+1} \left(c - \frac{\eta}{n} \right). \quad (3.36)$$

Since $\varepsilon_{n+1} = 1$, (3.36) shows that $H^2 = \left(c - \frac{\eta}{n} \right)$ and $\eta < nc$.

- For the form (II), since $\varepsilon_1\varepsilon_2 = -1$, a direct calculation then gives

$$\sum_i^2 \varepsilon_i \langle A_3(e_i), A_3(e_i) \rangle = 2((a^{n+1})^2 - (b^{n+1})^2).$$

Then we have from (3.35) that

$$(a^{n+1})^2 - (b^{n+1})^2 = \varepsilon_3 \left(c - \frac{\eta}{2} \right). \quad (3.37)$$

Note that $a^{n+1} = H$, we know that b^{n+1} is a nonzero constant, denoted by b . Together with $\varepsilon_3 = 1$, we have from (3.37) that $H^2 = \left(c - \frac{\eta}{2} \right) + b^2$.

- For the form (III) or (IV), we have from (3.35) that

$$\sum_i^n (\lambda_i^{n+1})^2 = \varepsilon_{n+1}(nc - \eta), \quad (3.38)$$

where λ_i^{n+1} s are the principal curvatures of A_{n+1} .

We claim that $nc \neq \eta$. If this is not in the case, then (3.2) and (3.38) lead to $H = 0$, a contradiction. So $\eta > nc$ or $\eta < nc$.

When $\eta < nc$, then $\varepsilon_{n+1} = 1$, i.e., $\vec{H}$ is spacelike, and

$$\sum_i^n (\lambda_i^{n+1})^2 = (nc - \eta). \quad (3.39)$$

- If the shape operator of M_r^n is the form (III), then, according to the definition of mean curvature, (3.39) leads to

$$H^2 = c - \frac{\eta}{n}.$$

- If the shape operator of M_r^n is the form (IV), then, using Cauchy inequality, and combining with (3.2), (3.9), and (3.39), we obtain

$$(nH)^2 = \left(\sum_i^n \lambda_i^{n+1} \right)^2 < n \sum_i^n (\lambda_i^{n+1})^2 = n(nc - \eta),$$

which shows that $H^2 < \left(c - \frac{\eta}{n} \right)$.

Those two cases show that $H \leq \sqrt{c - \frac{\eta}{n}}$.

When $\eta > nc$, then $\varepsilon_{n+1} = -1$, i.e., $\vec{H}$ is timelike, and

$$\sum_i^n (\lambda_i^{n+1})^2 = -(nc - \eta).$$

Using similar discussion of $\eta < nc$, we know that $H \leq \sqrt{-\left(c - \frac{\eta}{n} \right)}$.

Summing up, we obtain that $H \leq \max \left\{ \sqrt{\left| c - \frac{\eta}{n} \right|}, \sqrt{\left(c - \frac{\eta}{2} \right) + b^2} \right\}$. Thus, the proof of Theorem 3.2 is complete. $\square$

3.2 Classification of f -biharmonic submanifolds

Using similar methods to that in the proof of Lemma 3.1, we have from (2.9) that M_r^n is f -biharmonic with f being a function if and only if

$$\begin{cases} \text{(i)} & -\Delta H - \varepsilon_{n+1} H \sum_i^n \varepsilon_i \langle A_{n+1}(e_i), A_{n+1}(e_i) \rangle + ncH = fH, \\ \text{(ii)} & \sum_i^n \varepsilon_i \langle A_{n+1}(e_i), A_\beta(e_i) \rangle = 0, \quad \forall \beta > n+1, \\ \text{(iii)} & 2A_{n+1}(\nabla H) + n\varepsilon_{n+1} H \nabla H = 0. \end{cases} \quad (3.40)$$

A nonminimal submanifold is called *pseudo-umbilical*, if it is umbilical with respect to the direction of $\vec{H}$ (cf. [25, p. 63]), i.e.,

$$\langle A_{\vec{H}}(X), Y \rangle = \langle \vec{H}, \vec{H} \rangle \langle X, Y \rangle. \quad (3.41)$$

In this section, we will classify completely pseudo-umbilical f -biharmonic submanifolds with parallel normalized mean curvature vector field in Lorentz space forms. More precisely,

Theorem 3.3. *Let M_r^n be a pseudo-umbilical submanifold with parallel normalized mean curvature vector field in Lorentz space $\mathbb{L}^{n+p}$. Assume that M_r^n is f -biharmonic, then f is a nonzero constant. Furthermore,*

- (i) *If $f > 0$, then $\vec{H}$ is timelike with $H = \sqrt{\frac{f}{n}}$, and M_r^n is a minimal submanifold in hyperbolic $\mathbb{H}^{n+p-1}(\vec{x}_0, -\frac{f}{n})$;*
- (ii) *If $f < 0$, then $\vec{H}$ is spacelike with $H = \sqrt{-\frac{f}{n}}$, and M_r^n is a minimal submanifold in pseudo-hypersphere $\mathbb{S}_1^{n+p-1}(\vec{x}_0, -\frac{f}{n})$,*

where $\vec{x}_0$ is a constant vector in $\mathbb{L}^{n+p}$.

Proof. We choose a local orthonormal frame field $\{e_1, e_2, \dots, e_{n+p}\}$ on $\mathbb{L}^{n+p}$ such that $e_1, \dots, e_n$ are tangent to M_r^n , and $e_{n+1} = \frac{\vec{H}}{H}$, $e_{n+2}, \dots, e_{n+p}$ are normal to M_r^n , then (3.1), (3.2), and (3.3) hold. Since M_r^n is pseudo-umbilical,

$$A_{n+1} = \varepsilon_{n+1} H I. \quad (3.42)$$

We claim that H is a constant. Suppose on the contrary that H is not constant, then we have from (iii) in (3.40) that $-\frac{n\varepsilon_{n+1}H}{2}$ is the principal curvature of M_r^n . This together with (3.42) which shows that $H = 0$, a contradiction to the hypothesis that M_r^n is nonminimal. Then it follows from (i) in (3.40) that

$$H^2 = -\varepsilon_{n+1} \frac{f}{n}, \quad (3.43)$$

which implies that f is a nonzero constant. In the following, we study $f > 0$ or $f < 0$.

- If $f > 0$, we have from (3.43) that $\varepsilon_{n+1} = -1$, i.e., $\vec{H}$ is timelike, and $H^2 = \frac{f}{n}$, then (3.6) implies that $D\vec{H} = 0$. Note that M_r^n is pseudo-umbilical, then we conclude from [25, Proposition 3.9] that M_r^n is a minimal submanifold in hyperbolic $\mathbb{H}^{n+p-1}(\vec{x}_0, -\frac{f}{n})$ with $\vec{x}_0$ being a constant vector.

Conversely, using Proposition 3.9 in [25, p. 63], we know that M_r^n has parallel mean curvature vector field, i.e., $D\vec{H} = 0$, and

$$A_{\vec{H}} = \langle \vec{H}, \vec{H} \rangle I. \quad (3.44)$$

Since $H^2 = \frac{f}{n}$ (a nonzero constant), it is not difficult to check that (i) and (iii) in (3.40) hold. Also, (3.3) and (3.44) lead to

$$\sum_{i=1}^n \varepsilon_i \langle A_{\beta}(e_i), A_{n+1}(e_i) \rangle = \varepsilon_{n+1} H \sum_{i=1}^n \varepsilon_i \langle A_{\beta}(e_i), e_i \rangle = 0, \quad \beta > n+1,$$

i.e., (ii) in (3.40) holds. Thus, M_r^n is f -biharmonic.

- If $f < 0$, it follows from (3.43) that $\varepsilon_{n+1} = 1$, i.e., $\vec{H}$ is spacelike and $H^2 = -\frac{f}{n}$, then (3.6) implies that $D\vec{H} = 0$. Note that M_r^n is pseudo-umbilical, then we know from [25, Proposition 3.9] that M_r^n is a minimal submanifold in pseudo-hypersphere $\mathbb{S}_1^{n+p-1}(\vec{x}_0, -\frac{f}{n})$ with $\vec{x}_0$ being a constant vector.

Conversely, using Proposition 3.9 in [25, p. 63], we know that $D\vec{H} = 0$ and $A_{\vec{H}} = \langle \vec{H}, \vec{H} \rangle I$. Note that

H is a nonzero constant, then, be similar to the discussion of Case $f > 0$, we prove that (3.40) holds, i.e., M_r^n is f -biharmonic.

We complete the proof of Theorem 3.3. $\square$

When $N_1^{n+p}(c) = \mathbb{S}_1^{n+p}(1)$, we can prove the following.

Theorem 3.4. *Let M_r^n be a pseudo-umbilical submanifold with parallel normalized mean curvature vector field in de-Sitter space $\mathbb{S}_1^{n+p}(1)$. Assume that M_r^n is f -biharmonic, then f is a constant which is not equal to n . Furthermore,*

- (i) *If $f > n$, then $\vec{H}$ is spacelike with $H = \sqrt{-\frac{(n-f)}{n}}$, and M_r^n lies in a non-totally geodesic, totally umbilical hypersurface of $\mathbb{S}_1^{n+p}(1)$ as a minimal submanifold;*
- (ii) *If $f < n$, then $\vec{H}$ is timelike with $H = \sqrt{\frac{n-f}{n}}$, and M_r^n lies in a flat totally umbilical hypersurface of $\mathbb{S}_1^{n+p}(1)$ as a minimal submanifold.*

Proof. Using the similar processing to the proof of Theorem 3.3, we obtain that f is a constant, which is not equal to n , and satisfies

$$H^2 = -\varepsilon_{n+1} \frac{(n-f)}{n}. \quad (3.45)$$

- If $f > n$, it follows from (3.45) that $\varepsilon_{n+1} = 1$, i.e., $\vec{H}$ is a spacelike vector field, and $H^2 = -\frac{(n-f)}{n}$, then it follows from (3.6) that $D\vec{H} = 0$, which together with the hypothesis that M_r^n is pseudo-umbilical, we know from [25, Proposition 3.10] that M_r^n lies in a non-totally geodesic, totally umbilical hypersurface of $\mathbb{S}_1^{n+p}(1)$ as a minimal submanifold.

Conversely, we know from Proposition 3.10 in [25] that $A_{\vec{H}} = \langle \vec{H}, \vec{H} \rangle I$ and $D\vec{H} = 0$. Applying the same arguments as in the proof of Theorem 3.3, we know that M_r^n is f -biharmonic.

- If $f < n$, we have from (3.45) that $\varepsilon_{n+1} = -1$, i.e., $\vec{H}$ is timelike, and $H^2 = \frac{n-f}{n}$. By the similar way followed in the proof of Case $f > n$, we know from [25, Corollary 3.2] that M_r^n lies in a flat totally umbilical hypersurface of $\mathbb{S}_1^{n+p}(1)$ as a minimal submanifold.

We complete the proof of Theorem 3.4. $\square$

Similarly, we have

Theorem 3.5. *Let M_r^n be a pseudo-umbilical submanifold with parallel normalized mean curvature vector field in anti de-Sitter space $\mathbb{H}_1^{n+p}(-1)$. Assume that M_r^n is f -biharmonic, then f is a constant which is not equal to $-n$. Furthermore,*

- (i) *If $f > -n$, then $\vec{H}$ is spacelike with $H = \sqrt{\frac{n+f}{n}}$, and M_r^n lies in a flat totally umbilical hypersurface of $\mathbb{H}_1^{n+p}(-1)$ as a minimal submanifold;*
- (ii) *If $f < -n$, then $\vec{H}$ is timelike with $H = \sqrt{\frac{-(n+f)}{n}}$, and M_r^n lies in a non-totally geodesic, totally umbilical hypersurface of $\mathbb{H}_1^{n+p}(-1)$ as a minimal submanifold.*

Remark 3.1. According to Theorems 3.3–3.5, we know from (3.6) that a pseudo-umbilical f -biharmonic submanifold with parallel normalized mean curvature vector field in Lorentz space forms has parallel mean curvature vector field.

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