Research Article

Mohammed Kbiri Alaoui*

Parabolic inequalities in Orlicz spaces with data in L^1

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Abstract: In this paper, we provide existence and uniqueness of entropy solutions to a general nonlinear parabolic problem on a general convex set with merely integrable data and in the setting of Orlicz spaces.

Keywords: nonlinear parabolic inequalities, entropy solution, Orlicz-Sobolev

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1 Introduction

In this paper, we deal with the boundary value problems

$$\begin{cases} u \in \mathcal{K}, \\ \frac{\partial u}{\partial t} + A(u) = f & \text{in } Q, \\ u = 0 & \text{on } \partial Q = \partial\Omega \times (0, T), \\ u(x, 0) = u_0 & \text{in } \partial\Omega, \end{cases} \quad (\text{P})$$

where

$$A(u) = -\operatorname{div}(a(\cdot, t, \nabla u)),$$

$Q = \Omega \times [0, T]$, $T > 0$ and Ω is a bounded domain of $\mathbb{R}^N$, with the segment property. $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function such that for all $\xi, \xi^* \in \mathbb{R}^N$, $\xi \neq \xi^*$,

$$a(x, t, \xi)\xi \geq \alpha M(|\xi|), \quad (1.1)$$

$$[a(x, t, \xi) - a(x, t, \xi^*)][\xi - \xi^*] > 0, \quad (1.2)$$

$$|a(x, t, \xi)| \leq c(x, t) + k_1 \overline{M}^{-1} M(k_2 |\xi|), \quad (1.3)$$

where $c(x, t)$ belongs to $E_{\overline{M}}(Q)$, $c \geq 0$ and k_i ($i = 1, 2$) to $\mathbb{R}^+$ and α to $\mathbb{R}_*^+$.

$$f \in L^1(Q), \quad f \geq 0, \quad (1.4)$$

$$u_0 \in L^1(\Omega), \quad u_0 \geq 0. \quad (1.5)$$

There exists a real $\gamma > 1$ such that

$$\sup_{t>0} \frac{t^{1+\frac{1}{\gamma}}}{M(t)} < +\infty. \quad (1.6)$$

$\mathcal{K}$ is a given closed convex subset of $\mathcal{M}(Q)$. $\mathcal{M}(Q)$ is the space of measure with the finite total mass.

* Corresponding author: Mohammed Kbiri Alaoui, Department of Mathematics, King Khalid University, Abha, Kingdom of Saudia Arabia, e-mail: mka_la@yahoo.fr

The problem (P) has several applications in engineering, game theory, finance, and economics. For example, one of the most important problems in finance is the optimal investment problem of a constant relative risk aversion. This problem leads to an obstacle parabolic problem with free boundaries (see [1]). Other important cases are the obstacle problem for parabolic minimizers studied in [2], where the obstacle is irregular, the pricing of American options (see [3]), as well as the models of pricing a double defaultable interest rate swap for which the solutions converge to a solution of a PDE coupled with two-obstacle problem.

On a physical area, the use of PDE in the convex set takes considerable importance, for example, the Boltzmann equation in a strictly convex domain with the specular, bounce-back, and diffuse (see [4]), dissipation inequalities for nonlinear PDEs which can be applied according to the choice of the so-called supply rate [5], Korteweg-de Vries (KdV), Kadomtsev-Petviashvili (KP) equation, etc.

It is well known that (P) admits at least one solution (see Leray and Lions [6], Browder and Brézis [7], and Puel [8]). In those papers, the function $a(x, t, \xi)$ was assumed to satisfy a polynomial growth condition with respect to ∇u . When trying to generalize the last condition of $a(\cdot, \xi)$ to the non polynomial one, we are led to replace the space $L^p(0, T; W^{1,p}(\Omega))$ by an inhomogeneous Sobolev space $W^{1,x}L_M$ built from an Orlicz space L_M instead of L^p , where the N -function M , which defines L_M , defines the new growth of the operator. Such type of extension of the growth condition is more realistic and appears in several physical phenomena.

Partial differential equations with data-only integrable received special attention. The cornerstone of the theory was initially developed by DiPerna and Lions [9], where they introduced the notion of renormalized solution for the Boltzmann equation. It was also developed by Boccardo et al. [10] and Murat [11]. Other notions of solutions solving the problem with L^1 -data are SOLA (solutions obtained as a limit of approximation), from the study by Boccardo and Gallouët [12,13]. Finally, entropy solutions are introduced by Benilan et al. [14], Boccardo et al. [15].

Our purpose in this paper is to prove existence results and uniqueness, of the entropy solution, of the problem (P) in the setting of the inhomogeneous Sobolev space $W^{1,x}L_M$ with data $f \in L^1(Q)$. In these types of problems the proofs are essentially based on the good choice of the test functions. The new idea of this paper is considering the problem in nonreflexive Banach space, namely, the inhomogeneous Orlicz-Sobolev space, and in a general convex set.

The simplest model of our problem (P) is the case $A(u) = -\operatorname{div}\left(a(x, t) \frac{m(|\nabla u|)}{|\nabla u|} \nabla u\right)$, where m is a function such that $M(t) = \int_0^t m(s)ds$ is an N -function and $\alpha < a(x, t) \leq \beta$, $\alpha > 0$, $\beta > 0$.

2 Preliminaries

Let us recall the following definitions of spaces and topologies that will be used later (for the detail, we refer the reader to the rich literature in [16–19]).

2.1 N -function and Orlicz space

$M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be an N -function, i.e., M is continuous, convex, with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \rightarrow 0$ as $t \rightarrow 0$ and $\frac{M(t)}{t} \rightarrow \infty$ as $t \rightarrow \infty$.

The Orlicz space $L_M(\Omega)$ is defined as the equivalence classes of real-valued measurable functions u on Ω such that: $\int_{\Omega} M\left(\frac{u(x)}{\lambda}\right)dx < +\infty$ for some $\lambda > 0$. Note that $L_M(\Omega)$ is a Banach space under the norm $\|u\|_{M,\Omega} = \inf\left\{\lambda > 0 : \int_{\Omega} M\left(\frac{u(x)}{\lambda}\right)dx \leq 1\right\}$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\bar{\Omega}$ is denoted by $E_M(\Omega)$.

Let us recall that two equivalent N -functions defined the same Orlicz space.

2.2 Inhomogeneous Orlicz-Sobolev spaces

The inhomogeneous Orlicz-Sobolev spaces are defined as follows: $W^{1,x}L_M(Q) = \{u \in L_M(Q) : D_x^\alpha u \in L_M(Q) \forall |\alpha| \leq 1\}$, and $W^{1,x}E_M(Q) = \{u \in E_M(Q) : D_x^\alpha u \in E_M(Q) \forall |\alpha| \leq 1\}$. The last space is a subspace of the first one, and both are Banach spaces under the norm $\|u\| = \sum_{|\alpha| \leq 1} \|D_x^\alpha u\|_{M,Q}$. These spaces are considered as subspaces of the product space $\Pi L_M(Q)$, which have as many copies as there are α -order derivatives, $|\alpha| \leq 1$. We shall also consider the weak topologies $\sigma(\Pi L_M, \Pi E_M)$ and $\sigma(\Pi L_M, \Pi L_M)$. The space $W_0^{1,x}E_M(Q)$ is defined as the (norm) closure in $W^{1,x}E_M(Q)$ of $\mathcal{D}(Q)$. We can easily show as in [17] that when Ω has the segment property, then each element u of the closure of $\mathcal{D}(Q)$ with respect to the weak $*$ topology $\sigma(\Pi L_M, \Pi E_M)$ is limit, in $W^{1,x}L_M(Q)$, of some subsequence $(u_i) \subset \mathcal{D}(Q)$ for the modular convergence, i.e., there exists $\lambda > 0$ such that for all $|\alpha| \leq 1$, $\int_Q M\left(\frac{D_x^\alpha u_i - D_x^\alpha u}{\lambda}\right) dx dt \rightarrow 0$ as $i \rightarrow \infty$, and this implies that (u_i) converges to u in $W^{1,x}L_M(Q)$ for the weak topology $\sigma(\Pi L_M, \Pi L_M)$. Consequently, $\overline{\mathcal{D}(Q)}^{\sigma(\Pi L_M, \Pi E_M)} = \overline{\mathcal{D}(Q)}^{\sigma(\Pi L_M, \Pi L_M)}$, and this space will be denoted by $W_0^{1,x}L_M(Q)$. Furthermore, Poincaré's inequality also holds in $W_0^{1,x}L_M(Q)$.

The dual space of $W_0^{1,x}L_M(Q)$ is defined as $W^{-1,x}L_{\overline{M}}(Q) = \left\{f = \sum_{|\alpha| \leq 1} D_x^\alpha f_\alpha : f_\alpha \in L_{\overline{M}}(Q)\right\}$ and equipped with the usual quotient norm. We also denote $W^{-1,x}E_{\overline{M}}(Q) := \left\{f = \sum_{|\alpha| \leq 1} D_x^\alpha f_\alpha : f_\alpha \in E_{\overline{M}}(Q)\right\}$.

3 Main results

The following lemmas will be of interest in the proof of our main results.

Let us denote $\chi_N = NC_N^{1/N}$, C_N is the measure of the unit ball of $\mathbb{R}^N$, and for a fixed $t \in [0, T]$, $\mu(\theta) = \text{meas}\{(x, t) : |u(x, t)| > \theta\}$.

Lemma 3.1. [20] Let $u \in W_0^{1,x}L_M(Q)$, and let fixed $t \in [0, T]$, then we have

$$-\mu'(\theta) \geq -\frac{1}{\chi_N \mu(\theta)^{1-\frac{1}{N}}} S \left(-\frac{1}{\chi_N \mu(\theta)^{1-\frac{1}{N}}} \frac{d}{d\theta} \int_{\{|u|>\theta\}} M(|\nabla u|) dx \right), \quad \forall \theta > 0,$$

where S is defined by $\frac{1}{S(s)} = \sup\{t : B(t) \leq s\}$, $B(s) = \frac{M(s)}{s}$.

Lemma 3.2. Under the hypotheses (1.1)–(1.3), if f, u_0 are regular functions and $f, u_0 \geq 0$, then there exists at least one positive weak solution of the problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \text{div } a(x, t, u, \nabla u) = f & \text{in } Q, \\ u = 0 & \text{on } \partial Q = \partial \Omega \times (0, T), \\ u(x, 0) = u_0 & \text{in } \partial \Omega, \end{cases}$$

such that

$$\frac{\partial u}{\partial t} \geq 0, \quad \text{a.e. } t \in (0, T).$$

Proof. Let u be a continuous function, we say that u satisfies (β) condition if there exists a continuous and increasing function β such that $\|u(t) - u(s)\|_2 \leq \beta(\|u_0\|_2)|t - s|$, where $u_0(x) = u(x, 0)$.

Let $X := \{u \in W_0^{1,x}L_M(Q) \cap L^2(Q), u \text{ satisfies } (\beta) \text{ condition and } \frac{du}{dt} \in L^\infty(0, T, L^2(Q))\}$.

Let us consider the set $C = \{v \in X : v(t) \in C, \frac{\partial v}{\partial t} \geq 0 \text{ a.e. } t \in (0, T)\}$, where C is a closed convex of $W_0^1L_M(\Omega)$. It is easy to see that C is a closed convex (since all its elements satisfy (β) condition). $\square$

We claim that the problem

$$\begin{cases} u \in C \cap L^2(Q), \\ \frac{\partial u}{\partial t} - \operatorname{div} a(x, t, u, \nabla u) = f & \text{in } Q, \\ u = 0 & \text{on } \partial Q = \partial\Omega \times (0, T), \\ u(x, 0) = u_0 & \text{in } \partial\Omega \end{cases} \quad (P')$$

has a weak solution, which is unique in the sense defined in [21].

Indeed, let us consider the approximate problem:

$$\begin{cases} \frac{\partial u_n}{\partial t} + A(u_n) + nT_n(\Phi(u_n)) = f & \text{in } Q, \\ u_n(\cdot, 0) = u_0 & \text{in } \Omega, \end{cases}$$

where the functional Φ is defined by $\Phi : X \rightarrow \mathbb{R} \cup \{+\infty\}$ such that

$$\Phi(v) := \begin{cases} 0 & \text{if } v \in C, \\ +\infty & \text{otherwise.} \end{cases}$$

The existence of such $u_n \in X$ was ensured by Kacur [22].

Following the same proof as in [21], we can prove the existence of a solution u of the problem (P') as limit of u_n .

Theorem 3.1. *Under hypotheses (1.1)–(1.6), the problem (P) has at least one entropy solution in the following sense:*

$$\begin{cases} u \in \mathcal{K}, T_k(u) \in W_0^{1,x}L_M(Q), \forall k > 0, \\ \int_{\Omega} S_k(u(T) - v(T))dx + \left\langle \frac{\partial v}{\partial t}, T_k(u - v) \right\rangle + \int_Q a(\cdot, \nabla u) \nabla T_k(u - v) dx dt \leq \langle f, T_k(u - v) \rangle + \int_{\Omega} S_k(u_0 - v(0))dx. \end{cases}$$

for all $v \in \mathcal{K} \cap L^\infty(Q) \cap D$, where

$$D := \{v \in W_0^{1,x}L_M(Q) : \frac{\partial v}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q) + L^1(Q)\}, S_k(t) = \int_0^t T_k(s)ds.$$

Proof. Let us define the indicator functional, $\Phi : \mathcal{M}(Q) \rightarrow \mathbb{R} \cup \{+\infty\}$ such that:

$$\Phi(v) := \begin{cases} 0 & \text{if } v \in \mathcal{K}, \\ +\infty & \text{otherwise.} \end{cases}$$

Φ is weakly lower semicontinuous.

1. A priori estimate

Let us consider the following approximate problem:

$$\begin{cases} \frac{\partial u_n}{\partial t} + A(u_n) + nT_n(\Phi(u_n)) = f_n & \text{in } Q, \\ u_n(\cdot, 0) = u_{0n} & \text{in } \Omega, \end{cases} \quad (P_n)$$

where $(u_{0n}) \subset \mathcal{D}(\Omega)$ such that $u_{0n} \rightarrow u_0$ strongly in $L^1(\Omega)$, $(f_n) \subset \mathcal{D}(\Omega)$ such that $|f_n| \leq |f|$ a.e., in Q and $f_n \rightarrow f$ strongly in $L^1(Q)$.

For the existence of a weak solution $u_n \in W_0^{1,x}L_M(Q)$, $u_n \geq 0$ of the aforementioned problem, see [23], and also (u_n) satisfies $\frac{\partial u_n}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q) + L^1(Q)$.

Let $v = T_k(u_n)$ be a test function in (P_n) , then

$$\left\langle \frac{\partial u_n}{\partial t}, T_k(u_n) \right\rangle + \int_Q a(\cdot, \nabla u_n) \nabla T_k(u_n) dx dt + \int_Q n T_n(\Phi(u_n)) T_k(u_n) dx dt = \langle f, T_k(u_n) \rangle.$$

We deduce easily that

$$\int_Q a(\cdot, \nabla u_n) \nabla T_k(u_n) dx dt \leq Ck, \quad (3.1)$$

$$\int_Q n T_n(\Phi(u_n)) T_k(u_n) dx dt \leq Ck. \quad (3.2)$$

So, $T_k(u_n)$ is bounded in $W_0^{1,x} L_M(Q)$. Then, there exist a subsequence (also denoted (u_n)) and a measurable function u such that

$$T_k(u_n) \rightharpoonup T_k(u), \text{ weakly in } W_0^{1,x} L_M(Q) \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{M}}), \text{ strongly in } E_M(Q) \text{ and a.e. in } Q. \quad (3.3)$$

Coming back to the inequality (3.2), we have $\int_Q n T_n(\Phi(u_n)) \frac{T_k(u_n)}{k} dx dt \leq C$, and by letting k to infinity and using Fatou lemma, one has

$$\int_Q T_n(\Phi(u_n)) dx dt \leq \frac{C}{n}.$$

Suppose there exists a subsequence (u_n) such that $u_n \notin \mathcal{K}$ for all n , then

$$n^2 |Q| = \int_Q n T_n(\Phi(u_n)) dx dt \leq C,$$

which is a contradiction. Then, there exists a subsequence that we denote as also (u_n) such that $u_n \in \mathcal{K}$ for all n .

In what follows, we only consider this subsequence.

To prove that $u \in \mathcal{K}$, we need to prove that $u \in \mathcal{M}(Q)$. For this reason, let us consider φ as the truncation defined by

The function $\varphi(\zeta)$

for all $\theta, h > 0$.

Using $v = \varphi(u_n)$ as a test function in the approximate problem (P_n) , we obtain

$$-\frac{d}{d\theta} \int_{\{|u_n|>\theta\}} M(|\nabla u_n|) dx \leq C \int_{\{|u_n|\geq\theta\}} \left(f_n - \frac{\partial u_n}{\partial t} \right) dx, \quad (3.4)$$

since $n T_n(\Phi(u_n)) = 0$.

By using Lemma 3.1, we obtained by following the same way as in [20], we have for a good N -function D

$$-\frac{d}{d\theta} \int_{\{|u_n|>\theta\}} D(|\nabla u_n|) dx \leq (-\mu'(\theta)) D o B^{-1} \left(\left(-\frac{1}{\chi_N \mu(\theta)^{1-\frac{1}{N}}} \frac{d}{d\theta} \int_{\{|u_n|>\theta\}} M(|\nabla u_n|) dx \right) \right). \quad (3.5)$$

Let us denote $k(t, s) := \int_0^s u_{n*}(t, \rho) d\rho$, then $\frac{\partial k}{\partial t}(t, s) = \int_0^s \frac{\partial u_{n*}(t, \rho)}{\partial t} d\rho$, $\int_{u_n > \theta} \frac{\partial u_n}{\partial t} dx = \frac{\partial k}{\partial t}(t, \mu(\theta))$, and $F(t, \mu(\theta)) := \int_0^{\mu(\theta)} (f_{n*})(\rho) d\rho$. Using Lemma 3.1, one has

$$1 \leq \frac{-\mu'(\theta)}{\chi_N \mu(\theta)^{1-\frac{1}{N}}} B^{-1} \left(\frac{1}{\chi_N \mu(\theta)^{1-\frac{1}{N}}} \left[F(t, \mu(\theta)) - \frac{\partial k}{\partial t}(t, \mu(\theta)) \right] \right). \quad (3.6)$$

Since $F(t, s) \geq \frac{\partial k}{\partial t}(t, s)$ and using Lemma 3.2, we have $\left| \frac{\partial k}{\partial t}(t, s) \right| \leq F(t, s)$. Combining (3.5)–(3.6) we obtain,

$$-\frac{d}{d\theta} \int_{\{|u_n| > \theta\}} D(|\nabla u_n|) dx \leq (-\mu'(\theta)) D \circ B^{-1} \left(-\frac{1}{\chi_N \mu(\theta)^{1-\frac{1}{N}}} \left[F(t, \mu(\theta)) - \frac{\partial k}{\partial t}(t, \mu(\theta)) \right] \right), \quad (3.7)$$

$$\int_{\Omega} D(|\nabla u_n|) dx = \int_0^{+\infty} \left(-\frac{d}{d\theta} \int_{\{|u_n| > \theta\}} D(|\nabla u_n|) dx \right) d\theta \leq \frac{1}{C'} \int_0^{C'|\Omega|} D \circ B^{-1} \left(\frac{C}{s^{1-1/N}} \right) ds.$$

Case 1: If there exists D an N -function such that $\int_0^{+\infty} D \circ B^{-1} \left(\frac{C}{s^{1-1/N}} \right) ds < +\infty$.

The sequence (u_n) is bounded in $W_0^{1,N} L_D(Q)$ and also $u \in W_0^{1,N} L_D(Q)$.

Case 2: If $\int_0^{+\infty} B^{-1} \left(\frac{C}{s^{1-1/N}} \right) ds < +\infty$.

Then, if we take $D(t) = t$, the sequence (u_n) is bounded in $W_0^{1,1}(Q)$. Then, there exists a measurable function $u \in L^1(Q)$ ($1^* = \frac{N}{N-1}$) such that $u_n \rightharpoonup u$ in $L^1(Q)$. By using [24] (see Proposition 2.3), it follows that

$$\|\nabla u\|_{\mathcal{M}} \leq \lim_n \|\nabla u_n\|_{L^1(Q)} \leq C,$$

where $\|\nabla u\|_{\mathcal{M}} := \sup \left\{ \int_{\Omega} u \operatorname{div} v : v \in (\mathcal{D}(Q))^N \text{ and } \|v\|_{\infty} \leq 1 \right\} < \infty$.

Then, $u \in BV_1(Q) := \{u \in L^1(Q) : \nabla u \in (\mathcal{M}(Q))^N\}$.

Case 3: General case.

Let $0 < \lambda \leq k$, then

$$\operatorname{meas}\{|u_n| > \lambda\} = \operatorname{meas}\{|T_k(u_n)| > \lambda\} \leq \frac{1}{M(\lambda)} \int_{\Omega} M(|T_k(u_n)|) dx \leq \frac{Ck}{M(\lambda)}.$$

Let us take $\lambda = k$, we have

$$\lambda \operatorname{meas}\{|u_n| > \lambda\}^{\frac{1}{\gamma}} \leq C \frac{\lambda^{1+\frac{1}{\gamma}}}{M(\lambda)^{\frac{1}{\gamma}}} \leq C \sup_{t>0} \frac{t^{1+\frac{1}{\gamma}}}{M(t)^{\frac{1}{\gamma}}} =: H.$$

Then, (u_n) is bounded in $M^{\gamma}(Q) \subset L^1(Q)$ (since $\gamma > 1$). Note that $M^{\gamma}(Q)$ is the Marcinkiewicz space. Then, $u \in \mathcal{M}(Q)$.

Finally, since $\int_Q T_1(\Phi(u_n)) dx dt \leq \int_Q T_n(\Phi(u_n)) dx dt \leq \frac{C}{n}$. Then, we deduce, $\int_Q \Phi(u) dx dt = 0$, which ensure $u \in \mathcal{K}$.

II. Almost everywhere convergence of the gradients

The main tool in this step proves

$$\lim_{n \rightarrow \infty} \int_Q (a(\cdot, \nabla T_k(u_n)) - a(\cdot, \nabla T_k(u))) (\nabla T_k(u_n) - \nabla T_k(u)) dx dt = 0,$$

which gives by the same argument as in [25] and adapted to the parabolic case, $\nabla u_n \rightarrow \nabla u$ a.e. in Q .

This is possible by using the following regularization principle $\omega_{\mu,j}^i = (T_k(v_j))_\mu + e^{-\mu t} T_k(\psi_i)$, where $v_j \in D(Q)$ such that $v_j \rightarrow T_k(u)$ with the modular convergence in $W_0^{1,x}L_M(Q)$, ψ_i is a smooth function such that $\psi_i \rightarrow T_k(u_0)$ strongly in $L^1(Q)$, and ω_μ is the mollifier function defined by Landes [26], $\omega_\mu(x, t) = \mu \int_{-\infty}^t \varpi(x, s) \exp(s - t) ds$, where ϖ is the zero extension of ω for $s > T$. The function $\omega_{\mu,j}^i$ have the following properties:

$$\begin{cases} \frac{\partial \omega_{\mu,j}^i}{\partial t} = \mu(T_k(v_j) - \omega_{\mu,j}^i), \omega_{\mu,j}^i(0) = \psi_i, |\omega_{\mu,j}^i| \leq k, \\ \omega_{\mu,j}^i \rightarrow T_k(u)_\mu + e^{-\mu t} T_k(\psi_i) \text{ in } W_0^{1,x}L_M(Q) \text{ for the modular convergence with respect to } j, \\ T_k(u)_\mu + e^{-\mu t} T_k(\psi_i) \rightarrow T_k(u) \text{ in } W_0^{1,x}L_M(Q) \text{ for the modular convergence with respect to } \mu. \end{cases}$$

Consider, for $m > k$, the following truncation

$$\rho_m(s) = \begin{cases} 1 & |s| \leq m, \\ m+1-|s| & m < |s| < m+1, \\ 0 & |s| \geq m+1, \end{cases}$$

and let $R_m(s) = \int_0^s \rho_m(t) dt$.

Consider $v = (T_k(u_n) - \omega_{\mu,j}^i) \rho_m(u_n)$ as a test function in the approximate problem (P_n) , then we have

$$\left\langle \frac{\partial u_n}{\partial t}, v \right\rangle + \int_Q a(., \nabla u_n) (\nabla T_k(u_n) - \nabla \omega_{\mu,j}^i) \rho_m(u_n) \quad (3.8)$$

$$+ \int_Q a(., \nabla u_n) \nabla u_n (T_k(u_n) - \omega_{\mu,j}^i) \rho_m'(u_n) = \int_Q f_n v dx dt. \quad (3.9)$$

We will be interested to estimate the elements of the aforementioned equation.

Since $u_n \in W_0^{1,x}L_M(Q)$, there exists a smooth function $u_{n\sigma}$ (see [23]) such that,

$$\begin{aligned} u_{n\sigma} &\rightarrow u_n \text{ for the modular convergence in } W_0^{1,x}L_M(Q) \text{ and} \\ \frac{\partial u_{n\sigma}}{\partial t} &\rightarrow \frac{\partial u_n}{\partial t} \text{ for the modular convergence in } W^{-1,x}L_{\overline{M}}(Q) + L^1(Q). \end{aligned}$$

$$\begin{aligned} \left\langle \frac{\partial u_n}{\partial t}, v \right\rangle &= \lim_{\sigma \rightarrow 0^+} \int_Q (u_{n\sigma})' (T_k(u_{n\sigma}) - \omega_{\mu,j}^i) \rho_m(u_{n\sigma}) dx dt \\ &= \lim_{\sigma \rightarrow 0^+} \left(\int_Q (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))' (T_k(u_{n\sigma}) - \omega_{\mu,j}^i) + \int_Q (T_k(u_{n\sigma}))' (T_k(u_{n\sigma}) - \omega_{\mu,j}^i) dx dt \right) \\ &= \lim_{\sigma \rightarrow 0^+} \left(\left[\int_\Omega (R_m(u_{n\sigma}) - T_k(u_{n\sigma})) (T_k(u_{n\sigma}) - \omega_{\mu,j}^i) \right]_0^T - \int_Q (R_m(u_{n\sigma}) - T_k(u_{n\sigma})) (T_k(u_{n\sigma}) - \omega_{\mu,j}^i)' \right. \\ &\quad \left. + \int_Q (T_k(u_{n\sigma}))' (T_k(u_{n\sigma}) - \omega_{\mu,j}^i) dx dt \right) \\ &= \lim_{\sigma \rightarrow 0^+} I_1(\sigma) + I_2(\sigma) + I_3(\sigma). \end{aligned}$$

We deal now with the terms $I_1(\sigma)$, $I_2(\sigma)$, and $I_3(\sigma)$ to prove that $\left\langle \frac{\partial u_n}{\partial t}, v \right\rangle \geq \varepsilon(n, j, i, \mu, s, m)$.

Claim 1: $\lim_{\sigma \rightarrow 0^+} I_1(\sigma) \geq \varepsilon(n, j, i, \mu)$.

$$\begin{aligned}
 I_1(\sigma) &= \int_{\Omega} (R_m(u_{n\sigma})(T) - T_k(u_{n\sigma})(T))(T_k(u_{n\sigma})(T) - \omega_{\mu,j}^i(T)) dx \\
 &\quad - \int_{\Omega} (R_m(u_{n\sigma})(0) - T_k(u_{n\sigma})(0))(T_k(u_{n\sigma})(0) - \omega_{\mu,j}^i(0)) dx \\
 &= \int_{u_{n\sigma}(T) \leq k} (R_m(u_{n\sigma})(T) - T_k(u_{n\sigma})(T))(T_k(u_{n\sigma})(T) - \omega_{\mu,j}^i(T)) dx \\
 &\quad + \int_{u_{n\sigma}(T) > k} (R_m(u_{n\sigma})(T) - T_k(u_{n\sigma})(T))(T_k(u_{n\sigma})(T) - \omega_{\mu,j}^i(T)) dx \\
 &\quad - \int_{u_{n\sigma}(0) \leq k} (R_m(u_{n\sigma})(0) - T_k(u_{n\sigma})(0))(T_k(u_{n\sigma})(0) - \psi_i) dx \\
 &\quad - \int_{u_{n\sigma}(0) > k} (R_m(u_{n\sigma})(0) - T_k(u_{n\sigma})(0))(T_k(u_{n\sigma})(0) - \psi_i) dx \\
 &:= J_1 + J_2 + J_3 + J_4.
 \end{aligned}$$

If $u_{n\sigma} \leq k$, we have $R_m(u_{n\sigma}) = T_k(u_{n\sigma})$ (since $m > k$), and if $u_{n\sigma} \geq k$, we have $R_m(u_{n\sigma}) > k \geq |\omega_{\mu,j}^i|$. Then,

$$\begin{aligned}
 J_1 &= \int_{u_{n\sigma}(T) \leq k} (R_m(u_{n\sigma})(T) - T_k(u_{n\sigma})(T))(T_k(u_{n\sigma})(T) - \omega_{\mu,j}^i(T)) dx = 0, \\
 J_2 &= \int_{u_{n\sigma}(T) > k} (R_m(u_{n\sigma})(T) - k)(k - \omega_{\mu,j}^i(T)) dx \geq 0, \\
 J_3 &= - \int_{u_{n\sigma}(0) \leq k} (R_m(u_{n\sigma})(0) - T_k(u_{n\sigma})(0))(T_k(u_{n\sigma})(0) - \psi_i) dx = 0.
 \end{aligned}$$

About J_4 , we have $\lim_{\sigma \rightarrow 0^+} J_4 = - \int_{u_n(0) \geq k} (R_m(u_n)(0) - u_n(0))(T_k(u_n)(0) - \psi_i) dx dt = \varepsilon(n, i)$.

We conclude that

$$\lim_{\sigma \rightarrow 0^+} I_1(\sigma) \geq \varepsilon(n, j, i, \mu).$$

Claim 2: $\lim_{\sigma \rightarrow 0^+} I_2(\sigma) \geq \varepsilon(n, j, i, \mu)$.

It is easy to remark that $T_k(u_{n\sigma})' = 0$, if $u_{n\sigma} > k$, and $(R_m(u_{n\sigma}) - T_k(u_{n\sigma}))(T_k(u_{n\sigma}) - \omega_{\mu,j}^i)' \chi_{u_{n\sigma} > k} \geq 0$.

Then, $I_2(\sigma) = - \int_{u_{n\sigma} \leq k} (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))(T_k(u_{n\sigma}) - \omega_{\mu,j}^i)' dx dt + \int_{u_{n\sigma} > k} (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))(T_k(u_{n\sigma}) - \omega_{\mu,j}^i)' dx dt$.

For the first term and as for I_1 , $-\int_{u_{n\sigma} \leq k} (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))(T_k(u_{n\sigma}) - \omega_{\mu,j}^i)' dx dt \geq \varepsilon(n, j, \mu)$.

For the second term, we have

$$\int_{u_{n\sigma} > k} (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))(T_k(u_{n\sigma}) - \omega_{\mu,j}^i)' dx dt \geq \mu \int_{u_{n\sigma} > k} (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))(T_k(v_j) - \omega_{\mu,j}^i) dx dt.$$

Then,

$$\lim_{\sigma \rightarrow 0^+} I_2(\sigma) \geq \varepsilon(n, j, i, \mu).$$

Claim 3: $\lim_{\sigma \rightarrow 0^+} I_3(\sigma) \geq \varepsilon(n, j, i, \mu)$.

Since,

$$\begin{aligned}
 I_3(\sigma) &= \int_Q (T_k(u_{n\sigma}))' (T_k(u_{n\sigma}) - \omega_{\mu,j}^i) dx dt \\
 &= \int_Q (T_k(u_{n\sigma}) - \omega_{\mu,j}^i)' (T_k(u_{n\sigma}) - \omega_{\mu,j}^i) dx dt + \int_Q (\omega_{\mu,j}^i)' (T_k(u_{n\sigma}) - \omega_{\mu,j}^i) dx dt
 \end{aligned}$$

$$\begin{aligned}
&= \left[\int_{\Omega} \frac{(T_k(u_{n\sigma}) - \omega_{\mu,j}^i)^2}{2} dx \right]_0^T + \mu \int_Q (T_k(v_j) - \omega_{\mu,j}^i)(T_k(u_{n\sigma}) - \omega_{\mu,j}^i) dx dt \\
&\geq \varepsilon(n, j, i, \mu) - \int_{\Omega} \frac{(T_k(u_{n\sigma}(0)) - \psi_i)^2}{2} dx + \mu \int_Q (T_k(v_j) - \omega_{\mu,j}^i)(T_k(u_{n\sigma}) - \omega_{\mu,j}^i) dx dt.
\end{aligned}$$

Then,

$$\begin{aligned}
\lim_{\sigma \rightarrow 0^+} I_3(\sigma) &\geq \varepsilon(n, j, i, \mu) - \int_{\Omega} \frac{(T_k(u_{0n}) - \psi_i)^2}{2} dx + \mu \int_Q (T_k(v_j) - T_k(u_n))(T_k(u_n) - \omega_{\mu,j}^i) dx dt \\
&= \varepsilon(n, j, i, \mu) - \int_{\Omega} \frac{(T_k(u_0) - \psi_i)^2}{2} dx + \mu \int_Q (T_k(v_j) - T_k(u))(T_k(u) - \omega_{\mu,j}^i) dx dt \\
&= \varepsilon(n, j, i, \mu).
\end{aligned}$$

Finally, we have $\left\langle \frac{\partial u_n}{\partial t}, (T_k(u_n) - \omega_{\mu,j}^i) \rho_m(u_n) \right\rangle \geq \varepsilon(n, j, i, \mu)$.

We will now treat the terms (3.8)–(3.9). Before that, we will give some convergence results.

Let $s > 0$, $Q_s = \{(x, t) \in Q : |\nabla T_k(u)| \leq s\}$, $Q_j^s = \{(x, t) \in Q : |\nabla T_k(v_j)| \leq s\}$, and χ^s, χ_j^s be their characteristic function, respectively.

Using (1.1) and (1.3), there exist some measurable function h_k such that

$$a(\cdot, \nabla T_k(u_n)) \rightharpoonup h_k \text{ in } (L_{\overline{M}}(Q))^N \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{M}}),$$

and we also have $\nabla T_k(v_j) \chi_j^s \rightarrow \nabla T_k(u) \chi^s$ strongly in $(L_{\overline{M}}(Q))^N$ and $a(\cdot, \nabla T_k(v_j) \chi_j^s) \rightarrow a(\cdot, \nabla T_k(u) \chi^s)$ strongly in $(E_{\overline{M}}(Q))^N$.

Concerning (3.9)

On the hand, we have

$$\int_Q a(\cdot, \nabla u_n) \nabla u_n (T_k(u_n) - \omega_{\mu,j}^i) \rho_m'(u_n) = \int_Q a(\cdot, \nabla T_{m+1}(u_n)) \nabla T_{m+1}(u_n) (k - \omega_{\mu,j}^i) dx dt \geq 0.$$

On the other hand, since $|f_n| \leq |f|$ and by using Lebesgue theorem with respect to n, j, i , and μ , one get

$$\int_Q f_n (T_k(u_n) - \omega_{\mu,j}^i) \rho_m(u_n) = \varepsilon(n, j, i, \mu).$$

Concerning the second term of (3.8)

$$\begin{aligned}
&\int_Q a(\cdot, \nabla u_n) (\nabla T_k(u_n) - \nabla \omega_{\mu,j}^i) \rho_m(u_n) dx dt \\
&= \int_Q (a(\cdot, \nabla T_k(u_n)) - a(\cdot, \nabla T_k(v_j) \chi_j^s)) (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) dx dt + \int_Q a(\cdot, \nabla T_k(v_j) \chi_j^s) (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) dx dt \\
&\quad + \int_Q a(\cdot, \nabla T_k(u_n)) \nabla T_k(v_j) \chi_j^s dx dt - \int_Q a(\cdot, \nabla u_n) \nabla \omega_{\mu,j} \rho_m(u_n) dx dt \\
&=: J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

$$J_2 = \int_Q a(\cdot, \nabla T_k(v_j) \chi_j^s) (\nabla T_k(u) - \nabla T_k(v_j) \chi_j^s) dx dt + \varepsilon(n) = \varepsilon(n, j).$$

For J_3 , we have

$$J_3 = \varepsilon(n) + \int_Q h_k \nabla T_k(v_j) \chi_j^s = \varepsilon(n, j) + \int_Q h_k \nabla T_k(u) \chi^s.$$

For J_4 , recall that $\rho_m(s) = 0$ if $|s| \geq m + 1$, then

$$\begin{aligned}
 J_4 &= - \int_Q a(., \nabla u_n) \nabla \omega_{\mu, j}^i \rho_m(u_n) dx dt \\
 &= - \int_{|u_n| \leq m+1} a(., \nabla u_n) \nabla \omega_{\mu, j}^i \rho_m(u_n) dx dt \\
 &= - \int_{|u_n| \leq k} a(., \nabla u_n) \nabla \omega_{\mu, j}^i \rho_m(u_n) dx dt - \int_{k < |u_n| \leq m+1} a(., \nabla u_n) \nabla \omega_{\mu, j}^i \rho_m(u_n) dx dt \\
 &= - \int_Q a(., \nabla T_k(u_n)) \nabla \omega_{\mu, j}^i dx dt - \int_{k < |u_n| \leq m+1} a(., \nabla T_{m+1} u_n) \nabla \omega_{\mu, j}^i \rho_m(u_n) dx dt \\
 &= - \int_Q h_k \nabla T_k(u) dx dt - \int_{k \leq |u| \leq m+1} h_{m+1} \nabla T_k(u) \rho_m(u) dx dt + \varepsilon(n, j, i, \mu) \\
 &= - \int_Q h_k \nabla T_k(u) dx dt + \varepsilon(n, j, i, \mu).
 \end{aligned}$$

Using the aforementioned results, we obtain

$$\begin{aligned}
 &\int_Q a(., \nabla u_n) (\nabla T_k(u_n) - \nabla \omega_{\mu, j}^i) \rho_m(u_n) dx dt \\
 &= \int_Q (a(., \nabla T_k(u_n)) - a(., \nabla T_k(v_j) \chi_j^s)) (\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s) dx dt + \varepsilon(n, j, \mu, i, s).
 \end{aligned}$$

Combining (3.8)–(3.9), we obtain the almost everywhere convergence of the gradients.

III. Modular convergence of the gradients

For all $k > 0$, $\nabla T_k(u_n) \rightarrow \nabla T_k(u)$ for the modular convergence in $(L_M(Q))^N$. Indeed, we have proved that

$$\int_Q (a(., \nabla T_k(u_n)) - a(., \nabla T_k(u)) \chi^s) (\nabla T_k(u_n) - \nabla T_k(u) \chi^s) dx dt \leq \varepsilon(n, j, \mu, s, m).$$

Then,

$$\begin{aligned}
 &\int_Q a(., \nabla T_k(u_n)) \nabla T_k(u_n) dx dt \\
 &\leq \int_Q a(., \nabla T_k(u)) \nabla T_k(u) \chi^s dx dt + \int_Q a(., \nabla T_k(u) \chi^s) (\nabla T_k(u_n) - \nabla T_k(u) \chi^s) dx dt + \varepsilon(n, j, \mu, s, m), \\
 &\lim_n \int_Q a(., \nabla T_k(u_n)) \nabla T_k(u_n) dx dt \leq \int_Q a(., \nabla T_k(u)) \nabla T_k(u) \chi^s dx dt + \lim_n \varepsilon(n, j, \mu, s, m).
 \end{aligned}$$

Also we have,

$$\lim_n \int_Q a(., \nabla T_k(u_n)) \nabla T_k(u_n) \leq \int_Q a(., \nabla T_k(u)) \nabla T_k(u) \leq \lim_n \int_Q a(., \nabla T_k(u_n)) \nabla T_k(u_n).$$

Then, we deduce that $a(., \nabla T_k(u_n)) \nabla T_k(u_n) \rightarrow a(., \nabla T_k(u)) \nabla T_k(u) \chi^s$ in $L^1(Q)$.

As mentioned earlier, we obtain $a(., \nabla T_k(u_n)) \nabla T_k(u_n) \rightarrow a(., \nabla T_k(u)) \nabla T_k(u)$ in $L^1(Q)$.

Using Vitali's theorem and (3.1) gives $\nabla T_k(u_n) \rightarrow \nabla T_k(u)$ for the modular convergence in $(L_M(Q))^N$.

IV. Passage to the limit

The passage to the limit is an easy task by taking $v \in \mathcal{K} \cap L^\infty(Q) \cap D$ and $T_k(u_n - v)$ as test function in (P_n) .

V. Uniqueness

Following the same way as Theorem 5.1 [14] for the parabolic case, we obtain the uniqueness.

4 Conclusion

In this paper, we have focused on the existence, uniqueness, and regularity of a class of inequalities in a general convex set and in a nonstandard functional framework, which is the Sobolev Orlicz spaces. The techniques used are not standard and require a very particular handling of the test functions and the approximated problems.

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