Research Article

Dengming Liu* and Miaoju Yu

On the extinction problem for a p -Laplacian equation with a nonlinear gradient source

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Abstract: We deal with the extinction properties of the weak solutions for a p -Laplacian equation with a gradient nonlinearity. The critical extinction exponent is specified and the decay estimates of the extinction solutions are given.

Keywords: extinction, non-extinction, p -Laplacian equation, nonlinear gradient source

MSC 2020: 35K20, 35K55

1 Introduction

The main aim of this paper is devoted to studying the extinction properties of the weak solutions for the following p -Laplacian equation

$$\begin{cases} u_t = \operatorname{div}(|\nabla u|^{p-2} \nabla u) + \mu u^l |\nabla u|^q, & (x, t) \in \Omega \times (0, +\infty), \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times (0, +\infty), \\ u(x, 0) = u_0(x), & x \in \overline{\Omega}, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbf{R}^N$, $N \geq 1$, is an open bounded domain with smooth boundary $\partial\Omega$, $1 < p < 2$, the parameters μ , l and q are positive, and $u_0 \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$ is a nonzero nonnegative function.

Nonlinear partial differential equations arise in various fields of science and have attracted the attention of many scholars (see [1–6]). Model (1.1) can be used to describe the combustion process in combustion theory where $u(x, t)$ represents the temperature of the combustible substance (see [7]). Model (1.1) can also be used to describe the evolution of the population density of a kind of biological species under the effect of certain natural mechanism in population dynamics where $u(x, t)$ is the density of the species (see [8]). In particular, model (1.1) with $l = 0$ is often referred to as a generalized viscous Hamilton-Jacobi equation. Model (1.1) with $l = 0$ is also related to the Kardar-Parisi-Zhang equation in the physical theory of growth and roughening of surfaces (see [9,10]).

One of the particular features of problem (1.1) is that the equation is singular at the points where $\nabla u = 0$. Hence, generally there is no classical solution and we introduce the definition of the weak solution for problem (1.1) as follows.

* **Corresponding author: Dengming Liu**, School of Mathematics and Computational Science, Hunan University of Science and Technology, Xiangtan, Hunan 411201, P. R. China, e-mail: liudengming@hnust.edu.cn

Miaoju Yu: School of Mathematics and Computational Science, Hunan University of Science and Technology, Xiangtan, Hunan 411201, P. R. China, e-mail: 3300523182@qq.com

Definition 1.1. By a local weak solution to problem (1.1), we understand a function $u \in C(0, T; L^1(\Omega))$ for some $T > 0$, which moreover satisfies the following assumptions:

- $|\nabla u|^p, u^l |\nabla u|^q \in L^1(\Omega \times (0, T))$;
- for any $\zeta \in C_0^\infty(\Omega \times (0, T))$ and $0 < t_1 < t_2 < T$, we have

$$\int_{\Omega} u(x, t_2) \zeta(x, t_2) dx + \int_{t_1}^{t_2} \int_{\Omega} [-u \zeta_t + |\nabla u|^{p-2} \nabla u \cdot \nabla \zeta] dx dt = \int_{\Omega} u(x, t_1) \zeta(x, t_1) dx + \mu \int_{t_1}^{t_2} \int_{\Omega} u^l |\nabla u|^q \zeta dx dt; \quad (1.2)$$

- $u(x, t) = 0$ for $x \in \partial\Omega$;
- $u(x, t) \rightarrow u_0(x)$ as $t \rightarrow 0$ with convergence in $L^1(\Omega)$.

Problem (1.1) exhibits various interesting qualitative properties, such as blow-up (or gradient blow-up), extinction, dead-core, and quenching, which reflect natural phenomena, according to various conditions on the parameters p, μ, l , and q , the initial data $u_0(x)$ and the domain Ω (see [11–25]). Because of this, problem (1.1) has attracted the attention of many mathematicians in the last decade. The authors of [9, 26–28] showed the local existence result and established the comparison principle of the weak solutions. When $\mu = 0$, DiBenedetto [1] and Yuan et al. [29] proved that the necessary and sufficient condition for the extinction to occur is $1 < p < 2$. The authors of [7, 30, 31] considered the extinction behaviors of the solutions for problem (1.1) with $q = 0$ and proved that the critical extinction exponent of the solution is $l = p - 1$. For the case $l = 0$ and $\mu = -1$, Iagar and Laurençot [32] concerned with the following Cauchy problem

$$\begin{cases} u_t = \operatorname{div}(|\nabla u|^{p-2} \nabla u) - |\nabla u|^q, & x \in \mathbf{R}^N, \quad t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbf{R}^N. \end{cases} \quad (1.3)$$

Based on comparison principle and gradient estimates of the solutions, they classify the behavior of the solutions for large time, obtaining either positivity as $t \rightarrow \infty$ for $q > p - \frac{N}{N+1}$, optimal decay estimates as $t \rightarrow \infty$ for $q \in [\frac{p}{2}, p - \frac{N}{N+1}]$, or extinction in finite time for $q \in (0, \frac{p}{2})$. Recently, under the restrictive condition $N > p$, Liu and Mu [33, 34] considered problem (1.1) with $l = 0$ and proved that $q = p - 1$ is the critical extinction exponent of the nonnegative weak solution.

To the best of our knowledge, there is no result on the extinction behaviors of the solutions to problem (1.1). From a physical point of view, $\operatorname{div}(|\nabla u|^{p-2} \nabla u)$ with $p \in (1, 2)$ is called a fast diffusion term, which may cause the extinction phenomenon of the weak solution; $\mu u^l |\nabla u|^q$ with $\mu > 0$ is called a nonlinear reaction term, which may prevent the extinction phenomenon of the weak solution. Motivated by the works above, our main attention will be focused on what role of the competition between the fast diffusion term and the coupled nonlinear hot source it plays in determining whether the extinction phenomenon occurs or not. Compared with some previous models which only have source term μu^l or $\mu |\nabla u|^q$, the coupled reaction term $\mu u^l |\nabla u|^q$ is more general, and it can reflect some complicated natural phenomena more exactly. Consequently, the coupled reaction term $\mu u^l |\nabla u|^q$ will bring more challenges while dealing with the integral norm estimates and require more skills.

The main results of this article are stated as follows.

Theorem 1.2. Assume that $0 < l \leq q < \frac{p}{p+2}$ and $p - 1 < q + l$, then the nonnegative weak solution of problem (1.1) vanishes in finite time provided that u_0 is sufficiently small. Furthermore, one has

$$\begin{cases} \|u\|_{\frac{N(2-p)}{p}} \leq \|u_0\|_{\frac{N(2-p)}{p}} \left[1 - \frac{p\Lambda_4}{N \|u_0\|_{\frac{N(2-p)}{p}}^{2-p}} t \right]^{\frac{1}{2-p}}, & t \in [0, T_1), \\ \|u\|_{\frac{N(2-p)}{p}} \equiv 0, & t \in [T_1, +\infty), \end{cases}$$

for $N \geq 2$ and $1 < p < \frac{2N}{N+2}$,

$$\begin{cases} \|u\|_2 \leq \|u_0\|_2 \left[1 - \frac{(2-p)\Lambda_6}{2\|u_0\|_2^{2-p}} t \right]^{\frac{1}{2-p}}, & t \in [0, T_2), \\ \|u\|_2 \equiv 0, & t \in [T_2, +\infty), \end{cases}$$

for $N \geq 2$ and $\frac{2N}{N+2} \leq p < 2$, and

$$\begin{cases} \|u\|_2 \leq \|u_0\|_2 \left[1 - \frac{(2-p)\Lambda_7}{2\|u_0\|_2^{2-p}} t \right]^{\frac{1}{2-p}}, & t \in [0, T_3), \\ \|u\|_2 \equiv 0, & t \in [T_3, +\infty), \end{cases}$$

for $N = 1$, where Λ_4 , T_1 , Λ_6 , T_2 , Λ_7 , and T_3 are positive constants, given by (2.7), (2.8), (2.13), (2.14), (2.18), and (2.19), respectively.

Theorem 1.3. Assume that $0 < q + l < p - 1$, then for any nonzero nonnegative initial data u_0 , problem (1.1) admits at least one non-extinction solution provided that μ is sufficiently large.

Theorem 1.4. Assume that $q + l = p - 1$.

(1) Assume that $0 < l \leq q < \frac{p}{p+2}$, then the nonnegative weak solution of problem (1.1) vanishes in finite time provided that μ is sufficiently small. Furthermore, one has

$$\begin{cases} \|u\|_{\frac{N(2-p)}{p}} \leq \|u_0\|_{\frac{N(2-p)}{p}} \left[1 - \frac{p\Lambda_8}{N\|u_0\|_{\frac{N(2-p)}{p}}^{2-p}} t \right]^{\frac{1}{2-p}}, & t \in [0, T_4), \\ \|u\|_{\frac{N(2-p)}{p}} \equiv 0, & t \in [T_4, +\infty), \end{cases}$$

for $N \geq 2$ and $1 < p < \frac{2N}{N+2}$,

$$\begin{cases} \|u\|_2 \leq \|u_0\|_2 \left[1 - \frac{(2-p)\Lambda_9}{2\|u_0\|_2^{2-p}} t \right]^{\frac{1}{2-p}}, & t \in [0, T_5), \\ \|u\|_2 \equiv 0, & t \in [T_5, +\infty), \end{cases}$$

for $N \geq 2$ and $\frac{2N}{N+2} \leq p < 2$, and

$$\begin{cases} \|u\|_2 \leq \|u_0\|_2 \left[1 - \frac{(2-p)\Lambda_{10}}{2\|u_0\|_2^{2-p}} t \right]^{\frac{1}{2-p}}, & t \in [0, T_6), \\ \|u\|_2 \equiv 0, & t \in [T_6, +\infty), \end{cases}$$

for $N = 1$, where Λ_8 , T_4 , Λ_9 , T_5 , Λ_{10} , and T_6 are positive constants, given by (2.24), (2.25), (2.27), (2.28), (2.30), and (2.31), respectively.

(2) For any nonzero nonnegative initial data u_0 , problem (1.1) admits at least one non-extinction solution provided that μ is sufficiently large.

Remark 1.5. Theorems 1.2, 1.3, and 1.4 tell us that $q + l = p - 1$ is the critical extinction exponent of the weak solution of problem (1.1). On the other hand, Theorems 1.2, 1.3, and 1.4 generalize and extend the previous results in [7,30,31,33,34] to a more general case.

2 Proof of the main results

In this section, we will give the conditions on the occurrence of the extinction phenomenon by using integral norm estimate approach. Meanwhile, the proof of the non-extinction results will be given by using super-solution and sub-solution methods.

Proof of Theorem 1.2. Multiplying the first equation in (1.1) by u^s with $s > 0$, and then integrating the resulting identity over Ω , one obtains

$$\frac{1}{s+1} \frac{d}{dt} \int_{\Omega} u^{s+1} dx + s \left(\frac{p}{p-1+s} \right)^p \int_{\Omega} \left| \nabla u^{\frac{p-1+s}{p}} \right|^p dx = \mu \left(\frac{p}{p-1+s} \right)^q \int_{\Omega} u^{\frac{p(s+l)-q(s-1)}{p}} \left| \nabla u^{\frac{p-1+s}{p}} \right|^q dx. \quad (2.1)$$

Since $0 < l \leq q < \frac{p}{p+2} < p$, it is easily seen that

$$0 < \frac{p(s+l)-q(s-1)}{(s+1)(p-q)} \leq \frac{p(s+q)-q(s-1)}{(s+1)(p-q)} < 1.$$

Then by Young's inequality and Hölder's inequality, it holds that

$$\begin{aligned} \int_{\Omega} u^{\frac{p(s+l)-q(s-1)}{p}} \left| \nabla u^{\frac{p-1+s}{p}} \right|^q dx &\leq \varepsilon \int_{\Omega} \left| \nabla u^{\frac{p-1+s}{p}} \right|^p dx + C(\varepsilon) \int_{\Omega} u^{\frac{p(s+l)-q(s-1)}{p-q}} dx \\ &\leq \varepsilon \int_{\Omega} \left| \nabla u^{\frac{p-1+s}{p}} \right|^p dx + C(\varepsilon) |\Omega|^{1-\frac{p(s+l)-q(s-1)}{(s+1)(p-q)}} \left(\int_{\Omega} u^{s+1} dx \right)^{\frac{p(s+l)-q(s-1)}{(s+1)(p-q)}}, \end{aligned} \quad (2.2)$$

where $\varepsilon \in \left(0, s\mu^{-1} \left(\frac{p}{p-1+s} \right)^{p-q} \right)$. Substituting (2.2) into (2.1) leads to

$$\frac{d}{dt} \int_{\Omega} u^{s+1} dx + \Lambda_1 \int_{\Omega} \left| \nabla u^{\frac{p-1+s}{p}} \right|^p dx \leq \Lambda_2 \left(\int_{\Omega} u^{s+1} dx \right)^{\frac{p(s+l)-q(s-1)}{(s+1)(p-q)}}, \quad (2.3)$$

where

$$\Lambda_1 = (s+1) \left(\frac{p}{p-1+s} \right)^q \left[s \left(\frac{p}{p-1+s} \right)^{p-q} - \mu \varepsilon \right]$$

and

$$\Lambda_2 = (s+1) \mu C(\varepsilon) \left(\frac{p}{p-1+s} \right)^q |\Omega|^{1-\frac{p(s+l)-q(s-1)}{(s+1)(p-q)}}.$$

Now, we will divide the proof of Theorem 1.2 into three cases according to the ranges of N and p .

(I) $N \geq 2$ and $1 < p < \frac{2N}{N+2}$. For this case, choosing $s = \frac{N(2-p)-p}{p} > 1$ in (2.3), then by Sobolev embedding inequality, it holds that

$$\left(\int_{\Omega} u^{s+1} dx \right)^{\frac{p-1+s}{p(s+1)}} = \left(\int_{\Omega} u^{\frac{N(p-1+s)}{N-p}} dx \right)^{\frac{N-p}{Np}} \leq \kappa_1 \left(\int_{\Omega} \left| \nabla u^{\frac{p-1+s}{p}} \right|^p dx \right)^{\frac{1}{p}},$$

that is,

$$\kappa_1^{-p} \left(\int_{\Omega} u^{s+1} dx \right)^{\frac{p-1+s}{s+1}} \leq \int_{\Omega} \left| \nabla u^{\frac{p-1+s}{p}} \right|^p dx, \quad (2.4)$$

where κ_1 is the embedding constant, depending only on p and N . Combining (2.3) with (2.4) yields

$$\frac{d}{dt} \int_{\Omega} u^{s+1} dx + \Lambda_3 \left(\int_{\Omega} u^{s+1} dx \right)^{\frac{p-1+s}{s+1}} \leq \Lambda_2 \left(\int_{\Omega} u^{s+1} dx \right)^{\frac{p(s+l)-q(s-1)}{(s+1)(p-q)}},$$

in other words,

$$\frac{d}{dt} \int_{\Omega} u^{s+1} dx \leq \left(\int_{\Omega} u^{s+1} dx \right)^{\frac{p-1+s}{s+1}} \left[\Lambda_2 \left(\int_{\Omega} u^{s+1} dx \right)^{\frac{p[q+l-(p-1)]}{(s+1)(p-q)}} - \Lambda_3 \right], \quad (2.5)$$

where $\Lambda_3 = \Lambda_1 \kappa_1^{-p}$. Noting that $q + l > p - 1$, one can take u_0 so small that

$$\left(\int_{\Omega} u_0^{s+1} dx \right)^{\frac{p[q+l-(p-1)]}{(s+1)(p-q)}} < \Lambda_2^{-1} \Lambda_3.$$

Then from (2.5), it follows that

$$\frac{d}{dt} \int_{\Omega} u^{s+1} dx \leq -\Lambda_4 \left(\int_{\Omega} u^{s+1} dx \right)^{\frac{p-1+s}{s+1}}, \quad (2.6)$$

here

$$\Lambda_4 = \Lambda_3 - \Lambda_2 \left(\int_{\Omega} u^{s+1} dx \right)^{\frac{p[q+l-(p-1)]}{(s+1)(p-q)}} > 0. \quad (2.7)$$

Integrating both sides of (2.6) with respect to the time variable from 0 to t , one can conclude that

$$\left(\int_{\Omega} u^{s+1} dx \right)^{\frac{2-p}{s+1}} \leq \left(\int_{\Omega} u_0^{s+1} dx \right)^{\frac{2-p}{s+1}} - \frac{(2-p)\Lambda_4}{s+1} t,$$

namely,

$$\|u\|_{\frac{N(2-p)}{p}} \leq \|u_0\|_{\frac{N(2-p)}{p}} \left[1 - \frac{p\Lambda_4}{N \|u_0\|_{\frac{N(2-p)}{p}}^{2-p}} t \right]_+^{\frac{1}{2-p}},$$

which means that $u(x, t)$ vanishes in finite time

$$T_1 = N(p\Lambda_4)^{-1} \|u_0\|_{\frac{N(2-p)}{p}}^{\frac{2-p}{p}}. \quad (2.8)$$

(II) $N \geq 2$ and $\frac{2N}{N+2} \leq p < 2$. For this case, taking $s = 1$ in (2.3), then (2.3) becomes

$$\frac{d}{dt} \int_{\Omega} u^2 dx + \Lambda_1 \int_{\Omega} |\nabla u|^p dx \leq \Lambda_2 \left(\int_{\Omega} u^2 dx \right)^{\frac{p(l+1)}{2(p-q)}}. \quad (2.9)$$

On the other hand, Hölder's inequality and Sobolev embedding inequality lead us to the following estimate:

$$\int_{\Omega} u^2 dx \leq |\Omega|^{1-\frac{2(N-p)}{Np}} \left(\int_{\Omega} u^{\frac{Np}{N-p}} dx \right)^{\frac{2(N-p)}{Np}} \leq \kappa_2 |\Omega|^{1-\frac{2(N-p)}{Np}} \left(\int_{\Omega} |\nabla u|^p dx \right)^{\frac{2}{p}},$$

which suggests that

$$\left(\kappa_2 |\Omega|^{1-\frac{2(N-p)}{Np}} \right)^{-\frac{p}{2}} \left(\int_{\Omega} u^2 dx \right)^{\frac{p}{2}} \leq \int_{\Omega} |\nabla u|^p dx, \quad (2.10)$$

where κ_2 is the embedding constant, depending only on p and N . Inserting (2.10) into (2.9), one has

$$\frac{d}{dt} \int_{\Omega} u^2 dx + \Lambda_5 \left(\int_{\Omega} u^2 dx \right)^{\frac{p}{2}} \leq \Lambda_2 \left(\int_{\Omega} u^2 dx \right)^{\frac{p(l+1)}{2(p-q)}},$$

which is equivalent to

$$\frac{d}{dt} \int_{\Omega} u^2 dx \leq \left(\int_{\Omega} u^2 dx \right)^{\frac{p}{2}} \left[\Lambda_2 \left(\int_{\Omega} u^2 dx \right)^{\frac{p(q+l-(p-1))}{2(p-q)}} - \Lambda_5 \right], \quad (2.11)$$

where

$$\Lambda_5 = \Lambda_1 \left(\kappa_2 |\Omega|^{1-\frac{2(N-p)}{Np}} \right)^{-\frac{p}{2}}.$$

By taking $u_0(x)$ so small that

$$\left(\int_{\Omega} u_0^2 dx \right)^{\frac{p(q+l-(p-1))}{2(p-q)}} \leq \Lambda_5 \Lambda_2^{-1},$$

then from (2.11), it follows that

$$\frac{d}{dt} \int_{\Omega} u^2 dx \leq -\Lambda_6 \left(\int_{\Omega} u^2 dx \right)^{\frac{p}{2}}, \quad (2.12)$$

where

$$\Lambda_6 = \Lambda_5 - \Lambda_2 \left(\int_{\Omega} u_0^2 dx \right)^{\frac{p(q+l-(p-1))}{2(p-q)}} > 0. \quad (2.13)$$

Integrating both sides of (2.12) with respect to the time variable on $(0, t)$, one arrives at

$$\left(\int_{\Omega} u^2 dx \right)^{\frac{2-p}{2}} \leq \left(\int_{\Omega} u_0^2 dx \right)^{\frac{2-p}{2}} - \frac{(2-p)\Lambda_6}{2} t,$$

that is,

$$\|u\|_2 \leq \|u_0\|_2 \left[1 - \frac{(2-p)\Lambda_6}{2 \|u_0\|_2^{2-p}} t \right]_+^{\frac{1}{2-p}},$$

which suggests that $u(x, t)$ vanishes in finite time

$$T_2 = 2[(2-p)\Lambda_6]^{-1} \|u_0\|_2^{2-p}. \quad (2.14)$$

(III) $N = 1$ and $1 < p < 2$. Since $1 < p < 2$, there exists an embedding constant κ_3 such that

$$\left(\int_{\Omega} u^2 dx \right)^{\frac{1}{2}} \leq \kappa_3 \left(\int_{\Omega} |\nabla u|^p dx \right)^{\frac{1}{p}},$$

namely,

$$\kappa_3^{-p} \left(\int_{\Omega} u^2 dx \right)^{\frac{p}{2}} \leq \int_{\Omega} |\nabla u|^p dx. \quad (2.15)$$

With the help of (2.9) and (2.15), one observes

$$\frac{d}{dt} \int_{\Omega} u^2 dx \leq \left(\int_{\Omega} u^2 dx \right)^{\frac{p}{2}} \left[\Lambda_2 \left(\int_{\Omega} u^2 dx \right)^{\frac{p[q+l-(p-1)]}{2(p-q)}} - \Lambda_1 \kappa_3^{-p} \right]. \quad (2.16)$$

By choosing $u_0(x)$ so small that

$$\left(\int_{\Omega} u_0^2 dx \right)^{\frac{p[q+l-(p-1)]}{2(p-q)}} \leq \Lambda_1 \Lambda_2^{-1} \kappa_3^{-p},$$

then it follows from (2.16) that

$$\frac{d}{dt} \int_{\Omega} u^2 dx \leq -\Lambda_7 \left(\int_{\Omega} u^2 dx \right)^{\frac{p}{2}}, \quad (2.17)$$

where

$$\Lambda_7 = \Lambda_1 \kappa_3^{-p} - \Lambda_2 \left(\int_{\Omega} u_0^2 dx \right)^{\frac{p[q+l-(p-1)]}{2(p-q)}} > 0. \quad (2.18)$$

Integrating (2.17), we arrive at the following inequality:

$$\left(\int_{\Omega} u^2 dx \right)^{\frac{2-p}{2}} \leq \left(\int_{\Omega} u_0^2 dx \right)^{\frac{2-p}{2}} - \frac{(2-p)\Lambda_7}{2} t.$$

This means that

$$\|u\|_2 \leq \|u_0\|_2 \left[1 - \frac{(2-p)\Lambda_7}{2 \|u_0\|_2^{2-p}} t \right]_+^{\frac{1}{2-p}},$$

which implies that $u(x, t)$ vanishes in finite time

$$T_3 = 2[(2-p)\Lambda_7]^{-1} \|u_0\|_2^{2-p}. \quad (2.19)$$

The proof of Theorem 1.2 is complete. $\square$

Proof of Theorem 1.3. Before starting the proof of Theorem 1.3, let λ_1 be the first eigenvalue and $\psi(x)$ be the corresponding eigenfunction of the following problem

$$\begin{cases} -\operatorname{div}(|\nabla \mathcal{U}|^{p-2} \nabla \mathcal{U}) = \lambda \mathcal{U} |\mathcal{U}|^{p-2}, & x \in \Omega, \\ \mathcal{U}(x) = 0, & x \in \partial\Omega. \end{cases} \quad (2.20)$$

In what follows, we assume that $\psi(x) > 0$. In addition, we normalize $\psi(x)$ in L^∞ norm, namely, $\max_{x \in \Omega} \psi(x) = 1$. Let us define a function $f(t)$ for $t \geq 0$ by

$$f(t) = d^{\frac{1}{p-1-(q+l)}} (1 - e^{-ct})^{\frac{1}{1-(q+l)}},$$

where $d \in (0, 1)$ and $c \in \left(0, (p-1-(q+l))d^{\frac{q+l-1}{p-1-(q+l)}}\right)$. Then it is easily seen that

$$f(0) = 0 \quad \text{and} \quad f(t) \in (0, 1) \quad \text{for } t > 0. \quad (2.21)$$

By virtue of

$$(1 - \alpha)^\beta + \alpha\beta < 1, \quad \text{for } \alpha, \beta \in (0, 1),$$

it holds that

$$f'(t) + \frac{1}{d} f^{p-1}(t) - f^{q+l}(t) < 0. \quad (2.22)$$

Let

$$\mathcal{V}_1(x, t) = f(t)\psi(x).$$

It remains to prove that $\mathcal{V}_1(x, t)$ is a non-extinction weak sub-solution of problem (1.1). By a series of straightforward computations, noting that (2.22) and the definition of $\psi(x)$, it holds that

$$\begin{aligned} I &:= \int_0^t \int_\Omega \mathcal{V}_{1s} \zeta + |\nabla \mathcal{V}_1|^{p-2} \nabla \mathcal{V}_1 \cdot \nabla \zeta - \mu \mathcal{V}_1^l |\nabla \mathcal{V}_1|^q \zeta \, dx \, ds \\ &= \int_0^t \int_\Omega [f_s(s)\psi(x) - \mu f^{q+l}(s)\psi^l(x) |\nabla \psi|^q + \lambda_1 f^{p-1}(s)\psi^{p-1}(x)] \zeta(x, s) \, dx \, ds \\ &\leq \int_0^t \int_\Omega \left[\left(f^{q+l} - \frac{1}{d} f^{p-1} \right) \psi - \mu f^{q+l} \psi^l |\nabla \psi|^q + \lambda_1 f^{p-1} \psi^{p-1} \right] \zeta \, dx \, ds \\ &< \int_0^t \int_\Omega f^{q+l} [\psi + \lambda_1 f^{p-1-(q+l)} \psi^{p-1} - \mu \psi^l |\nabla \psi|^q] \zeta \, dx \, ds \\ &< \int_0^t f^{q+l} \sup_{x \in \Omega} \zeta \, ds \int_\Omega [1 + \lambda_1 - \mu \psi^l |\nabla \psi|^q] \, dx. \end{aligned}$$

If

$$\mu > (\lambda_1 + 1)|\Omega| \left[q^{-1}(q+l) \left\| \nabla \psi^{\frac{q+l}{q}} \right\|_q^{-1} \right]^q,$$

then one can immediately claim that $I < 0$, which suggests that $\mathcal{V}_1(x, t)$ is a strict non-extinction weak sub-solution of problem (1.1).

On the other hand, one can prove that $\mathcal{V}_2(x, t) = K \geq \max\{1, \|u_0(x)\|_{L^\infty}\}$ is a non-extinction super-solution of problem (1.1), and $\mathcal{V}_1(x, t) \leq \mathcal{V}_2(x, t)$. Therefore, by an iteration process, one can conclude that there is at least a non-extinction solution $u(x, t)$ of problem (1.1), which satisfies $\mathcal{V}_1(x, t) \leq u(x, t) \leq \mathcal{V}_2(x, t)$. The proof of Theorem 1.3 is complete. $\square$

Proof of Theorem 1.4.

(1) For $N \geq 2$ and $1 < p < \frac{2N}{N+2}$. From (2.5) it follows that

$$\frac{d}{dt} \int_{\Omega} u^{s+1} dx \leq -\Lambda_8 \left(\int_{\Omega} u^{s+1} dx \right)^{\frac{p-1+s}{s+1}}, \quad (2.23)$$

where

$$\Lambda_8 = \Lambda_3 - \Lambda_2, \quad (2.24)$$

and Λ_8 is greater than zero if one takes

$$\mu \in \left(0, \frac{[N(2-p) - p]p^{2(p-q)-1}}{[(N-p)(2-p)]^{p-q}} \left(\varepsilon + C(\varepsilon)\kappa_1^p |\Omega|^{\frac{p[p(1-l)-2q]}{N(2-p)(p-q)}} \right)^{-1} \right).$$

Integrating both sides of (2.23) with respect to the time variable from 0 to t , one arrives at

$$\|u\|_{\frac{N(2-p)}{p}} \leq \|u_0\|_{\frac{N(2-p)}{p}} \left[1 - \frac{p\Lambda_8}{N\|u_0\|_{\frac{N(2-p)}{p}}^{2-p}} t \right]_+^{\frac{1}{2-p}},$$

which suggests that $u(x, t)$ vanishes in finite time

$$T_4 = N(p\Lambda_8)^{-1} \|u_0\|_{\frac{N(2-p)}{p}}^{2-p}. \quad (2.25)$$

For $N \geq 2$ and $\frac{2N}{N+2} \leq p < 2$. By (2.11), it holds that

$$\frac{d}{dt} \int_{\Omega} u^2 dx \leq -\Lambda_9 \left(\int_{\Omega} u^2 dx \right)^{\frac{p}{2}}, \quad (2.26)$$

where

$$\Lambda_9 = \Lambda_5 - \Lambda_2 \quad (2.27)$$

is a positive constant if one takes

$$\mu \in \left(0, \left(\varepsilon + C(\varepsilon)\kappa_2^{\frac{p}{2}} |\Omega|^{\frac{p[N(p-q-l-1)+2(p-q)]}{2N(p-q)}} \right)^{-1} \right).$$

Integrating both sides of (2.26) with respect to the time variable on $(0, t)$, one can claim that

$$\|u\|_2 \leq \|u_0\|_2 \left[1 - \frac{(2-p)\Lambda_9}{2\|u_0\|_2^{2-p}} t \right]_+^{\frac{1}{2-p}},$$

which implies that $u(x, t)$ vanishes in finite time

$$T_5 = 2((2-p)\Lambda_9)^{-1} \|u_0\|_2^{2-p}. \quad (2.28)$$

For $N = 1$ and $1 < p < 2$. From (2.16) it follows that

$$\frac{d}{dt} \int_{\Omega} u^2 dx \leq -\Lambda_{10} \left(\int_{\Omega} u^2 dx \right)^{\frac{p}{2}}, \quad (2.29)$$

where

$$\Lambda_{10} = \Lambda_1 \kappa_3^{-p} - \Lambda_2 \quad (2.30)$$

is a positive constant if one takes

$$\mu \in \left(0, \left(\varepsilon + C(\varepsilon) \kappa_3^p |\Omega|^{\frac{2(p-q)-p(l+1)}{2(p-q)}} \right)^{-1} \right).$$

Integrating (2.29), one can conclude that

$$\|u\|_2 \leq \|u_0\|_2 \left[1 - \frac{(2-p)\Lambda_{10}}{2\|u_0\|_2^{2-p}} t \right]_+^{\frac{1}{2-p}},$$

which suggests that $u(x, t)$ vanishes in finite time

$$T_6 = 2((2-p)\Lambda_{10})^{-1} \|u_0\|_2^{2-p}. \quad (2.31)$$

(2) Let

$$\mathcal{V}_3(x, t) = [(1-q-l)t]^{-\frac{1}{1-q-l}} \psi(x).$$

Then using the similar manners to the proof of Theorem 1.3, one can easily check that $\mathcal{V}_3(x, t)$ is a strict non-extinction weak sub-solution of problem (1.1) provided that

$$\mu > (\lambda_1 + 1) |\Omega| \|\nabla \psi\|_q^{-q}.$$

On the other hand, $\mathcal{V}_4(x, t) = Ae^t$ with $A > \max\{1, \|u_0(x)\|_{L^\infty}\}$ is a non-extinction super-solution of problem (1.1) and $\mathcal{V}_3(x, t) \leq \mathcal{V}_4(x, t)$. Therefore, one can show that problem (1.1) admits at least a non-extinction solution $u(x, t)$, which satisfies $\mathcal{V}_3(x, t) \leq u(x, t) \leq \mathcal{V}_4(x, t)$.

The proof of Theorem 1.4 is complete. $\square$

3 Conclusion

In the present article, we mainly concern with the critical extinction exponent of the weak solution to a fast diffusive p -Laplacian equation with a nonlinear reaction term $\mu u^l |\nabla u|^q$. By using the integral norm estimate method and super-solution and sub-solution methods, the conditions on the occurrence of the extinction phenomenon of the weak solution are given. We complete classifying the extinction and non-extinction phenomenon of the weak solution by the exponent $p = q + l + 1$. Properly speaking, we prove that if $p - 1 < q + l$, the weak solution of problem (1.1) will vanish in finite time for appropriate small initial data, while if $q + l < p - 1$, problem (1.1) admits at least one non-extinction weak solution for any nonzero nonnegative initial data. When $q + l = p - 1$, the size of the parameter μ plays a crucial role in the occurrence of the extinction phenomenon. To sum up, the present study is a natural extension result of the previous ones in [7,30,31,33,34], where the reaction term is just a power function of u or $|\nabla u|$.

Our future work includes the following two aspects:

- (1) The extinction behaviors of the weak solutions to some classes of inhomogeneous parabolic problems.
- (2) The study of the numerical extinction phenomena of the weak solution to some kinds of parabolic equations like (1.1).

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