Research Article

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Structure of coincidence isometry groups

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Abstract: Let L be a lattice of rank n in an n -dimensional Euclidean space. We show that the coincidence isometry group of L is generated by coincidence reflections if and only if L contains an orthogonal subset of order n .

Keywords: coincidence isometry group, lattice, orthogonal subset, multiplicative noise

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1 Introduction

The theory of coincidence site lattice (CSL) gives partial answers to some questions that appeared in the physics of interfaces and grain boundaries, see [1,2]. The CSL theory mainly studies the coincidence problem between two lattices in a finite-dimensional Euclidean space. Several mathematical approaches were used in the study of this problem, including matrix theory, number theory, and geometric algebra, see [3–12]. In this paper, we focus on the structure of the coincidence isometry group of a lattice in a finite-dimensional Euclidean space.

Let V be an n -dimensional Euclidean space and let $\alpha_1, \dots, \alpha_n$ be a basis of V . Let $L = \oplus_{i=1}^n \mathbb{Z}\alpha_i$ be the lattice generated by $\alpha_1, \dots, \alpha_n$. A linear isometry $\mathcal{A}$ of V is called a coincidence isometry of L if $L \cap \mathcal{A}(L)$ is a sublattice of L with finite index. The coincidence isometry group of L was introduced by Baake [1], which consists of all coincidence isometries of L . It is clear that an isometry $\mathcal{A}$ is a coincidence isometry of L if and only if the matrix of $\mathcal{A}$ with respect to $\alpha_1, \dots, \alpha_n$ is a rational matrix. It follows that the set of all coincidence isometries of L is a group which we denote by $OC(L)$.

Aragón et al. [3] and Rodríguez et al. [13] used geometric algebra as a tool in the study of the coincidence isometry group. In particular, the coincidence problem was completely solved in the planar case. They also found that coincidence reflections play an important role and conjectured that if $L = \mathbb{Z}^n$ is the lattice spanned by the canonical basis in $\mathbb{R}^n$, then any coincidence isometry is a product of coincidence reflections. Zou in [14] showed that if the reflection defined by an arbitrary nonzero vector of L is a coincidence isometry of L , then any coincidence isometry of L is a product of coincidence reflections defined by the vectors of L . This result includes the conjecture as a special case and an algorithm to decompose a coincidence isometry into coincidence reflections was obtained. Zou also gave example to show that this result is not valid for any lattice. Huck [15] generalized this result to free modules over some subrings of $\mathbb{R}$. In this paper, we show that for any lattice L of rank n in $\mathbb{R}^n$, any coincidence isometry of L is a product of coincidence reflections defined by the vectors of L if and only if L contains an orthogonal subset of order n .

In Section 2, we recall some relevant definitions and known results. In Section 3, we obtain many interesting properties of the coincidence reflections of a lattice. The main result is proved in Section 4.

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We also give a rough classification of coincidence isometry groups in dimension two of three. Section 5 is a concluding section.

2 Preliminaries

The set of integers, rational numbers and real numbers are denoted by $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$, respectively. For a ring R , the set of all $m \times n$ matrices over R is denoted by $M_{m \times n}(R)$. Let $O(V)$ denote the group of orthogonal transformations of an Euclidean space V .

We recall some notations in linear algebra and the definition of lattice.

Definition 2.1. Let V be an n -dimensional Euclidean space.

- (1) Two vectors $\alpha, \beta \in V$ are orthogonal, written $\alpha \perp \beta$, if $(\alpha, \beta) = 0$;
- (2) Two subsets $X, Y \subset V$ are orthogonal, written $X \perp Y$, if $(\alpha, \beta) = 0$ for all $\alpha \in X$ and $\beta \in Y$. We write $\alpha \perp Y$ in place of $\{\alpha\} \perp Y$;
- (3) The orthogonal complement of a subset X is the set

$$X^\perp := \{\beta \in V : \beta \perp X\}.$$

We also write $\alpha^\perp$ instead of $\{\alpha\}^\perp$;

- (4) A nonempty subset X of V is called an orthogonal set if $0 \notin V$ and $(\alpha, \beta) = 0$ for pairwise distinct vectors in X .

Since any n -dimensional Euclidean space is isomorphic to $\mathbb{R}^n$ with the standard inner product, we only consider lattices in $\mathbb{R}^n$ and we write vectors in $\mathbb{R}^n$ as column vectors.

Definition 2.2. A discrete subset L of $\mathbb{R}^n$ is called a lattice of dimension d if it is a free abelian group spanned by a linearly independent set $\{\alpha_1, \dots, \alpha_d\}$ of $\mathbb{R}^n$. We always write it in the form $\oplus_{i=1}^d \mathbb{Z}\alpha_i$. A sublattice L' of L is also a free abelian group. Let $[L : L']$ denote the index of L' in L . Two lattices L_1, L_2 of $\mathbb{R}^n$ are called commensurate if both the indices $[L_1 : L_1 \cap L_2]$ and $[L_2 : L_1 \cap L_2]$ are finite.

Here we do not require that $[L : L']$ is finite for a sublattice L' of L . This is because we will study the relationship between the group of coincidence isometries of a lattice L and the group of coincidence isometries of a sublattice L' of L having lower rank. Now we introduce the coincidence isometry group of a lattice.

Definition 2.3. Let L be a lattice of $\mathbb{R}^n$ and let V be the Euclidean subspace of $\mathbb{R}^n$ spanned by the elements in L . An orthogonal transformation $\mathcal{A}$ of V is called a coincidence isometry of L if $[L : L \cap \mathcal{A}L] < \infty$. The set of all coincidence isometries of L is a subgroup of $O(V)$ and is denoted by $OC(L)$.

Remark 2.1. Each subspace V of $\mathbb{R}^n$ has a unique orthogonal complement $V^\perp = \{\alpha \in \mathbb{R}^n : \alpha \perp V\}$ such that $\mathbb{R}^n = V \oplus V^\perp$. There exists a canonical way to extend an isometry $\mathcal{A}$ of V to an isometry of $\mathbb{R}^n$:

$$\mathcal{A}(\alpha + \beta) = \mathcal{A}(\alpha) + \beta, \quad \alpha \in V, \quad \beta \in V^\perp.$$

Hence, the coincidence isometry group $OC(L)$ of a lattice in V can be viewed as a subgroup of $O(\mathbb{R}^n)$.

Theorem 2.1. [16, Theorem 1] Let $L_1 = \oplus_{i=1}^n \mathbb{Z}\alpha_i$ and $L_2 = \oplus_{i=1}^n \mathbb{Z}\beta_i$ be two lattices in $\mathbb{R}^n$. Then the following are equivalent.

- (1) L_1 and L_2 are commensurate;
- (2) The transition matrix from $\alpha_1, \dots, \alpha_n$ to $\beta_1, \dots, \beta_n$ is a rational matrix;
- (3) $\oplus_{i=1}^n \mathbb{Q}\alpha_i = \oplus_{i=1}^n \mathbb{Q}\beta_i$.

Lemma 2.1. Let $L_1 = \oplus_{i=1}^n \mathbb{Z}\alpha_i$, $L_2 = \oplus_{i=1}^n \mathbb{Z}\beta_i$ be two lattices of $\mathbb{R}^n$. If $(\beta_1, \dots, \beta_n) = (\alpha_1, \dots, \alpha_n)P$ and P is a non-singular rational matrix, then $OC(L_1) = OC(L_2)$.

Proof. It follows immediately from Theorem 2.1 and Lemma 7.2 of [5]. $\square$

By Grimmer's theorem, we see that the coincidence isometry group of lattice with basis $\alpha_1, \dots, \alpha_n$ consists precisely of the isometries whose matrices with respect to the basis $\alpha_1, \dots, \alpha_n$ are rational matrices.

Lemma 2.2. Suppose that a lattice L of $\mathbb{R}^n$ is the orthogonal direct sum of sublattices L_1 and L_2 . Then $OC(L_1) \oplus OC(L_2)$ is isomorphic to a subgroup of $OC(L)$.

Proof. Without loss of generality, we may assume that L is n -dimensional, $L_1 = \oplus_{i=1}^r \mathbb{Z}\alpha_i$ and $L_2 = \oplus_{i=r+1}^n \mathbb{Z}\alpha_i$. Let $V_1 = \oplus_{i=1}^r \mathbb{R}\alpha_i$ and $V_2 = \oplus_{i=r+1}^n \mathbb{R}\alpha_i$ be the respective $\mathbb{R}$ -subspaces of $\mathbb{R}^n$ spanned by the basis vectors of L_1 and L_2 . It is routine to check that the following map

$$(\mathcal{A}_1, \mathcal{A}_2) \mapsto \mathcal{A}, \quad \mathcal{A}(\alpha_1 + \alpha_2) = \mathcal{A}_1(\alpha_1) + \mathcal{A}_2(\alpha_2)$$

is an injective group homomorphism from $OC(L_1) \oplus OC(L_2)$ to $O(\mathbb{R}^n)$. It remains to show that $\mathcal{A} \in OC(L)$. Let A_1 be the matrix of $\mathcal{A}_1$ (viewed as an element of $O(V_1)$) with respect to the basis $\alpha_1, \dots, \alpha_r$ and A_2 be the matrix of $\mathcal{A}_2$ (viewed as an element of $O(V_2)$) with respect to the basis $\alpha_{r+1}, \dots, \alpha_n$. Then the matrix of $\mathcal{A}$ with respect to the basis $\alpha_1, \dots, \alpha_n$ is the quasi-diagonal matrix $\text{diag}(A_1, A_2)$. Since $\mathcal{A}_i$ is a coincidence isometry, A_i is a rational matrix and is also A . Hence, $\mathcal{A}$ is a coincidence isometry of L . $\square$

We need the following result proved by Zou.

Theorem 2.2. [14, Theorem 3.1 and Theorem 3.2] Let $L = \oplus_{i=1}^n \mathbb{Z}\alpha_i$ be a lattice of $\mathbb{R}^n$. Then every non-zero vector of L defines a coincidence reflection of L if and only if $\frac{(\alpha_i, \alpha_j)}{(\alpha_k, \alpha_k)}$ are all rational for all $1 \leq i, j, k \leq n$. Moreover, in this case any coincidence isometry of L is a product of at most n reflections defined by the vectors in L .

3 Reflections in the coincidence isometry group

The well-known Cartan-Dieudonné theorem states that any isometry in $O(\mathbb{R}^n)$ is the product of at most n reflections. It was found that the coincidence reflections of a lattice L play an important role in the structure of $OC(L)$, see [10] and [14]. In this section, we obtain many interesting properties of a coincidence reflection in the coincidence isometry group. For any nonzero vector $v \in V$, let $\mathcal{R}_v$ denote the reflection defined by v , i.e. $\mathcal{R}_v(\alpha) = \alpha - \frac{2(v, \alpha)}{(v, v)}v$.

Lemma 3.1. Let $L = \oplus_{i=1}^n \mathbb{Z}\alpha_i$ be a lattice of $\mathbb{R}^n$. Suppose that $\mathcal{R}_v \in OC(L)$. Then there exists a linearly independent subset $\{\beta_1, \dots, \beta_n\}$ of L such that β_1 is parallel to v and $(\beta_1, \beta_i) = 0$ for $i > 1$.

Proof. We put the proof in the appendix. $\square$

Corollary 3.1. Let $L = \oplus_{i=1}^n \mathbb{Z}\alpha_i$ be a lattice of $\mathbb{R}^n$ and let $S = \{v \in \oplus_{i=1}^n \mathbb{Q}\alpha_i : v \neq 0, \mathcal{R}_v \in OC(L)\}$. Then S is an invariant set of any $\mathcal{A} \in OC(L)$, that is, $\mathcal{A}(v) \in S$ for any $\mathcal{A} \in OC(L)$ and $v \in S$.

Proof. By Lemma 3.1, there exists a linearly independent subset $\{\beta_1, \dots, \beta_n\}$ of L such that β_1 is parallel to $v \in S$ and $(\beta_1, \beta_i) = 0$ for $i > 1$. It suffices to show that $\mathcal{A}(\beta_1) \in S$. In fact, let $\gamma_i = \mathcal{A}(\beta_i)$, $L_1 = \oplus_{i=1}^n \mathbb{Z}\beta_i$ and $L_2 = \oplus_{i=1}^n \mathbb{Z}\gamma_i$. Since $\mathcal{A}$ is an isometry, one has $(\gamma_i, \gamma_i) = 0$ for $i > 1$. It follows that $\mathcal{R}_{\gamma_1}(\gamma_1) = -\gamma_1$ and $\mathcal{R}_{\gamma_1}(\gamma_i) = \gamma_i$

for $i > 1$. Hence, $\mathcal{R}_{v_1}$ is a coincidence reflection of L_2 . Since L and L_2 are commensurate, we have $\mathcal{R}_{v_1} \in OC(L_2) = OC(L)$ by Lemma 2.1. This finishes the proof. $\square$

Lemma 3.2. Let $L = \oplus_{i=1}^n \mathbb{Z}\alpha_i$ be a lattice of $\mathbb{R}^n$. Let $V_1 = \oplus_{i=1}^r \mathbb{R}\alpha_i$ and $V_2 = \oplus_{i=r+1}^n \mathbb{R}\alpha_i$. Suppose that $V_1 \perp V_2$. Let $v = v_1 + v_2$, $0 \neq v_i \in V_i$. If $\mathcal{R}_v \in OC(L)$, then $\mathcal{R}_{v_i} \in OC(L)$.

Proof. We put the proof in the appendix. $\square$

Lemma 3.3. Let $L = \oplus_{i=1}^n \mathbb{Z}\alpha_i$ be a lattice of $\mathbb{R}^n$ and let $V = \oplus_{i=1}^n \mathbb{Q}\alpha_i$. Suppose that $\{v_1, \dots, v_r\}$ is a linearly independent subset of V such that $\mathcal{R}_{v_i} \in OC(L)$. Applying the Gram-Schmidt orthogonalization process to $v_1, \dots, v_r$ gives that

$$w_1 = v_1, \quad w_i = v_i - \sum_{j=1}^{i-1} \frac{(v_i, w_j)}{(w_j, w_j)} w_j, \quad \text{for } i = 2, \dots, r.$$

Then $w_i \in V$ and $\mathcal{R}_{w_i} \in OC(L)$ for $i = 1, \dots, r$.

Proof. We put the proof in the appendix. $\square$

The following theorem is part of the main result.

Theorem 3.1. Let $L = \oplus_{i=1}^n \mathbb{Z}\alpha_i$ be a lattice of $\mathbb{R}^n$ and let r be dimension of the $\mathbb{Q}$ -space spanned by $\{v \in L : \mathcal{R}_v \in OC(L)\}$. Then

- (1) $r \neq n - 1$.
- (2) If $1 \leq r \leq n - 2$, then there exist two orthogonal sublattices L_1 of dimension r and L_2 of dimension $n - r$ such that $OC(L) = OC(L_1) \oplus OC(L_2)$, and $OC(L)$ is not generated by the reflections.

Proof. Let $V = \oplus_{i=1}^n \mathbb{Q}\alpha_i$ and V_1 be the $\mathbb{Q}$ -subspace of V spanned by $\{v \in L : \mathcal{R}_v \in OC(L)\}$. Suppose that $w_1, \dots, w_r$ is a basis of V_1 . By Lemmas 3.3 and 3.1, we may assume that $\{w_1, \dots, w_r\}$ is orthogonal subset of L . By Lemma 3.1 again, $W_j = \{\alpha \in V : (\alpha, w_j) = 0\}$ is an $(n - 1)$ -dimensional subspace of V . Let $V_2 = \cap_{j=1}^r W_j$. Then V_2 is the orthogonal complement of V_1 in the $\mathbb{Q}$ -space V and $V = V_1 \oplus V_2$. Choosing a basis $\{w_{r+1}, \dots, w_n\} \subset L$ of V_2 . Then $L_1 = \oplus_{i=1}^r \mathbb{Z}w_i$ and $L_2 = \oplus_{i=r+1}^n \mathbb{Z}w_i$ are orthogonal sublattices of L . Since $\{w_1, \dots, w_n\}$ is a linearly independent subset of $V = \oplus_{i=1}^n \mathbb{Q}\alpha_i$, we have $V = \oplus_{i=1}^n \mathbb{Q}w_i$ and L is commensurate to $L' = L_1 \oplus L_2$ by Theorem 2.1. So we have $OC(L) = OC(L')$ by Lemma 2.1.

If $r = n - 1$, then $\{w_1, \dots, w_{n-1}, w_n\}$ is an orthogonal basis of L' . Thus, $\mathcal{R}_{w_n} \in OC(L') = OC(L)$, which is a contradiction. This proves (1).

Let $\mathcal{A} \in OC(L)$. By Corollary 3.1, the reflection defined by $\mathcal{A}(w_j)$ is also a coincidence isometry of L for $j = 1, \dots, r$. It follows that $\mathcal{A}(w_j) \in V_1$ and V_1 is an $\mathcal{A}$ -invariant subspace of V . Since $V_1 \perp V_2$ and $\mathcal{A}$ is an isometry, V_2 is also an $\mathcal{A}$ -invariant subspace of V . It follows that the group homomorphism defined in Lemma 2.2 is an isomorphism in this case.

If $\mathcal{R}_v$ is a coincidence isometry of L defined by a vector $v \in L$, then $v \in V_1$ by hypothesis. Since $V_1 \perp V_2$, it follows that $\mathcal{R}_v|_{V_2} = I$, and is also for any coincidence isometry which is a product of reflections. Hence, $-I$ is not contained in the subgroup of $OC(L)$ generated by reflections. $\square$

Corollary 3.2. Let L be an n -dimensional lattice of $\mathbb{R}^n$. Suppose that $0 \notin X \subset L$ and $OC(L)$ is generated by the reflections defined by the vectors of X . Then $|X| \geq n$.

Proof. Suppose to the contrary that $|X| < n$. Let $\{v_1, \dots, v_r\}$ be a maximal linearly independent subset of X , $r \leq |X| < n$. Since $OC(L)$ is generated by reflections, by Theorem 3.1 there exist vectors $v_{r+1}, \dots, v_n \in \{v \in L : \mathcal{R}_v \in OC(L)\}$ such that $\{v_1, \dots, v_n\}$ is linearly independent. By Lemma 3.3, applying the Gram-Schmidt orthogonalization process to $v_1, \dots, v_n$, we obtain an orthogonal set $\{w_1, \dots, w_n\}$ such that $\mathcal{R}_{w_i} \in OC(L)$. Since $X \subset \oplus_{i=1}^r \mathbb{Q}v_i = \oplus_{i=1}^r \mathbb{Q}w_i$, so $w_n \perp X$. It follows that $\mathcal{R}_v(w_n) = w_n$ for any $v \in X$, and is also for any coincidence isometry of L . This contradicts $\mathcal{R}_{w_n} \in OC(L)$. $\square$

4 Main results

In this section, we present our main result. We first give a criterion so that a reflection is a coincidence isometry.

Lemma 4.1. *Let $\alpha \in \mathbb{R}^n$ be a nonzero vector. Then $\alpha^\perp$ has a basis consisting of vectors in $\mathbb{Q}^n$ if and only if α is parallel to a nonzero vector of $\mathbb{Q}^n$.*

Proof. We put the proof in the appendix. $\square$

Theorem 4.1. *Let $L = \oplus_{i=1}^n \mathbb{Z}\alpha_i$ be a lattice in $\mathbb{R}^n$ and let $G = ((\alpha_i, \alpha_j))$ be the Gram matrix of $\alpha_1, \dots, \alpha_n$. Let $v = \sum_{i=1}^n k_i \alpha_i$ be a non-zero vector of L . Then $\mathcal{R}_v \in OC(L)$ if and only if $G(k_1, \dots, k_n)^T$ is parallel to a nonzero vector in $\mathbb{Q}^n$.*

Proof. Let $B = (k_1, \dots, k_n)^T$. First assume that $\mathcal{R}_v \in OC(L)$. By Lemma 3.1, there exists a linearly independent subset $\{\beta_1, \dots, \beta_n\}$ of $\oplus_{i=1}^n \mathbb{Q}\alpha_i$ such that $\beta_1 = v$ and $(\beta_1, \beta_i) = 0$ for $i > 1$. Suppose $\beta_i = \sum_{j=1}^n b_{ij} \alpha_j$. By the property of Gram matrix, we have for $i > 1$,

$$B_i G B = (\beta_i, \beta_1) = 0,$$

where $B_i = (b_{i1}, \dots, b_{in})$. By Lemma 4.1, GB is parallel to a nonzero vector in $\mathbb{Q}^n$.

Now suppose that $GB = rC$ and $C \in \mathbb{Q}^n$. By Lemma 4.1, there exist $n-1$ linearly independent row vectors $C_1 = (a_{11}, \dots, a_{1n}), \dots, C_{n-1} = (a_{n-1,1}, \dots, a_{n-1,n})$ such that $C_i \in \mathbb{Q}^n$ and $C_i G B = 0$. Set $v_i = \sum_{j=1}^n a_{ij} \alpha_j \in \oplus_{i=1}^n \mathbb{Q}\alpha_i$. Then $\{v_1, \dots, v_{n-1}\}$ is also linearly independent and

$$(v_i, v) = C_i G B = 0$$

by the property of Gram matrix again. We observe that $v_1, \dots, v_{n-1}, v_n = v$ is a basis of $\oplus_{i=1}^n \mathbb{Q}\alpha_i$, and $\mathcal{R}_v \in OC(\oplus_{i=1}^n \mathbb{Z}v_i) = OC(L)$ by Lemma 2.1. $\square$

The following result is an immediate consequence of Theorem 4.1.

Corollary 4.1. *Let $L = \oplus_{i=1}^n \mathbb{Z}\alpha_i$ be a lattice in $\mathbb{R}^n$ with an orthogonal basis $\alpha_1, \dots, \alpha_n$. Suppose that $v = \sum_{i=1}^n k_i \alpha_i \in L$ is a nonzero vector and $S = \{i : k_i \neq 0\}$. Then $\mathcal{R}_v \in OC(L)$ if and only if $\frac{(\alpha_i, \alpha_i)}{(\alpha_j, \alpha_j)} \in \mathbb{Q}$ for any $i, j \in S$.*

We define a relation $\sim_{\mathbb{Q}}$ on the set of nonzero vectors in $\mathbb{R}^n$ such that $\alpha \sim_{\mathbb{Q}} \beta$ if $\frac{(\alpha, \alpha)}{(\beta, \beta)} \in \mathbb{Q}$. This is clearly an equivalence relation. Now we state and prove our main result.

Theorem 4.2. *Let $L = \oplus_{i=1}^n \mathbb{Z}\alpha_i$ be a lattice in $\mathbb{R}^n$ and let $L_{\mathcal{R}}$ be the sublattice of L spanned by a maximal orthogonal subset of $\{v \in L : \mathcal{R}_v \in OC(L)\}$. Then the following are equivalent.*

- (1) L contains an orthogonal subset of order n ;
- (2) $\dim L_{\mathcal{R}} = n$;
- (3) Any coincidence isometry of L can be decomposed as a product of at most n reflections defined by the vectors in L .

Proof. (1) $\Rightarrow$ (2) Suppose that $\{w_1, \dots, w_n\}$ is an orthogonal subset of L . Then $\{w_1, \dots, w_n\}$ is linearly independent and L is commensurate to $L' = \oplus_{i=1}^n \mathbb{Z}w_i$ by Theorem 2.1. Hence, $\mathcal{R}_{w_i} \in OC(L') = OC(L)$ by Lemma 2.1 and we have $\dim L_{\mathcal{R}} = n$.

(2) $\Rightarrow$ (1) is trivial.

(3) $\Rightarrow$ (1) follows from Theorem 3.1 and Lemma 3.3.

(1) $\Rightarrow$ (3) Without loss of generality we may assume that $\alpha_1, \dots, \alpha_n$ is an orthogonal basis of L . Let $V = \oplus_{i=1}^n \mathbb{Q}\alpha_i$. Suppose that $\{\alpha_1, \dots, \alpha_n\} = \cup_{i=1}^r X_i$ be the partition corresponding to the equivalence relation $\sim_{\mathbb{Q}}$

restricted on $\{\alpha_1, \dots, \alpha_n\}$. Let $L_i = \oplus_{\gamma \in X_i} \mathbb{Z}\gamma$, $V_i = \oplus_{\gamma \in X_i} \mathbb{Q}\gamma$ and $W_i = \oplus_{\gamma \in X_i} \mathbb{R}\gamma$ be the sublattice of L , the $\mathbb{Q}$ -subspace of V and the $\mathbb{R}$ -subspace of $\mathbb{R}^n$ generated by the elements of X_i , respectively. Then $L = \oplus_{i=1}^r L_i$. Let $\alpha, \beta \in V$ be two nonzero vectors.

We claim that:

- (1) If $\alpha, \beta \in V_i$ for some $i \in \{1, \dots, r\}$, then $\frac{(\alpha, \alpha)}{(\beta, \beta)} \in \mathbb{Q}$;
- (2) If $\alpha \in V_i, \beta \in V_j$ and $i \neq j$, then $\frac{(\alpha, \alpha)}{(\beta, \beta)} \notin \mathbb{Q}$.

Suppose $\alpha = \sum_{\alpha_k \in X_i} l_k \alpha_k \in V_i$ and $\beta = \sum_{\alpha_k \in X_j} m_k \alpha_k \in V_j$, where $l_k, m_k \in \mathbb{Q}$. Fix $\alpha' \in X_i$ and $\beta' \in X_j$. If $i = j$, then

$$\frac{(\alpha, \alpha)}{(\beta, \beta)} = \frac{\sum_{\alpha_k \in X_i} l_k^2 (\alpha_k, \alpha_k)}{\sum_{\alpha_k \in X_i} m_k^2 (\alpha_k, \alpha_k)} = \frac{\sum l_k^2 s_k}{\sum m_k^2 s_k} \in \mathbb{Q},$$

where $s_k = \frac{(\alpha_k, \alpha_k)}{(\alpha', \alpha')} \in \mathbb{Q}$. This proves claim (1). If $i \neq j$, then $\alpha' \neq_Q \beta'$ by the construction of the partition of $\{\alpha_1, \dots, \alpha_n\} = \bigcup_{i=1}^r X_i$. By claim (1), we have $\alpha \sim_Q \alpha'$ and $\beta \sim_Q \beta'$. Since $\sim_Q$ is an equivalence relation, one has $\alpha \neq_Q \beta$. This proves claim (2).

By Corollary 4.1, for any nonzero vector $w \in L$, $\mathcal{R}_w \in OC(L)$ if and only if $w \in L_i$ for some i . Now choose a nonzero vector $v \in L_i$ and $\mathcal{A} \in OC(L)$. By Corollary 3.1, there exists an index j such that $\mathcal{A}(v) \in V_j$. Since $\mathcal{A}$ is an isometry, one has $(v, v) = (\mathcal{A}(v), \mathcal{A}(v))$. By claim (2), we have $i = j$. This shows that $\mathcal{A}(V_i) = V_i$, $\mathcal{A}(W_i) = W_i$ and $\mathcal{A}|_{W_i}$ is a coincidence isometry of the lattice L_i considered in the subspace W_i . It follows that

$$OC(L) \simeq OC(L_1) \oplus OC(L_2) \oplus \dots \oplus OC(L_r),$$

and $\mathcal{A} = \mathcal{A}_1 \mathcal{A}_2 \dots \mathcal{A}_r$, where $\mathcal{A}_i|_{W_i} = \mathcal{A}|_{W_i}$ and $\mathcal{A}_i|_{W_j} = I$ for $i \neq j$.

By Theorem 2.2, each $\mathcal{A}_i$ is a product of at most $|X_i|$ reflections defined by the vectors in L_i , and hence $\mathcal{A}$ is a product of at most $\sum_{i=1}^r |X_i| = n$ reflections defined by the vectors in L . The proof is complete. $\square$

Corollary 4.2. *With the same notations in the above theorem. The subgroup of $OC(L)$ generated by coincidence reflections is isomorphic to $OC(L_{\mathcal{R}})$.*

Proof. It follows immediately from Theorems 4.2 and 2.2. $\square$

Remark 4.1. Now we compare our main result and Theorem 2.2 by Zou. Let L be a lattice of rank n in $\mathbb{R}^n$. Set

- (1) $\frac{(\alpha, \alpha)}{(\beta, \beta)} \in \mathbb{Q}$ for any nonzero vectors $\alpha, \beta \in L$;
- (2) Every nonzero vector of L defines a coincidence reflection of L ;
- (3) Any coincidence isometry of L is a product of at most n reflections defined by the vectors in L ;
- (4) L contains an orthogonal subset of order n .

Then Theorem 2.2 states that

$$(1) \Leftrightarrow (2) \Rightarrow (3),$$

and Theorem 4.2 states that

$$(3) \Leftrightarrow (4) \not\Leftrightarrow (2).$$

Let $L = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2$ be the lattice, where $\alpha_1 = (a, 0)^T$, $\alpha_2 = (1, b)^T$. Zou [14] showed that one of the following is satisfied:

- (i) If $a, b^2 \in \mathbb{Q}$, then every non-zero vector of L defines a coincidence reflection of L and $OC(L)$ is generated by these reflections;
- (ii) If $a \in \mathbb{Q}$ but $b^2 \notin \mathbb{Q}$, then $OC(L) = \{\pm I, \pm \mathcal{R}_{\alpha_1}\} \simeq \mathbb{Z}_2^2$;
- (iii) If $a \notin \mathbb{Q}$ but $\frac{a}{1+b^2} \in \mathbb{Q}$, then $OC(L) = \{\pm I, \pm \mathcal{R}_{\alpha_2}\} \simeq \mathbb{Z}_2^2$;
- (iv) If $a, \frac{a}{1+b^2} \notin \mathbb{Q}$, then $OC(L) = \{\pm I\} \simeq \mathbb{Z}_2$.

This example also shows that (3) $\nRightarrow$ (2) in Remark 4.1. It should be mentioned that the coincidence problem was completely solved in the planar case by using Clifford Algebra, see [13]. We give a rough classification of the coincidence isometry groups in two and three dimensional based on our main result. We also put the proof of the following two examples in the appendix.

Example 4.1. Let $L = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2$ be a lattice in $\mathbb{R}^2$. Then one of the following is satisfied.

- (1) There are no reflections in $OC(L)$ and $OC(L) = \{\pm I\} \simeq \mathbb{Z}_2$;
- (2) There are exactly two reflections $\mathcal{R}_{v_1}, \mathcal{R}_{v_2}$ in $OC(L)$ and $OC(L) = \{\pm I, \mathcal{R}_{v_1}, \mathcal{R}_{v_2}\} \simeq \mathbb{Z}_2^2$;
- (3) Each vector in L defines a coincidence reflection of L and $OC(L)$ is generated by coincidence reflections.

Example 4.2. Let $L = \oplus_{i=1}^3 \mathbb{Z}\alpha_i$ be a lattice in $\mathbb{R}^3$. Suppose that X is a maximal orthogonal subset of $\{v \in L : \mathcal{R}_v \in OC(L)\}$. Then exactly one of the following is satisfied.

- (1) $X = \emptyset$ and $OC(L) = \{\pm I\} \simeq \mathbb{Z}_2$;
- (2) $X = \{v\}$ and $OC(L) = \{\pm I, \pm \mathcal{R}_v\} \simeq \mathbb{Z}_2^2$;
- (3) $X = \{v_1, v_2, v_3\}$ and one of the following holds:
 - (3.1) Neither $\frac{(v_1, v_1)}{(v_2, v_2)}$ nor $\frac{(v_2, v_2)}{(v_3, v_3)}$ is rational, $OC(L) = \{\pm I, \pm \mathcal{R}_{v_1}, \pm \mathcal{R}_{v_2}, \pm \mathcal{R}_{v_3}\} \simeq \mathbb{Z}_2^3$;
 - (3.2) $\frac{(v_1, v_1)}{(v_2, v_2)} \in \mathbb{Q}$ and $\frac{(v_2, v_2)}{(v_3, v_3)} \notin \mathbb{Q}$, $OC(L) \simeq OC(\mathbb{Z}v_1 \oplus \mathbb{Z}v_2) \oplus \mathbb{Z}_2$, where every nonzero vector of $L' = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$ defines a coincidence reflection of L' ;
 - (3.3) Both $\frac{(v_1, v_1)}{(v_2, v_2)}$ and $\frac{(v_2, v_2)}{(v_3, v_3)}$ are rational, and every nonzero vector of L defines a coincidence reflection of L .

5 Concluding remark

In this work, we focus on the structure of coincidence isometry groups of lattices in $\mathbb{R}^n$. We obtain a necessary and sufficient condition for an arbitrary coincidence isometry of certain lattices L to be a product of at most n coincidence reflections defined by the vectors of L . In particular, we show that if L has an orthogonal basis, then there exists an orthogonal decomposition of $L = \oplus_{i=1}^r L_i$ into its sublattices such that $OC(L) \simeq \oplus_{i=1}^r OC(L_i)$, where $\mathcal{R}_v \in OC(L)$ if and only if $v \in L_i$ for some i .

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References

- [1] M. Baake, M. Birkner, and U. Grimm, *Non-Periodic Systems with Continuous Diffraction Measures*, in: The Mathematics of Aperiodic Order, R. V. Moody, (ed.), 9–44, Kluwer Academic Publishers, Dordrecht, 1997.
- [2] W. Bollmann, *Crystal Defects and Crystalline Interfaces*, Berlin: Springer, 1970.
- [3] G. Aragón, J. L. Aragón, F. Dávila, Z. Gómez, and M. A. Rodríguez, *Modern geometric calculations in crystallography*, in: E. Bayro Corrochano, G. Sobczyk (eds.), Geometric Algebra with Applications in Science and Engineering, Birkhäuser, Boston, 2001, pp. 371–386.

- [4] M. Baake, U. Grimm, M. Heuer, and P. Zeiner, *Coincidence rotations of the root lattice A_4* , European J. Combin. **29** (2008), no. 8, 1808–1819, DOI: <https://doi.org/10.1016/j.ejc.2008.01.012>.
- [5] M. Baake and P. Zeiner, *Geometric enumeration problems for lattices and embedded $\mathbb{Z}$ -modules*, in: M. Baake and U. Grimm (eds.), *Aperiodic Order* (Encyclopedia of Mathematics and its Applications), Cambridge University Press, Cambridge, 2017, pp. 73–172, DOI: <https://doi.org/10.1017/9781139033862.005>.
- [6] M. A. Fortes, *Crystallographic applications of the elementary divisor theorem*, Acta Cryst. **39** (1983), 348–350, DOI: <https://doi.org/10.1107/S0108767383000781>.
- [7] H. Grimmer, *The generating function for coincidence site lattices in the cubic system*, Acta Cryst. **A40** (1984), 108–112, DOI: <https://doi.org/10.1107/S0108767384000246>.
- [8] P. A. B. Pleasants, M. Baake, and J. Roth, *Planar coincidence for N -fold symmetry*, J. Math. Phys. **37** (1996), 1029–1058.
- [9] S. Ranganathan, *On the geometry of coincidence-site lattices*, Acta Cryst. **21** (1966), 197–199, DOI: <https://doi.org/10.1107/S0365110X66002615>.
- [10] M. A. Rodríguez-Andrade, G. Aragón-González, J. L. Aragón, A. Gómez-Rodríguez, and D. Romeu, *The coincidence site Lattices in 2D hexagonal lattices using Clifford algebra*, Adv. Appl. Clifford Algebr. **25** (2015), 425–440, DOI: <https://doi.org/10.1007/s00006-014-0508-7>.
- [11] P. Zeiner, *Multiplicativity in the theory of coincidence site lattices*, J. Phys.: Conf. Ser. **226** (2010), 012025, DOI: <https://doi.org/10.1088/1742-6596/226/1/012025>.
- [12] P. Zeiner, *Similar submodules and coincidence site modules*, Acta Phys. Polon. A **126** (2014), no. 2, 641–644, DOI: <http://dx.doi.org/10.12693/APhysPolA.126.641>.
- [13] M. A. Rodríguez, J. L. Aragón, and L. Verde-Star, *Clifford algebra approach to the coincidence problem for planar lattices*, Acta Cryst. **A61** (2005), 173–184, DOI: <https://doi.org/10.1107/S0108767304025358>.
- [14] Y. Zou, *Structure of coincidence symmetry groups*, Acta Cryst. **A62** (2006), 109–114, DOI: <https://doi.org/10.1107/S0108767305033489>.
- [15] C. Huck, *A note on coincidence isometries of modules in Euclidean space*, Zeitschrift für Kristallographie **224** (2009), no. 7, 341–344, DOI: <https://doi.org/10.1524/zkri.2009.1148>.
- [16] H. Grimmer, *Coincidence-site lattices*, Acta Cryst. **A32** (1976), 783–785, DOI: <https://doi.org/10.1107/S056773947601231X>.
- [17] M. Artin, *Algebra*, Prentice Hall, New Jersey, 1991.
- [18] S. Roman, *Advanced Linear Algebra*, Third Edition, Springer-Verlag, New York, 2008.

Appendix

A.1 Proof of Lemma 3.1

Let $v = \sum_{i=1}^n x_i \alpha_i$, $x_i \in \mathbb{R}$. Since $\mathcal{R}_v(\alpha_j) = \alpha_j - \frac{2(\alpha_j, v)}{(v, v)} \sum_{i=1}^n x_i \alpha_i$, it follows that $\frac{x_i(\alpha_j, v)}{(v, v)} \in \mathbb{Q}$ for any $i, j \in \{1, \dots, n\}$. Without loss of generality, we may assume that $x_i(\alpha_1, v) \neq 0$. It follows that for $j \neq i$

$$\frac{x_j}{x_i} = \frac{x_j(\alpha_1, v)}{(v, v)} \cdot \frac{(v, v)}{x_i(\alpha_1, v)} \in \mathbb{Q}.$$

Choose $m \in \mathbb{Z}$ such that $\frac{mx_j}{x_i} \in \mathbb{Z}$ for all $j \in \{1, \dots, n\}$, then $\beta_1 = \frac{m}{x_i} v \in L$.

We also have

$$\frac{(\alpha_j, v)}{(\alpha_1, v)} = \frac{x_j(\alpha_j, v)}{(v, v)} \cdot \frac{(v, v)}{x_i(\alpha_1, v)} = q_j \in \mathbb{Q},$$

and thus $\beta_1 \perp \alpha_j - q_j \alpha_1$ for $j = 2, \dots, n$. There exists a nonzero integer m_1 such that $m_1 q_j \in \mathbb{Z}$ for all $j \in \{2, \dots, n\}$. Let $\beta_j = m_1(\alpha_j - q_j \alpha_1)$. Then $\{\beta_1, \dots, \beta_n\}$ is a linearly independent subset of L and $\beta_1 \perp \beta_i$ for $i > 1$. The proof is complete.

A.2 Proof of Lemma 3.2

By Lemma 3.1, after a scalar multiplication we may assume that $v = \sum_{i=1}^n k_i \alpha_i \in L$. It suffices to show that $\mathcal{R}_{v_2} \in OC(L)$. Since $\mathcal{R}_v \in OC(L)$, one has $\frac{(\alpha_i, v)}{(v, v)} \in \mathbb{Q}$ for all $i \in \{1, \dots, n\}$. Since $V_1 \perp V_2$, we have

$$\frac{(v_2, v_2)}{(v, v)} = \frac{(v_2, v)}{(v, v)} = \sum_{i=r+1}^n k_i \frac{(\alpha_i, v)}{(v, v)} \in \mathbb{Q}.$$

Note that $\mathcal{R}_{v_2}(\alpha_j) = \alpha_j$ for $1 \leq j \leq r$ and for $j \geq r+1$,

$$\frac{(\alpha_j, v_2)}{(v_2, v_2)} = \frac{(\alpha_j, v)}{(v_2, v_2)} = \frac{(\alpha_j, v)}{(v, v)} \cdot \frac{(v, v)}{(v_2, v_2)} \in \mathbb{Q}.$$

It follows that $\mathcal{R}_{v_2} \in OC(L)$.

A.3 Proof of Lemma 3.3

We induct on i . Suppose that $w_j \in V$ and $\mathcal{R}_{w_j} \in OC(L)$ for any $j \leq i-1$. Then $\frac{(v_i, w_j)}{(w_j, w_j)} \in \mathbb{Q}$ and $w_i = v_i - u \in V$, where $u = \sum_{j=1}^{i-1} \frac{(v_i, w_j)}{(w_j, w_j)} w_j$. Since $\mathcal{R}_{w_j} \in OC(L)$, by Lemma 3.1, $W_j = \{\alpha \in V : (\alpha, w_j) = 0\}$ is an $(n-1)$ -dimensional $\mathbb{Q}$ -subspace of V . Set $V_1 = \oplus_{j=1}^{i-1} \mathbb{Q} w_j$ and $V_2 = \bigcap_{j=1}^{i-1} W_j$. Then $V = V_1 \oplus V_2$ is an orthogonal direct sum. We have $v_i = w_i + u$, where $u \in V_1$ and $w_i \in V_2$. By Lemma 3.2, we obtain that $\mathcal{R}_{w_i} \in OC(L)$.

B.1 Proof of Lemma 4.1

If $\alpha = r\beta$, $\beta \in \mathbb{Q}^n$, $r \in \mathbb{R} \setminus \{0\}$, then by the standard result of solution of system of linear equations there exists a $\mathbb{Q}$ -linearly independent subset $\{\beta_1, \dots, \beta_{n-1}\}$ of $\mathbb{Q}^n$ with that $(\alpha, \beta_i) = 0$, and it is also linearly independent over $\mathbb{R}$ and consists of a basis of $\alpha^\perp$.

Conversely, suppose $\beta_1, \dots, \beta_{n-1}$ is a basis of $\alpha^\perp$ such that $\beta_i \in \mathbb{Q}^n$. Without loss of generality we may assume that $\alpha = (p_1, \dots, p_n)$ with $p_n \neq 0$. It suffices to show that $\frac{p_i}{p_n} \in \mathbb{Q}$. Since

$$\gamma_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -\frac{p_1}{p_n} \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ -\frac{p_2}{p_n} \end{pmatrix}, \dots, \quad \gamma_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ -\frac{p_{n-1}}{p_n} \end{pmatrix}$$

is also a basis of $\alpha^\perp$, there exists an invertible matrix $A = (a_{ij}) \in M_{n-1}(\mathbb{R})$ such that

$$(\beta_1, \dots, \beta_{n-1}) = (\gamma_1, \dots, \gamma_{n-1})A = \begin{pmatrix} I_{n-1} \\ P \end{pmatrix} A = \begin{pmatrix} A \\ PA \end{pmatrix},$$

where $P = \left(-\frac{p_1}{p_n}, \dots, -\frac{p_{n-1}}{p_n}\right)$. Hence, $A \in M_{n-1}(\mathbb{Q})$, $PA \in M_{1 \times (n-1)}(\mathbb{Q})$, and we obtain $P = (PA)A^{-1} \in M_{1 \times (n-1)}(\mathbb{Q})$. This finishes the proof.

B.2 Proof of Example 4.1

Let $(\alpha_1, \alpha_2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$. It is well known that each linear isometry of $\mathbb{R}^2$ is either a reflection or a rotation, see [17]. Let $\mathcal{T}_\theta$ denote the rotation through the angle θ . If $\mathcal{T}_\theta \in OC(L)$ for some $\theta \neq 0, \pi$, then

$$B = A^{-1} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} A = \frac{1}{\det A} \begin{pmatrix} \det A \cos \theta - (ab + cd) \sin \theta & -(b^2 + d^2) \sin \theta \\ (a^2 + c^2) \sin \theta & \det A \cos \theta + (ab + cd) \sin \theta \end{pmatrix}$$

is a rational matrix. It follows that $\cos \theta \in \mathbb{Q}$ and

$$\frac{ab + cd}{a^2 + c^2} = \frac{(\alpha_1, \alpha_2)}{(\alpha_1, \alpha_1)} \in \mathbb{Q}, \quad \frac{a^2 + c^2}{b^2 + d^2} = \frac{(\alpha_1, \alpha_1)}{(\alpha_2, \alpha_2)} \in \mathbb{Q}.$$

By Theorem 2.2, any reflection defined by the vectors in L is a coincidence isometry, in this case Statement (3) holds.

Let us now assume that $\mathcal{T}_\theta \notin OC(L)$ for any $\theta \neq 0, \pi$. If in addition $OC(L)$ contains no reflection, then $OC(L) = \{\mathcal{T}_0, \mathcal{T}_\pi\} = \{\pm I\}$. If $OC(L)$ contains a reflection $\mathcal{R}_{v_1}$, $v_1 \in L$, then by Theorem 4.2 there exists $v_1, v_2 \in L$ with $(v_1, v_2) = 0$ and $\frac{(v_1, v_1)}{(v_2, v_2)} \notin \mathbb{Q}$. Hence, Statement (2) is satisfied.

B.3 Proof of Example 4.2

We have $|X| = 0, 1$, or 3 by Theorem 3.1. Statement (3) follows immediately from Theorem 2.2 and Theorem 4.2.

We claim that: if $\mathcal{A}$ is a coincidence isometry of L and $\mathcal{A} \neq \pm I$, then there exists an eigenvector v of $\mathcal{A}$ such that $\mathcal{R}_v \in OC(L)$.

Let $f(x) \in \mathbb{Q}[x]$ be the characteristic polynomial of $\mathcal{A}$. Since $\deg f(x) = 3$ and $\mathcal{A}$ is an orthogonal transformation, $f(x)$ has a real root $t \in \{-1, 1\}$. If $\mathcal{A}$ is diagonalizable, then $f(x) = (x + 1)(x - 1)^2$ or $(x - 1)^2(x + 1)$, if $\mathcal{A}$ is not diagonalizable, then $f(x) = (x - t)g(x)$, where $g(x)$ is irreducible over $\mathbb{R}$. In both cases, there is a factorization of $f(x) = (x - t)h(x)$ such that $\gcd(x - t, h(x)) = 1$. Let $V = \oplus_{i=1}^3 \mathbb{Q}\alpha_i$. By the primary decomposition of $\mathcal{A}|_V$, see [18], we have $V = V_1 \oplus V_2$, where $V_1 = \ker(\mathcal{A} - tI)|_V$ and $V_2 = \ker h(\mathcal{A})|_V$, and both V_1 and V_2 are $\mathcal{A}$ -invariant spaces. We will show that $V_1 \perp V_2$. Since $\mathcal{A}$ is an isometry,

we have $(v_1, v_2) = (\mathcal{A}(v_1), \mathcal{A}(v_2)) = (tv_1, \mathcal{A}(v_2)) = (v_1, t\mathcal{A}(v_2))$ for any $v_1 \in V_1$ and $v_2 \in V_2$. Since $t = \pm 1$, we have $(v_1, (t\mathcal{A} - I)(v_2)) = 0 = (v_1, (\mathcal{A} - tI)(v_2))$. Note that $(\mathcal{A} - tI)|_{V_2}$ is an invertible linear transformation, we obtain $v_1 \perp V_2$. Since $\dim V_1 = 1$, $\mathcal{R}_{v_1}$ is a coincidence reflection. This proves the claim.

Statement (1) is an immediate consequence of the claim. It remains to prove Statement (2). If $X = \{v\}$, then $OC(L)$ contains a unique reflection $\mathcal{R}_v$. By Lemma 3.1 and Theorem 2.1, there exists a linearly independent subset $\{w_1, w_2\}$ of L such that $(v, w_1) = (v, w_2) = 0$ and L is commensurate to $L' = \mathbb{Z}v \oplus \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$. Let $W = \mathbb{R}w_1 \oplus \mathbb{R}w_2$. By the claim, v is a common eigenvector of all coincidence isometries of L . Thus, $\mathcal{A}(v) = \pm v$ and $\mathcal{A}|_W$ is a coincidence isometry of the lattice $\mathbb{Z}w_1 \oplus \mathbb{Z}w_2$. Since $v \perp W$, we see that if $\mathcal{R}_\alpha|_W$ is a coincidence reflection of $\mathbb{Z}w_1 \oplus \mathbb{Z}w_2$ for some nonzero vector $\alpha \in W$, then $\mathcal{R}_\alpha$ is also a coincidence reflection of L' . Therefore, $OC(\mathbb{Z}w_1 \oplus \mathbb{Z}w_2)$ has no reflection by the hypothesis $X = \{v\}$ is a maximal orthogonal subset of $\{v \in L : \mathcal{R}_v \in OC(L) = OC(L')\}$. From Example 4.1, we have shown that $\mathcal{A}|_W = \pm I$ for any $\mathcal{A} \in OC(L)$. That is, $OC(L) = \{I, -I, \mathcal{R}_v, -\mathcal{R}_v\}$ and the matrices of these isometries with respect to the ordered basis v, w_1, w_2 are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

respectively.