Research Article

Houcine Sadraoui, Borhen Halouani*, and Mubariz T. Garayev

On hyponormality on a weighted annulus

<https://doi.org/10.1515/math-2021-0094>

received November 2, 2020; accepted August 25, 2021

Abstract: In this work, we consider the hyponormality of Toeplitz operators on the Bergman space of the annulus with a logarithmic weight. We give necessary conditions when the symbol is of the form $\varphi + \bar{\psi}$, where φ and ψ are analytic on the annulus $\{z \in \mathbb{C}; 1/2 < |z| < 1\}$.

Keywords: Toeplitz operator, weighted Bergman spaces, hyponormality, positive matrices

MSC 2020: 47B35, 47B20, 15B48

1 Introduction

A bounded operator T on a Hilbert space is hyponormal if $T^*T - TT^*$ is positive. The first work on hyponormality of Toeplitz operators on the Hardy space is that of Cowen [1,2]. Hyponormality on the Bergman space was first considered in [3] and a boundary derivative condition was given as a necessary condition. A function theory generalization of the necessary condition was given by Ahern and Cuckovic [4]. The necessary condition was recently improved in a special case by Curto and Cuckovic [5]. Other results deal mostly with special cases. Some of these results can be found in [6]. Sufficient conditions when the analytic part of the symbol is a monomial are provided in [7]. In this work, we complete our results given in [8] by showing that the derivative condition holds for other types of symbols. We begin with definitions and notations. Set $R_{1/2} = \{u \in \mathbb{C}; 1/2 < |u| < 1\}$. The space L_v^2 is the space of measurable functions f on $R_{1/2}$ such that $\int_{R_{1/2}} |f|^2 dv(u) < \infty$, where $dv(u) = \frac{8}{\pi(3-2\ln 2)} r |\log r| dr d\theta$. The subspace of L_v^2 consisting of analytic functions is denoted by A_v^2 . If f is analytic on $R_{1/2}$, we have $f = \sum_{n=-\infty}^{-1} c_n u^n + \sum_{n=0}^{\infty} c_n u^n$ and

$$\begin{aligned} \|f\|^2 &= \frac{1}{3-2\ln 2} \sum_{n=0}^{\infty} \frac{2^{2n+2} - 1 - (2n+2)\ln 2}{2^{2n}(n+1)^2} |c_n|^2 + \frac{8(\ln 2)^2}{3-2\ln 2} |c_{-1}|^2 \\ &\quad + \frac{4}{3-2\ln 2} \sum_{n=2}^{\infty} \frac{2^{2n-1}(n-1)\ln 2 - 2^{2n-2} + 1}{(n-1)^2} |c_{-n}|^2. \end{aligned}$$

The space A_v^2 has the following orthonormal basis:

$$\begin{aligned} \{e_n; n \geq 0\} \cup \{e_{-1}\} \cup \{e_{-n}; n \geq 1\} &= \left\{ \sqrt{3-2\ln 2} \frac{2^n(n+1)}{\sqrt{2^{2n+2} - 1 - (2n+2)\ln 2}} u^n, n \geq 0 \right\} \cup \left\{ \frac{\sqrt{3-2\ln 2}}{2\sqrt{2}\ln 2} \frac{1}{u} \right\} \\ &\quad \cup \left\{ \frac{\sqrt{3-2\ln 2}}{2} \frac{n-1}{\sqrt{2^{2n-1}(n-1)\ln 2 - 2^{2n-2} + 1}} \frac{1}{u^n}, n \geq 2 \right\}. \end{aligned}$$

* **Corresponding author: Borhen Halouani**, Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia, e-mail: halouani@ksu.edu.sa

Houcine Sadraoui: Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia, e-mail: sadrawi@ksu.edu.sa

Mubariz T. Garayev: Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia, e-mail: mgarayev@ksu.edu.sa

For h bounded measurable on $R_{1/2}$ we define the Toeplitz operator B_h by $B_h(f) = Q(hf)$, where Q is the orthogonal projection of L_v^2 onto A_v^2 . We also define Hankel operators by $H_h(f) = (I - Q)(hf)$. In this work, the Toeplitz operators considered have a symbol of the form $\varphi + \bar{\psi}$, where φ and ψ are bounded analytic on $R_{1/2}$. We give necessary conditions for the hyponormality of these operators. We start with some basic properties of these operators. These properties are easy to prove and the proof is omitted.

2 General properties of Toeplitz and Hankel operators

Lemma 2.1. *Let φ and ψ be bounded measurable on $R_{1/2}$. The following statements hold:*

- (a) $B_{\varphi+\psi} = B_\varphi + B_\psi$.
- (b) $B_\varphi^* = B_{\bar{\varphi}}$.
- (c) $B_\varphi B_\psi = B_{\varphi\psi}$ if ψ is analytic or φ is conjugate analytic.
- (d) $B_\varphi^* B_\varphi - B_\varphi B_\varphi^* = H_{\bar{\varphi}}^* H_\varphi$ if φ is analytic.

As in the case of the Bergman space of the unit disk [3], the following proposition gives equivalent forms of hyponormality.

Proposition 2.2. *For φ_1 and φ_2 bounded analytic on $R_{1/2}$ the following statements are equivalent:*

- (a) $B_{\varphi_1+\bar{\varphi}_2}$ is hyponormal.
- (b) $B_{\bar{\varphi}_2} B_{\varphi_2} - B_{\varphi_2} B_{\bar{\varphi}_2} \leq B_{\bar{\varphi}_1} B_{\varphi_1} - B_{\varphi_1} B_{\bar{\varphi}_1}$.
- (c) $H_{\bar{\varphi}_2}^* H_{\varphi_2} \leq H_{\bar{\varphi}_1}^* H_{\varphi_1}$.
- (d) $H_{\bar{\varphi}_2} = CH_{\bar{\varphi}_1}$, where C is bounded of norm less than or equal to one.

Computations involving the projection are given in the next lemma.

Lemma 2.3. *The orthogonal projection of L_v^2 onto A_v^2 satisfies the following properties:*

- (1) $Q(u^m \bar{u}^n) = \frac{2^{2m+2} - 1 - (2m+2)\ln 2}{2^{2m}(m+1)^2} \frac{2^{2(m-n)}(m-n+1)^2}{2^{2(m-n)+2} - 1 - (2(m-n)+2)\ln 2} u^{m-n}$ if $m \geq n$.
- (2) $Q(u^m \bar{u}^n) = \frac{1}{4} \frac{2^{2m+2} - 1 - (2m+2)\ln 2}{2^{2m}(m+1)^2} \frac{(n-m-1)^2}{2^{2(n-m)-1}((n-m)-1)\ln 2 - 2^{2(n-m)-2} + 1} \frac{1}{u^{n-m}}$ if $n - m \geq 2$.
- (3) $Q(u^m \bar{u}^{m+1}) = \frac{1}{8(\ln 2)^2} \frac{2^{2m+2} - 1 - (2m+2)\ln 2}{2^{2m}(m+1)^2} \frac{1}{u}$.
- (4) $Q\left(\frac{1}{u^m} \bar{u}^n\right) = \frac{2^{2m-1}(m-1)\ln 2 - 2^{2m-2} + 1}{(m-1)^2} \frac{(m+n-1)^2}{2^{2(m+n)-1}(m+n-1)\ln 2 - 2^{2(m+n)-2} + 1} \frac{1}{u^{m+n}}$, $m \geq 2$.
- (5) $Q\left(\frac{1}{u} \bar{u}^n\right) = 2(\ln 2)^2 \frac{n^2}{2^{2n+1}n\ln 2 - 2^{2n} + 1} \frac{1}{u^{n+1}}$, $n \geq 1$.
- (6) $Q\left(\frac{1}{u^m} \bar{u}^n\right) = \frac{2^{2n+2} - 1 - (2n+2)\ln 2}{2^{2n}(n+1)^2} \frac{2^{2(m+n)}(m+n+1)^2}{2^{2(m+n)+2} - 1 - (2(m+n)+2)\ln 2} u^{m+n}$.
- (7) $Q\left(\frac{1}{u^m} \bar{u}^n\right) = 4 \frac{2^{2n-1}(n-1)\ln 2 - 2^{2n-2} + 1}{(n-1)^2} \frac{2^{2(m-n)}(m-n+1)^2}{2^{2(m-n)+2} - 1 - (2(m-n)+2)\ln 2} u^{m-n}$, $m \geq n$, $n \geq 2$.
- (8) $Q\left(\frac{1}{u^m} \bar{u}^1\right) = 2(\ln 2)^2 \frac{2^{2m}m^2}{2^{2m} - 1 - 2m\ln 2} u^{m-1}$, $m \geq 1$.
- (9) $Q\left(\frac{1}{u^m} \bar{u}^1\right) = \frac{2^{2n-1}(n-1)\ln 2 - 2^{2n-2} + 1}{(n-1)^2} \frac{(n-m-1)^2}{2^{2(n-m)-1}(n-m-1)\ln 2 - 2^{2(n-m)-2} + 1} \frac{1}{u^{n-m}}$, $n - m \geq 2$.
- (10) $Q\left(\frac{1}{u^m} \bar{u}^{m+1}\right) = \frac{1}{2(\ln 2)^2} \frac{m^2}{2^{2m+1}m\ln 2 - 2^{2m} + 1} \frac{1}{u}$.

3 Necessary conditions

We start by remarking that $\varphi = \sum_1^\infty c_n u^n$ bounded analytic on $R_{1/2}$ is equivalent to φ being analytic and bounded on the unit disk. We will need the following computation.

Lemma 3.1. Let $\varphi = \sum_1^\infty c_n u^n$ be bounded and analytic on $R_{1/2}$. For k and $l \geq 1$ we have

$$\begin{aligned} & \langle B_{\bar{\varphi}} B_{\varphi} - B_{\bar{\varphi}} B_{\bar{\varphi}}(e_k), e_l \rangle \\ &= \sum_{n \geq 1, n+k-l \geq 1}^{\infty} \overline{c_{n+k-l}} c_n \frac{2^k(k+1)2^l(l+1)}{\sqrt{2^{2k+2}-1-(2k+2)\ln 2}} \frac{(2^{2(n+k)+2}-1-(2(n+k)+2)\ln 2)}{\sqrt{2^{2l+2}-1-(2l+2)\ln 2}} \frac{2^{2(n+k)}(n+k+1)^2}{2^{2(n+k)}(n+k+1)^2} \\ &- \sum_{l \geq n \geq 1, n+k-l \geq 1}^{\infty} \overline{c_{n+k-l}} c_n \frac{\sqrt{2^{2k+2}-1-(2k+2)\ln 2}}{2^k(k+1)2^l(l+1)} \frac{\sqrt{2^{2l+2}-1-(2l+2)\ln 2}}{(2^{2(l-n)+2}-1-(2(l-n)+2)\ln 2)} \frac{2^{2(l-n)}(l-n+1)^2}{2^{2(l-n)}(l-n+1)^2} \\ &- \frac{1}{8(\ln 2)^2} \overline{c_{k+1}} c_{l+1} \frac{\sqrt{2^{2k+2}-1-(2k+2)\ln 2}}{2^k(k+1)} \frac{\sqrt{2^{2l+2}-1-(2l+2)\ln 2}}{2^l(l+1)} \\ &- \frac{1}{4} \sum_{l \leq n-2}^{\infty} \overline{c_{n+k-l}} c_n \frac{\sqrt{2^{2k+2}-1-(2k+2)\ln 2}}{2^k(k+1)2^l(l+1)} \frac{\sqrt{2^{2l+2}-1-(2l+2)\ln 2}}{2^{2(n-l)-1}((n-l-1)\ln 2 - 2^{2(n-l)-2} + 1)} \frac{(n-l-1)^2}{2^{2(n-l)-1}((n-l-1)\ln 2 - 2^{2(n-l)-2} + 1)}. \end{aligned}$$

Proof. We have

$$\begin{aligned} \langle B_{\bar{\varphi}} B_{\varphi}(e_k), e_l \rangle &= \sum_{m, n \geq 1}^{\infty} \overline{c_m} c_n \frac{(3-2\ln 2)2^k(k+1)}{\sqrt{2^{2k+2}-1-(2k+2)\ln 2}} \frac{2^l(l+1)\langle u^{n+k}, u^{m+l} \rangle}{\sqrt{2^{2l+2}-1-(2l+2)\ln 2}} \\ &= \sum_{n \geq 1, n+k-l \geq 1}^{\infty} \overline{c_{n+k-l}} c_n \frac{2^k(k+1)2^l(l+1)}{\sqrt{2^{2k+2}-1-(2k+2)\ln 2}} \frac{(2^{2(n+k)+2}-1-(2(n+k)+2)\ln 2)}{\sqrt{2^{2l+2}-1-(2l+2)\ln 2}} \frac{2^{2(n+k)}(n+k+1)^2}{2^{2(n+k)}(n+k+1)^2}. \end{aligned}$$

We also have

$$\begin{aligned} \langle B_{\bar{\varphi}} B_{\bar{\varphi}}(e_k), e_l \rangle &= \frac{1}{4} \sum_{n-l \geq 2}^{\infty} \overline{c_{n+k-l}} c_n \frac{\sqrt{2^{2k+2}-1-(2k+2)\ln 2}}{2^k(k+1)2^l(l+1)} \frac{\sqrt{2^{2l+2}-1-(2l+2)\ln 2}}{(2^{2(n-l)-1}((n-l-1)\ln 2 - 2^{2(n-l)-2} + 1))} \frac{(n-l-1)^2}{(2^{2(n-l)-1}((n-l-1)\ln 2 - 2^{2(n-l)-2} + 1))} \\ &+ \frac{1}{8(\ln 2)^2} \overline{c_{k+1}} c_{l+1} \frac{\sqrt{2^{2k+2}-1-(2k+2)\ln 2}}{2^k(k+1)} \frac{\sqrt{2^{2l+2}-1-(2l+2)\ln 2}}{2^l(l+1)} \\ &+ \sum_{l \geq n \geq 1, n+k-l \geq 1}^{\infty} \overline{c_{n+k-l}} c_n \frac{\sqrt{2^{2k+2}-1-(2k+2)\ln 2}}{2^k(k+1)2^l(l+1)} \frac{\sqrt{2^{2l+2}-1-(2l+2)\ln 2}}{(2^{2(l-n)+2}-1-(2(l-n)+2)\ln 2)} \frac{2^{2(l-n)}(l-n+1)^2}{2^{2(l-n)}(l-n+1)^2}. \end{aligned}$$

The result follows. $\square$

Put $\sigma_{l,k} = \langle B_{\bar{\varphi}} B_{\varphi} - B_{\bar{\varphi}} B_{\bar{\varphi}}(e_k), e_l \rangle$. We have with obvious notations

$$\sigma_{l,l+p} = \sum_{1 \leq n \leq l} \overline{c_{n+p}} c_n M_{n,p,l} + \overline{c_{l+p+1}} c_{l+1} N_{l,p} + \sum_{n \geq l+2}^{\infty} \overline{c_{n+p}} c_n L_{n,p,l}.$$

Recall that H^2 denotes the Hardy space and denote by T_{ω} the Toeplitz operator on H^2 with symbol ω . If we denote the matrix of T_{ω} in the usual basis of H^2 by $(\omega_{i,j})$, it is known that $\omega_{i,i+p} = \omega_{j,j+p}$ for any nonnegative integers i, j and p .

Lemma 3.2. Assume that $\varphi' \in H^2$. Then we have $\lim_{l \rightarrow \infty} l^2 \sigma_{l,l+p} = \lambda_{s,s+p}$, where $(\lambda_{r,s})_{r,s}$ is the matrix of the operator $T_{|\varphi'|^2}$.

Proof. A somewhat involved but elementary computation shows that:

$$\lim_{l \rightarrow \infty} l^2 M_{n,p,l} = n(n+p).$$

Set $g_l(n) = l^2 \chi_{\{0, \dots, l\}}(n) \overline{c_{n+p}} c_n M_{n,p,l}$, then

$$\lim_{l \rightarrow \infty} g_l(n) = n(n+p) \overline{c_{n+p}} c_n$$

and

$$\int g_l(n) d\rho(n) = \sum_{1 \leq n \leq l} l^2 \overline{c_{n+p}} c_n M_{n,p,l}.$$

It is not difficult to see that for l large

$$|l^2 \overline{c_{n+p}} c_n M_{n,p,l}| \leq 2n(n+p) |\overline{c_{n+p}} c_n| \leq (n+p)^2 |c_{n+p}|^2 + n^2 |c_n|^2.$$

Since $f' \in H^2$ it follows by the dominated convergence theorem that

$$\lim_{l \rightarrow \infty} \sum_{1 \leq n \leq l} l^2 \overline{c_{n+p}} c_n M_{n,p,l} = \sum_{n \geq 1} n(n+p) \overline{c_{n+p}} c_n.$$

Also, we can readily see that for l large

$$|l^2 \overline{c_{l+p+1}} c_{l+1} N_{l,p}| \leq C((l+p+1)^2 |\overline{c_{l+p+1}}|^2 + (l+1)^2 |c_{l+1}|^2)$$

for some constant C , so

$$\lim_{l \rightarrow \infty} l^2 \overline{c_{l+p+1}} c_{l+1} N_{l,p} = 0.$$

Finally, we have

$$|l^2 \overline{c_{n+p}} c_n L_{n,p,l}| \leq n(n+p) |c_n| |c_{n+p}| \leq \frac{1}{2} (n^2 |c_n|^2 + (n+p)^2 |c_{n+p}|^2)$$

for l large and $n \geq l+2$. It follows from the dominated convergence theorem that

$$\lim_{l \rightarrow \infty} \sum_{n \geq l+2} l^2 \overline{c_{n+p}} c_n L_{n,p,l} = 0.$$

We obtain that $\lim_{l \rightarrow \infty} l^2 \sigma_{l,p} = \sum_{n \geq 1} n(n+p) \overline{c_{n+p}} c_n = \lambda_{s,p}$. $\square$

We deduce one of our main results.

Theorem 3.3. Let $\varphi = \sum_1^\infty c_n u^n$ and $\psi = \sum_1^\infty d_n u^n$ be bounded and analytic on $R_{1/2}$ and assume that $\varphi' \in H^2$. If $B_{\varphi+\bar{\psi}}$ is hyponormal, then $\psi' \in H^2$ and $|\varphi'| \geq |\psi'|$ a.e. on the unit circle.

Proof. Denote by $(\Psi_{i,j})$ the matrix of $B_{\bar{\varphi}} B_\varphi - B_\varphi B_{\bar{\varphi}} - (B_{\bar{\psi}} B_\psi - B_\psi B_{\bar{\psi}})$, by $(\sigma_{i,j})$ the matrix of $B_{\bar{\varphi}} B_\varphi - B_\varphi B_{\bar{\varphi}}$ and $(\theta_{i,j})$ the matrix of $B_{\bar{\psi}} B_\psi - B_\psi B_{\bar{\psi}}$. Hyponormality of $B_{\varphi+\bar{\psi}}$ gives $\theta_{l,l} \leq \sigma_{l,l}$. We deduce that

$$\sum_{1 \leq n \leq l} l^2 |c_n|^2 M_{n,0,l} + l^2 |c_{l+1}|^2 N_{l,0} + \sum_{n \geq l+2} l^2 |c_n|^2 L_{n,0,l} \geq \sum_{1 \leq n \leq l} l^2 |d_n|^2 M_{n,0,l} + l^2 |d_{l+1}|^2 N_{l,0} + \sum_{n \geq l+2} l^2 |d_n|^2 L_{n,0,l}.$$

We have

$$\lim_{l \rightarrow \infty} l^2 |d_{l+1}|^2 N_{l,0} = 0 = \lim_{l \rightarrow \infty} l^2 |c_{l+1}|^2 N_{l,0}.$$

Using the fact that $\sum_{l \geq 1} l^2 |c_l|^2 < \infty$ and the dominated convergence theorem we see that

$$\lim_{l \rightarrow \infty} \sum_{n \geq l+2} l^2 |c_n|^2 L_{n,0,l} = 0.$$

Also, since $\sum_1^\infty |d_n|^2 < \infty$, by the dominated convergence theorem, we get

$$\lim_{l \rightarrow \infty} \sum_{n \geq l+2} l^2 |d_n|^2 L_{n,0,l} = 0.$$

Writing $\sum_{1 \leq n \leq l} l^2 |d_n|^2 M_{n,0,l}$ as an integral with respect to the counting measure, and using Fatou's lemma in the last inequality we get:

$$\sum_{1 \leq n} n^2 |c_n|^2 \geq \sum_{1 \leq n} n^2 |d_n|^2.$$

Thus $\psi' \in H^2$. From the previous Lemma 3.2 we obtain that

$$\lim_{l \rightarrow \infty} l^2(\sigma_{l,l+p} - \theta_{l,l+p}) = \Lambda_{s,s+p},$$

where $(\Lambda_{s,s+p})$ is the matrix of the H^2 Toeplitz operator $T_{|\varphi'|^2 - |\psi'|^2}$. Hyponormality and a property of Toeplitz forms [9] lead to $|\varphi'| \geq |\psi'|$ a.e. on the unit circle. $\square$

Corollary 3.4. Let $\varphi = \sum_1^\infty c_n u^n$ and $\psi = \sum_1^\infty d_n u^n$ be bounded and analytic and univalent on an open set containing the unit disk. Then $B_{\varphi+\bar{\psi}}$ is normal if and only if $\psi = a\varphi$ for some constant a with $|a| = 1$.

Proof. If $\psi = a\varphi$, then it is easy to show that $B_{\varphi+\bar{\psi}}$ is normal. Normality of $B_{\varphi+\bar{\psi}}$ leads to $|\varphi'| = |\psi'|$ on the unit circle. By the maximum modulus theorem, $\psi' = a\varphi'$ for some constant a with $|a| = 1$. Thus, $\psi = a\varphi$. $\square$

The following computation is needed.

Lemma 3.5. Let $\varphi = \sum_1^\infty c_n u^n$ be bounded and analytic on $R_{1/2}$. For k and l greater than or equal to 3 we have

$$\begin{aligned} & \langle B_{\bar{\varphi}} B_{\varphi} - B_{\varphi} B_{\bar{\varphi}}(e_{-k}), e_{-l} \rangle \\ &= \frac{1}{4} \sum_{n \geq k} \frac{\overline{c_{n+l-k}} c_n}{\sqrt{2^{2k-1}(k-1)\ln 2 - 2^{2k-2} + 1}} \frac{(k-1)(l-1)}{\sqrt{2^{2l-1}(l-1)\ln 2 - 2^{2l-2} + 1}} \frac{(2^{2(n-k)+2} - 1 - (2(n-k) + 2)\ln 2)}{2^{2(n-k)}(n-k+1)^2} \\ &+ 2(\ln 2)^2 \frac{\overline{c_{l-1}} c_{k-1}}{\sqrt{2^{2k-1}(k-1)\ln 2 - 2^{2k-2} + 1}} \frac{k-1}{\sqrt{2^{2l-1}(l-1)\ln 2 - 2^{2l-2} + 1}} \frac{l-1}{\sqrt{2^{2l-1}(l-1)\ln 2 - 2^{2l-2} + 1}} \\ &+ \sum_{1 \leq n+l-k, n \leq k-2} \frac{\overline{c_{n+l-k}} c_n}{\sqrt{2^{2k-1}(k-1)\ln 2 - 2^{2k-2} + 1}} \frac{(k-1)(l-1)}{\sqrt{2^{2l-1}(l-1)\ln 2 - 2^{2l-2} + 1}} \frac{2^{2(k-n)-1}(k-n-1)\ln 2 - 2^{2(k-n)-2} + 1}{\sqrt{2^{2l-1}(l-1)\ln 2 - 2^{2l-2} + 1}} \frac{1}{(k-n-1)^2} \\ &- \sum_{n \geq 1, n+k-l \geq 1} \frac{\overline{c_n} c_{n+k-l}}{\sqrt{2^{2k-1}(k-1)\ln 2 - 2^{2k-2} + 1}} \frac{(k-1)(l-1)}{(k-1)(l-1)} \frac{\sqrt{2^{2l-1}(l-1)\ln 2 - 2^{2l-2} + 1}}{(2^{2(n+k)-1}(n+k-1)\ln 2 - 2^{2(n+k)-2} + 1)}. \end{aligned}$$

Proof. Indeed, we have

$$\begin{aligned} & \langle B_{\bar{\varphi}} B_{\varphi}(e_{-k}), e_{-l} \rangle \\ &= \frac{1}{4} \sum_{m, n \geq 1} \frac{\overline{c_m} c_n}{\sqrt{2^{2k-1}(k-1)\ln 2 - 2^{2k-2} + 1}} \frac{(3 - 2\ln 2)(k-1)(l-1)}{\sqrt{2^{2l-1}(l-1)\ln 2 - 2^{2l-2} + 1}} \frac{\langle u^{n-k}, u^{m-l} \rangle}{\sqrt{2^{2l-1}(l-1)\ln 2 - 2^{2l-2} + 1}} \\ &= \frac{1}{4} \sum_{n \geq k} \frac{\overline{c_{n+l-k}} c_n}{2^{2(n-k)}(n-k+1)^2 \sqrt{2^{2k-1}(k-1)\ln 2 - 2^{2k-2} + 1}} \frac{(k-1)(l-1)(2^{2(n-k)+2} - 1 - (2(n-k) + 2)\ln 2)}{\sqrt{2^{2l-1}(l-1)\ln 2 - 2^{2l-2} + 1}} \\ &+ 2(\ln 2)^2 \frac{\overline{c_{l-1}} c_{k-1}}{\sqrt{2^{2k-1}(k-1)\ln 2 - 2^{2k-2} + 1}} \frac{(k-1)(l-1)}{\sqrt{2^{2l-1}(l-1)\ln 2 - 2^{2l-2} + 1}} \\ &+ \sum_{1 \leq n \leq k-2} \frac{\overline{c_{n+l-k}} c_n}{\sqrt{2^{2k-1}(k-1)\ln 2 - 2^{2k-2} + 1}} \frac{(k-1)(l-1)(2^{2(k-n)-1}(k-n-1)\ln 2 - 2^{2(k-n)-2} + 1)}{\sqrt{2^{2l-1}(l-1)\ln 2 - 2^{2l-2} + 1}} \frac{1}{(k-n-1)^2}, \\ & \langle B_{\varphi} B_{\bar{\varphi}}(e_{-k}), e_{-l} \rangle \\ &= \sum_{m, n \geq 1} \frac{\overline{c_n} c_m}{4 \sqrt{2^{2k-1}(k-1)\ln 2 - 2^{2k-2} + 1}} \frac{(3 - 2\ln 2)(k-1)(l-1) \langle P(\overline{u^n} \frac{1}{u^k}), P(\overline{u^m} \frac{1}{u^l}) \rangle}{\sqrt{2^{2l-1}(l-1)\ln 2 - 2^{2l-2} + 1}} \\ &= \sum_{n \geq 1, n+k-l \geq 1} \frac{\overline{c_n} c_{n+k-l}}{\sqrt{2^{2k-1}(k-1)\ln 2 - 2^{2k-2} + 1}} \frac{\sqrt{2^{2l-1}(l-1)\ln 2 - 2^{2l-2} + 1}}{(k-1)(l-1)(2^{2(n+k)-1}(n+k-1)\ln 2 - 2^{2(n+k)-2} + 1))}. \end{aligned} \quad \square$$

We can write

$$\begin{aligned}\sigma_{-l,-l-p} &= \frac{1}{4} \sum_{n \geq l} \overline{c_n} c_{n+p} \frac{(l+p-1)(l-1)(2^{2(n-l)+2} - 1 - (2(n-l)+2)\ln 2)}{\sqrt{2^{2(l+p)-1}(l+p-1)\ln 2 - 2^{2(l+p)-2} + 1} \sqrt{2^{2l-1}(l-1)\ln 2 - 2^{2l-2} + 1}} \frac{2^{2(n-l)}(n-l+1)^2}{2^{2(n-l)}(n-l+1)^2} \\ &\quad + \frac{2(\ln 2)^2 \overline{c_{l-1}} c_{l+p-1} (l+p-1)(l-1)}{\sqrt{2^{2(l+p)-1}(l+p-1)\ln 2 - 2^{2(l+p)-2} + 1} \sqrt{2^{2l-1}(l-1)\ln 2 - 2^{2l-2} + 1}} \\ &\quad + \sum_{1 \leq n \leq l-2} \frac{\overline{c_n} c_{n+p} (l+p-1)(l-1)(2^{2(l-n)-1}(l-n-1)\ln 2 - 2^{2(l-n)-2} + 1)}{\sqrt{2^{2(l+p)-1}(l+p-1)\ln 2 - 2^{2(l+p)-2} + 1} \sqrt{(2^{2l-1}(l-1)\ln 2 - 2^{2l-2} + 1)} (l-n-1)^2} \\ &\quad - \sum_{n \geq 1} \overline{c_n} c_{n+p} \frac{\sqrt{2^{2(l+p)-1}((l+p)-1)\ln 2 - 2^{2(l+p)-2} + 1}}{(l+p-1)(l-1)} \frac{\sqrt{2^{2l-1}(l-1)\ln 2 - 2^{2l-2} + 1}}{(2^{2(n+l+p)-1}(n+l+p-1)\ln 2 - 2^{2(n+l+p)-2} + 1)} (n+l+p-1)^2 \\ &= \sum_{n \geq l} \overline{c_n} c_{n+p} A_{n,l,p} + \overline{c_{l-1}} c_{l+p-1} B_{l,p} + \sum_{1 \leq n \leq l-2} \overline{c_n} c_{n+p} C_{n,l,p} + \sum_{n \geq l} \overline{c_n} c_{n+p} D_{n,p,l},\end{aligned}$$

where, in the last equality,

$$\begin{aligned}C_{n,l,p} &= \frac{(l+p-1)(l-1)(2^{2(l-n)-1}(l-n-1)\ln 2 - 2^{2(l-n)-2} + 1)}{\sqrt{2^{2(l+p)-1}(l+p-1)\ln 2 - 2^{2(l+p)-2} + 1} \sqrt{2^{2l-1}(l-1)\ln 2 - 2^{2l-2} + 1}} (l-n-1)^2 \\ &\quad - \frac{\sqrt{2^{2(l+p)-1}((l+p)-1)\ln 2 - 2^{2(l+p)-2} + 1} \sqrt{2^{2l-1}(l-1)\ln 2 - 2^{2l-2} + 1}}{(l+p-1)(l-1)(2^{2(n+l+p)-1}(n+l+p-1)\ln 2 - 2^{2(n+l+p)-2} + 1)} (n+l+p-1)^2\end{aligned}$$

and

$$D_{n,p,l} = - \frac{\sqrt{2^{2(l+p)-1}((l+p)-1)\ln 2 - 2^{2(l+p)-2} + 1} \sqrt{2^{2l-1}(l-1)\ln 2 - 2^{2l-2} + 1}}{(l+p-1)(l-1)(2^{2(n+l+p)-1}(n+l+p-1)\ln 2 - 2^{2(n+l+p)-2} + 1)} (n+l+p-1)^2.$$

For $\varphi = \sum_{n=1}^{\infty} c_n u^n$ bounded and analytic on $R_{1/2}$, we clearly have $\sum_{n \geq 1} \frac{n^2}{2^{2n}} |c_n|^2 < \infty$. Put $\varphi_{1/2}(u) = \sum_{n \geq 1} \overline{c_n} \frac{u^n}{2^n}$ and denote by $(\zeta_{i,j})$ the matrix of the Hardy space Toeplitz operator $T_{|\varphi_{1/2}|^2}$. As mentioned earlier, this Toeplitz matrix satisfies $\zeta_{i,i+p} = \zeta_{j,j+p}$.

Lemma 3.6. We have $\lim_{l \rightarrow \infty} l^2 \sigma_{-l,-l-p} = \zeta_{s,s+p}$.

Proof. We proceed as in the proof Lemma 3.2. A computation shows that

$$\lim_{l \rightarrow \infty} l^2 C_{n,l,p} = \frac{n(n+p)}{2^{2n+p}}.$$

We can also show that for l large

$$|l^2 \overline{c_n} c_{n+p} C_{n,l,p}| \leq \frac{C n(n+p) \overline{c_n} c_{n+p}}{2^{2n+p}} \leq \frac{C}{2} \left(\frac{|c_n|^2 n^2}{2^{2n}} + \frac{|c_{n+p}|^2 (n+p)^2}{2^{2n+2p}} \right)$$

for some constant C . Writing $\sum_{1 \leq n, n \leq l-2} l^2 \overline{c_n} c_{n+p} C_{n,l,p}$ as an integral with respect to the counting measure and using the dominated convergence theorem we get

$$\lim_{l \rightarrow \infty} \sum_{1 \leq n, n \leq l-2} l^2 \overline{c_n} c_{n+p} C_{n,l,p} = \sum_{n \geq 1} \frac{n(n+p) \overline{c_n} c_{n+p}}{2^{2n+p}}.$$

Clearly $\lim_{l \rightarrow \infty} \overline{c_{l-1}} c_{l+p-1} B_{l,p} = 0$. Finally, using the dominated convergence theorem and the fact that $\sum_{n \geq 1} \frac{n^2}{2^{2n}} |c_n|^2 < \infty$, it is not difficult to show that

$$\lim_{l \rightarrow \infty} \sum_{n \geq l} l^2 \overline{c_n} c_{n+p} A_{n,l,p} = \lim_{l \rightarrow \infty} \sum_{n \geq l} l^2 \overline{c_n} c_{n+p} D_{n,p,l} = 0.$$

Thus, we have $\lim_{l \rightarrow \infty} l^2 \sigma_{-l,-l-p} = \zeta_{s,s+p}$. □

Proceeding as in the proof of Theorem 3.3, we get the following result.

Proposition 3.7. *Let $\varphi = \sum_1^\infty c_n u^n$ and $\psi = \sum_1^\infty d_n u^n$ be bounded and analytic on $R_{1/2}$. If $B_{\varphi+\bar{\psi}}$ is hyponormal, then $|\psi'| \leq |\varphi'|$ a.e. on $\{z, |z| = 1/2\}$.*

We get an improvement of Corollary 3.4.

Theorem 3.8. *Let $\varphi = \sum_1^\infty c_n u^n$ and $\psi = \sum_1^\infty d_n u^n$ be bounded and analytic on $R_{1/2}$ and assume that φ and ψ are univalent in an open set containing $\{z, |z| < 1/2\}$. Then $B_{\varphi+\bar{\psi}}$ is normal if and only if $\psi = a\varphi$ for some constant a with $|a| = 1$.*

We conclude with the following theorem.

Theorem 3.9. *Let $\varphi = \sum_1^\infty c_n u^n$ and $\psi = \sum_1^\infty d_n u^n$ be bounded and analytic on $R_{1/2}$ and assume $\varphi' \in H^2$. If $B_{\varphi+\bar{\psi}}$ is hyponormal, then $\psi' \in H^2$ and $|\psi'| \leq |\varphi'|$ a.e. on $\{z, |z| = 1\} \cup \{z, |z| = 1/2\}$.*

Acknowledgements: The authors would like to extend their sincere appreciation to the Deanship of scientific research at King Saud University for its funding of this work through Research Group Project no RG-1439-68.

Conflict of interest: Authors state no conflict of interest.

References

- [1] C. Cowen, *Hyponormal and subnormal Toeplitz operators*, in: J. B. Conway, B. B. Morrel (Eds.), *Surveys of Some Results in Operator Theory*, Vol. I, Pitman Research Notes in Mathematics, vol. 171, Longman, 1988, pp. 155–167.
- [2] C. Cowen, *Hyponormality of Toeplitz operators*, *Proc. Amer. Math. Soc.* **103** (1988), 809–812.
- [3] H. Sadraoui, *Hyponormality of Toeplitz operators and composition operators*, Ph.D. thesis, Purdue University, 1992.
- [4] P. Ahern and Z. Cuckovic, *A mean value inequality with applications to Bergman space operators*, *Pacific J. Math.* **173** (1996), no. 2, 295–305.
- [5] Z. Cuckovic and R. Curto, *A new necessary condition for the hyponormality of Toeplitz operators on the Bergman space*, *J. Operator Theory* **79** (2018), no. 2, 287–300.
- [6] I. S. Hwang, *Hyponormality of Toeplitz operators on the Bergman space*, *J. Korean. Math. Soc.* **45** (2008), no. 4, 1027–1041, DOI: <https://doi.org/10.4134/JKMS.2008.45.4.1027>.
- [7] H. Sadraoui and M. Guediri, *Hyponormal Toeplitz operators on the Bergman space*, *Oper. Matrices* **11** (2017), no. 3, 669–677, DOI: <https://dx.doi.org/10.7153/oam-11-44>.
- [8] H. Sadraoui and B. Halouani, *Hyponormality on an annulus with a weight*, *Math. Methods Appl. Sci.* **2020** (2020), DOI: <https://doi.org/10.1002/mma.6640>.
- [9] U. Grenander and G. Szegő, *Toeplitz Forms and Their Applications*, University of California Press, Berkeley and Los Angeles, 1958.