Research Article

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A posteriori error estimates based on superconvergence of FEM for fractional evolution equations

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Abstract: In this paper, we consider an approximation scheme for fractional evolution equation with variable coefficient. The space derivative is approximated by triangular finite element and the time fractional derivative is evaluated by the $L1$ approximation. The main aim of this work is to provide convergence and superconvergence analysis and derive *a posteriori* error estimates. Some numerical examples are presented to demonstrate our theoretical results.

Keywords: *a posteriori* error estimates, convergence and superconvergence, finite element method, fractional evolution equations

MSC 2020: 65M30, 35R11

1 Introduction

Since the remarkable hereditary and memory properties, fractional partial differential equations (FPDEs) play a very important role in wave propagation, finance, physics, engineering and so on [1]. Since the exact solutions of most FPDEs are very difficult to obtain, numerical methods of FPDEs have been an active research area, such as finite difference methods [2–6], finite element methods (FEMs) [7–11], mixed FEMs [12,13], finite volume methods [14], spectral methods [15–17] and so on.

It is well known that there has been extensive research on the superconvergence of FEMs for partial differential equations (PDEs). A systematic introduction can be found in [18–23]. Generally speaking, there are three types of superconvergence. The first is pointwise superconvergence, namely in certain sampling points the values in derivatives of error have a higher convergent order than elsewhere [24]. The second is global superconvergence, namely the gradient in L^2 -norm of error between the numerical solution and the projection of exact solution have a greater accuracy than the optimal order of convergence. The third is superconvergence based on post-processing technique, namely reconstruct an improved accuracy gradient. Representative post-processing techniques are developed by interpolation [25], extrapolation [26] and gradient recovery, which include superconvergent patch recovery [27], polynomial preserving recovery [28,29] and superconvergent cluster recovery [30]. In recent years, there has been some research on superconvergence analysis of FEMs for FPDEs. Superconvergence of FEMs and mixed FEMs for FPDEs is inves-

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tigated in [31–33] and [34], respectively. In [35,36], superconvergence of nonconforming FEMs for fractional PDEs is established.

Adaptive FEMs are among the most important means to improve the accuracy and efficiency of finite element discretization. The pioneering work was carried out by Babuška and Rheinboldt in [37]. One of the key concepts in adaptive FEMs is *a posteriori* error estimates, which are computable quantities in terms of the discrete solution and can measure the actual discrete errors without the knowledge of exact solution. *A posteriori* error estimates of FEMs for elliptic problems are well-developed [38–40]. There are substantial research on *a posteriori* error estimates of FEMs for integer-order PDEs based on explicit residual [41], local problems [42,43], recovery [27,29,44,45], hierarchical basis [46–49] and equilibrated error [50,51]. However, to the best of our knowledge, *a posteriori* error estimates of FEMs for evolution equations are less developed, and only a few results can be found in [52–54].

The purpose of this work is to provide a fully discrete finite element approximation for fractional evolution equations and analyze its convergence and superconvergence. Then, we derive *a posteriori* error estimates based on the superconvergence results and construct an adaptive FEM algorithm for fractional evolution equations with variable coefficients.

We are interested in the following fractional evolution equation:

$$\begin{cases} \partial_t^\alpha y(t, x) - \operatorname{div}(A(x)\nabla y(t, x)) = f(t, x), & t \in J, \quad x \in \Omega, \\ y(t, x) = 0, & t \in J, \quad x \in \partial\Omega, \\ y(0, x) = y_0(x), & x \in \Omega. \end{cases} \quad (1)$$

Here $J = [0, T]$ ($0 < T < +\infty$) and Ω be a bounded open domain of R^d ($1 \leq d \leq 3$) with smooth boundary $\partial\Omega$ and $A(x) = (a_{ij}(x))_{d \times d} \in W^{1,\infty}(\bar{\Omega})^{d \times d}$ is a positive definite matrix, $f(t, x)$ and $y_0(x)$ are given smooth functions. ∂_t^α ($0 < \alpha < 1$) denotes the α -order left Caputo derivative defined by

$$\partial_t^\alpha y(t, x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-\tau)^\alpha} \frac{\partial y(\tau, x)}{\partial \tau} d\tau.$$

Throughout the paper, $L^s(J; W^{m,q}(\Omega))$ denotes all L^s integrable functions from J into $W^{m,q}(\Omega)$ with norm $\|v\|_{L^s(J; W^{m,q}(\Omega))} = \left(\int_0^T \|v\|_{W^{m,q}(\Omega)}^s dt \right)^{\frac{1}{s}}$ for $s \in [1, \infty)$ and the standard modification for $s = \infty$, where $W^{m,q}(\Omega)$ is Sobolev spaces on Ω . Similarly, one can define $H^l(J; W^{m,q}(\Omega))$ and $C^k(J; W^{m,q}(\Omega))$ (see e.g. [55]). In addition, c or C is a generic positive constant.

The rest of this paper is organized as follows. In Section 2, we give a fully discrete approximation scheme of (1). Convergence analysis results are presented in Section 3. In Section 4, we derive the superconvergence between the numerical solutions and the elliptic projection of exact solutions. In Section 5, we construct *a posteriori* error estimates based on the superconvergence results. Some numerical experiments are presented to support our theoretical results in Section 6.

2 Fully discrete finite element approximation

In this section, we present a fully discrete approximation scheme of (1). To begin with, we introduce triangular FEMs for the spatial discretization. For brevity, we denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ and drop Ω or J whenever possible, i.e.,

$$\begin{aligned} \|\cdot\|_{m,2,\Omega} &= \|\cdot\|_{m,2} = \|\cdot\|_m, & \|\cdot\|_0 &= \|\cdot\|, \\ \|\cdot\|_{L^p(J; W^{m,2}(\Omega))} &= \|\cdot\|_{L^p(H^m)}, & \|\cdot\|_{L^p(J; W^{m,q}(\Omega))} &= \|\cdot\|_{L^p(W^{m,q})}. \end{aligned}$$

Moreover, we set $H_0^1(\Omega) \equiv \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$, $W = H_0^1(\Omega)$, $U = L^2(\Omega)$. In addition,

$$\begin{aligned} a(v, w) &= \int_{\Omega} (A \nabla v) \cdot \nabla w, \quad \forall v, w \in W, \\ (f_1, f_2) &= \int_{\Omega} f_1 \cdot f_2, \quad \forall f_1, f_2 \in U. \end{aligned}$$

From the assumption of coefficient matrix $A(x)$, we have

$$a(v, v) \geq c \|v\|_1^2, \quad |a(v, w)| \leq C \|v\|_1 \|w\|_1, \quad \forall v, w \in W.$$

We recast (1) as the following weak formulation:

$$\begin{cases} (\partial_t^\alpha y, w) + a(y, w) = (f, w), & \forall w \in W, \quad t \in J, \\ y(0, x) = y_0(x), & x \in \Omega. \end{cases} \quad (2)$$

Let $\mathcal{T}^h$ be a family of quasi-uniform triangulations of Ω , such that $\bar{\Omega} = \bigcup_{\tau \in \mathcal{T}^h} \bar{\tau}$ and $h = \max_{\tau \in \mathcal{T}^h} \{h_\tau\}$, where h_τ is the diameter of the element τ . Furthermore, we set

$$V_h = \{v_h \in C(\bar{\Omega}) : v_h|_\tau \in \mathbb{P}_1, \quad \forall \tau \in \mathcal{T}^h\},$$

where $\mathbb{P}_1$ represents the space of all polynomials whose degree at most 1, and $W_h = V_h \cap H_0^1(\Omega)$.

Then a semi-discrete approximation scheme of (2) reads as

$$\begin{cases} (\partial_t^\alpha y_h, w_h) + a(y_h, w_h) = (f, w_h), & \forall w_h \in W_h, \quad t \in J, \\ y_h^0(x) = y_0^h(x), \end{cases} \quad (3)$$

where $y_0^h(x) \in W_h$ is a suitable approximation of $y_0(x)$.

In the second, we will consider the $L1$ scheme for the time discretization.

Let $0 = t_0 < t_1 < \dots < t_N = T$ be a uniform partition of J with time step $\tau = \frac{T}{N}$ and $t_n = n\tau$, $n = 0, 1, \dots, N$.

We set $\varphi^n = \varphi(t_n, x)$ and $d_t y^n = \frac{y^n - y^{n-1}}{\tau}$. The time fractional derivative can be approximated as follows:

$$\begin{aligned} \partial_t^\alpha y^n &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_n} \frac{1}{(t_n - s)^\alpha} \frac{\partial y(s, x)}{\partial s} ds \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \frac{1}{(t_n - s)^\alpha} \frac{\partial y(s, x)}{\partial s} ds \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} d_t y^{k+1} \int_{t_k}^{t_{k+1}} \frac{1}{(t_n - s)^\alpha} ds + r_\tau^n \\ &= \frac{1}{\tau^{\alpha-1} \Gamma(2-\alpha)} \sum_{k=0}^{n-1} b_k d_t y^{n-k} + r_\tau^n, \\ &= \frac{1}{\tau^\alpha \Gamma(2-\alpha)} \sum_{k=0}^n b_k^n y^k + r_\tau^n, \\ &:= L_t^\alpha y^n + r_\tau^n, \end{aligned} \quad (4)$$

where $b_k = (k+1)^{1-\alpha} - k^{1-\alpha}$, $b_0^n = (n-1)^{1-\alpha} - n^{1-\alpha}$, $b_n^n = 1$, $b_k^n = b_{n-k} - b_{n-k-1}$ and r_τ^n is the truncation error. It follows from [25] for $y \in W^{2,\infty}(L^2)$ that

$$|r_\tau^n| = |\partial_t^\alpha y^n - L_t^\alpha y^n| \leq C \tau^{2-\alpha}. \quad (5)$$

Then a fully discrete approximation scheme of (2) is as follows:

$$\begin{cases} (L_t^\alpha y_h^n, w_h) + a(y_h^n, w_h) = (f^n, w_h), & \forall w_h \in W_h, \quad n = 1, 2, \dots, N, \\ y_h^0 = y_0^h(x). \end{cases} \quad (6)$$

Usually, we set $y_0^h(x) = P_h y_0(x)$, where P_h will be specified later.

3 Convergence analysis

We will derive the convergence of the numerical scheme (6). Let $P_h : W \rightarrow W_h$ be the elliptic projection operator, for any $v \in W$ defined by

$$a(v - P_h v, w_h) = 0, \quad \forall w_h \in W_h.$$

It has the following approximation properties (see [8]):

$$\|v - P_h v\| + h \|\nabla(v - P_h v)\| \leq Ch^2 \|v\|_2. \quad (7)$$

The following conclusions will be used in the following error analysis.

Lemma 3.1. [13] *Let $\{\xi^n\}_{n=0}^N$ be a sequence of functions on Ω . Then*

$$\left(\xi^n, \sum_{k=0}^n b_k^n \xi^k \right) = \frac{1}{2} \left(\|\xi^n\|^2 + \sum_{k=0}^{n-1} b_k^n \|\xi^k\|^2 - \sum_{k=0}^{n-1} b_k^n \|\xi^k - \xi^n\|^2 \right).$$

Lemma 3.2. [33] *Let $\varepsilon^k \geq 0$, $k = 1, 2, \dots, L$, satisfy $\varepsilon^n \leq \sum_{k=1}^{n-1} (b_{k-1} - b_k) \varepsilon^{n-k} + \gamma$ with $\gamma > 0$. Then*

$$\varepsilon^n \leq C\tau^{-\alpha} \gamma, \quad n = 1, 2, \dots, L.$$

Theorem 3.1. *Let y and y_h be the solutions of (2) and (6), respectively. Suppose that $y \in W^{2,\infty}(L^2) \cap W^{1,\infty}(H^2)$. Then, for any integer $1 \leq n \leq N$, we have*

$$\|y^n - y_h^n\|_{L^2} \leq C(h^2 + \tau^{2-\alpha}), \quad (8)$$

$$\|y^n - y_h^n\|_{H^1} \leq C(h + \tau^{2-\alpha}). \quad (9)$$

Proof. From (4) and the definition of P_h , we can rewrite (2) at t_n as

$$(L_t^\alpha y^n, w_h) + a(P_h y^n, w_h) + (r_\tau^n, w_h) = (f^n, w_h), \quad \forall w_h \in W_h. \quad (10)$$

Setting $\theta^n = y_h^n - P_h y^n$ and subtracting (10) from (6), we get

$$(L_t^\alpha \theta^n, w_h) + a(\theta^n, w_h) = (r_\tau^n, w_h) + (L_t^\alpha (y^n - P_h y^n), w_h), \quad \forall w_h \in W_h. \quad (11)$$

Taking $w_h = \theta^n$ in (11), we have

$$(L_t^\alpha \theta^n, \theta^n) + a(\theta^n, \theta^n) = (r_\tau^n, \theta^n) + (L_t^\alpha (y^n - P_h y^n), \theta^n). \quad (12)$$

Note that $a(\theta^n, \theta^n) \geq 0$, from (5), (7) and (12), we can obtain

$$\begin{aligned} \|\theta^n\|^2 &\leq - \sum_{k=0}^{n-1} b_k^n \|\theta^k\|^2 + \sum_{k=0}^{n-1} b_k^n \|\theta^k - \theta^n\|^2 + 2\tau^\alpha \Gamma(2-\alpha) (\|r_\tau^n\| \|\theta^n\| + \|L_t^\alpha (y^n - P_h y^n)\| \|\theta^n\|) \\ &\leq - \sum_{k=0}^{n-1} b_k^n \|\theta^k\|^2 + C2\tau^\alpha \Gamma(2-\alpha) (\tau^{2-\alpha} + h^2) \|\theta^n\|. \end{aligned} \quad (13)$$

Hence, (8) follows from (7), (13) and triangle inequality.

Setting $w_h = L_t^\alpha \theta^n$ in (11), we get

$$(L_t^\alpha \theta^n, L_t^\alpha \theta^n) + a(\theta^n, L_t^\alpha \theta^n) = (r_\tau^n, L_t^\alpha \theta^n) + (L_t^\alpha (y^n - P_h y^n), L_t^\alpha \theta^n). \quad (14)$$

Similarly, from Hölder's inequality (5), (7) and (14), we derive

$$c \|\theta^n\|_1 \|L_t^\alpha \theta^n\|_1 \leq (\|r_\tau^n\|_{-1} + \|L_t^\alpha (y^n - P_h y^n)\|_{-1}) \|L_t^\alpha \theta^n\|_1 \leq C(\tau^{2-\alpha} + h^2) \|L_t^\alpha \theta^n\|_1. \quad (15)$$

Thus, (9) follows from (7), (15) and triangle inequality. $\square$

4 Superconvergence analysis

In this section, we will derive the global superconvergence results between the finite element solution and the elliptic projection of exact solution.

Theorem 4.1. *Let y and y_h be the solutions of (2) and (6), respectively. Assume all the conditions in Theorem 3.1 are valid. Then, for any integer $1 \leq n \leq N$, we have*

$$\|P_h y^n - y_h^n\|_{H^1} \leq C(h^2 + \tau^{2-\alpha}). \quad (16)$$

Proof. Set $y^n - y_h^n = y^n - P_h y^n + P_h y^n - y_h^n := \eta^n + \zeta^n$. Choosing $v = w_h$ in (2) and subtracting (2) from (6), we obtain the following error equation:

$$(L_t^\alpha (y^n - y_h^n), w_h) + a(y^n - y_h^n, w_h) + (r_\tau^n, w_h) = 0. \quad (17)$$

By using the definition of P_h and (17), we have

$$(L_t^\alpha \zeta^n, w_h) + a(\zeta^n, w_h) = -(L_t^\alpha \eta^n, w_h) - (r_\tau^n, w_h). \quad (18)$$

Let $w_h = L_t^\alpha \zeta^n$. Using Lemma 3.1, we get

$$\begin{aligned} (L_t^\alpha \zeta^n, L_t^\alpha \zeta^n) + a(\zeta^n, L_t^\alpha \zeta^n) &\geq (L_t^\alpha \zeta^n, L_t^\alpha \zeta^n) + c(\nabla \zeta, \nabla (L_t^\alpha \zeta^n)) \\ &= \|L_t^\alpha \zeta^n\|^2 + \frac{c}{2\tau^\alpha \Gamma(2-\alpha)} \left(\|\nabla \zeta^n\|^2 + \sum_{k=0}^{n-1} b_k^n \|\nabla \zeta^k\|^2 - \sum_{k=0}^{n-1} b_k^n \|\nabla (\zeta^k - \zeta^n)\|^2 \right). \end{aligned} \quad (19)$$

According to $y \in W^{1,\infty}(H^2)$ and (4), there holds

$$\begin{aligned} \|L_t^\alpha \eta^n\| &= \left\| \frac{1}{\tau^{\alpha-1} \Gamma(2-\alpha)} \sum_{k=0}^{n-1} b_k \partial_t \eta^{n-k} \right\| \\ &\leq \frac{1}{\tau^{\alpha-1} \Gamma(2-\alpha)} \sum_{k=0}^{n-1} b_k \|\partial_t \eta^{n-k}\| \\ &\leq \frac{1}{\tau^\alpha \Gamma(2-\alpha)} \sum_{k=0}^{n-1} b_k \int_{t_{n-k-1}}^{t_{n-k}} \|\eta_t\| dt \\ &\leq Ch^2 \|y_t\|_{L^\infty(H^2)}. \end{aligned} \quad (20)$$

Then, by applying Hölder's inequality, Young's inequality and (7), we arrive at

$$(L_t^\alpha \eta^n, L_t^\alpha \zeta^n) \leq \|L_t^\alpha \eta^n\| \|L_t^\alpha \zeta^n\| \leq \frac{1}{2} \|L_t^\alpha \zeta^n\|^2 + \frac{C}{2} h^4. \quad (21)$$

Likewise

$$(r_\tau^n, L_t^\alpha \zeta^n) \leq \|r_\tau^n\| \|L_t^\alpha \zeta^n\| \leq \frac{1}{2} \|L_t^\alpha \zeta^n\|^2 + \frac{C}{2} \tau^{4-2\alpha}. \quad (22)$$

Combining (18)–(22) and noting that $b_k^n < 0$ ($0 \leq k < n$), we obtain

$$\|\nabla \zeta^n\|^2 \leq -\sum_{k=0}^{n-1} b_k^n \|\nabla \zeta^k\|^2 + C\tau^\alpha (h^2 + \tau^{2-\alpha})^2. \quad (23)$$

It follows from (23), Lemma 3.2 and Poincaré's inequality that

$$\|\zeta^n\|_1 \leq C(h^2 + \tau^{2-\alpha}). \quad (24)$$

Hence, we complete the proof of Theorem 4.1. $\square$

5 A posteriori error estimates

We introduce recovery operators R_h and G_h . Similar to the Z–Z patch recovery in [27], $R_h v$ be a continuous piecewise linear function (without zero boundary constraint), the value of $R_h v$ on the nodes is defined by least-squares argument on element patches surrounding the nodes. Then the gradient recovery operator $G_h v = (R_h x_1, R_h x_2)$. The details can be found in [56].

Theorem 5.1. *Let y and y_h be the solutions of (2) and (6), respectively. Suppose that $y \in W^{2,\infty}(L^2) \cap W^{1,\infty}(H^3)$. Then for any integer $1 \leq n \leq N$, we have*

$$\|G_h y_h^n - \nabla y^n\|_{L^2} \leq C(h^2 + \tau^{2-\alpha}). \quad (25)$$

Proof. Let y_I^n be the piecewise linear Lagrange interpolation of y^n . According to Theorem 2.1.1 in [20], we have

$$\|P_h y^n - y_I^n\|_{H^1} \leq Ch^2 \|y^n\|_{H^3}. \quad (26)$$

From the interpolation error estimate [18], we get

$$\|G_h y_I^n - \nabla y^n\| \leq Ch^2 \|y^n\|_{H^3}. \quad (27)$$

From triangle inequality, (16) and (26)–(27), we obtain

$$\begin{aligned} \|G_h y_h^n - \nabla y^n\|_{L^2} &\leq \|G_h y_h^n - G_h P_h y^n\|_{L^2} + \|G_h P_h y^n - G_h y_I^n\|_{L^2} + \|G_h y_I^n - \nabla y^n\|_{L^2} \\ &\leq C\|y_h^n - P_h y^n\|_{H^1} + C\|P_h y^n - y_I^n\|_{H^1} + \|G_h y_I^n - \nabla y^n\|_{L^2} \\ &\leq C(h^2 + \tau^{2-\alpha}). \end{aligned} \quad (28)$$

Hence, we complete the proof of (25). $\square$

Combining the previous results, we obtain the following *a posteriori* error estimates of fully discrete finite element approximation for fractional evolution equations.

Theorem 5.2. *Assume that all the conditions in Theorem 4.1 and Theorem 5.1 are valid. Then*

$$\eta^n := \|G_h y_h^n - \nabla y_h^n\|_{L^2} = \|\nabla(y_h^n - y^n)\|_{L^2} + O(h^2 + \tau^{2-\alpha}). \quad (29)$$

Proof. According to (25) and triangle inequality, it is easy to get (29). $\square$

6 Numerical experiments

In this section, we present some different numerical examples to illustrate the correctness of the convergence and superconvergence results and the reliable and efficient *a posteriori* error estimates.

For an acceptable iteration error Tol , we present an uniformly refined FEM algorithm for the discrete problem (6) of fractional evolution equations.

Algorithm 6.1. FEM algorithm

- Step 1. Set $i = 1$, initialize mesh $\mathcal{T}_i^h$ and $y_{h,i}^0$.
 Step 2. Solve the following discrete equations:

$$(L_t^\alpha y_{h,i}^n, w_h) + a(y_{h,i}^n, w_h) = (f^n, w_h), \quad \forall w_h \in W_h, \quad n = 1, 2, \dots, N.$$

 Step 3. Uniformly refine the meshes obtain new meshes $\mathcal{T}_{i+1}^h$;
 Step 4. Calculate the iterative error: $E_i = \|y_{h,i}^n - y_{h,i-1}^n\|_{L^\infty(L^2)}$;
 Step 5. If $E_i > Tol$, $i = i + 1$, go to Step 2; else stop.
-

For an acceptable iteration error Tol , by selecting η^n in (29) as mesh refinement indicators, we construct the following adaptive FEM algorithm for the discrete problem (6):

Algorithm 6.2. Adaptive FEM algorithm

- Step 1. Set $i = 1$, initialize mesh $\mathcal{T}_i^h$ and $y_{h,i}^0$.
 Step 2. Solve the following discretized problems:

$$(L_t^\alpha y_{h,i}^n, w_h) + a(y_{h,i}^n, w_h) = (f^n, w_h), \quad \forall w_h \in W_h, \quad n = 1, 2, \dots, N.$$

 Obtain numerical solution $y_{h,i}^n$ ($n = 1, 2, \dots, N$) on the current meshes $\mathcal{T}_i^h$ and calculate the error estimators η_i^n ;
 Step 3. Adjust the meshes by using the estimators η_i^n obtain new meshes $\mathcal{T}_{i+1}^h$;
 Step 4. Calculate the iterative error: $E_i = \|y_{h,i}^n - y_{h,i-1}^n\|_{L^\infty(L^2)}$;
 Step 5. If $E_i > Tol$, $i = i + 1$, go to Step 2; else stop.
-

The following examples were dealt numerically with codes developed based on AFEPack, which is freely available and the details can be found in [56]. The discretization was described in Section 2. We denote $\|\cdot\|_{L^\infty(H^1)}$ and $\|\cdot\|_{L^\infty(L^2)}$ by $\|\cdot\|_{1,\infty}$ and $\|\cdot\|_{0,\infty}$, respectively. The convergence order rate is computed by the following formula:

$$\text{Rate} = \frac{\ln(e_{i+1}) - \ln(e_i)}{\ln(h_{i+1}) - \ln(h_i)},$$

where e_i (e_{i+1}) denotes the error when the spatial partition size is h_i (h_{i+1}). E is the 2-by-2 identity matrix.

Example 6.1. This is a 1D example. The data are as follows:

$$\begin{aligned} \Omega &= (0, 1), \quad T = 1, \quad A(x) = 1, \\ y(t, x) &= t \sin(2\pi x), \\ f(t, x) &= \left(\frac{1}{t^\alpha \Gamma(2 - \alpha)} + 4\pi^2 \right) y(t, x). \end{aligned}$$

This example is solved by Algorithm 6.1. For different α values, the errors on a sequence of uniformly refined meshes' size h and time step size τ are shown in Tables 1–3. It is easy to see $\|y - y_h\|_{0,\infty} = O(h^2 + \tau^{2-\alpha})$, $\|y - y_h\|_{1,\infty} = O(h + \tau^{2-\alpha})$ and $\|P_h y - y_h\|_{1,\infty} = O(h^2 + \tau^{2-\alpha})$. In Figure 1, we show the numerical solution y_h with $\alpha = 0.5$ when $h = \frac{1}{40}$ and $\tau = \frac{1}{90}$.

Figure 1: The numerical solution y_h when $h = \frac{1}{40}$, $\tau = \frac{1}{90}$, $\alpha = 0.5$, Example 6.1.

Table 1: Numerical results with $\alpha = 0.05$, Example 6.1

h	τ	$\ y - y_h\ _{0,\infty}$	Rate	$\ y - y_h\ _{1,\infty}$	Rate	$\ P_h y - y_h\ _{1,\infty}$	Rate
$\frac{1}{10}$	$\frac{1}{10}$	1.524550×10^{-3}	—	7.84280×10^{-1}	—	5.24997×10^{-2}	—
$\frac{1}{20}$	$\frac{1}{20}$	4.069121×10^{-4}	1.9056	4.01217×10^{-1}	0.9670	1.47561×10^{-2}	1.8310
$\frac{1}{40}$	$\frac{1}{40}$	1.021949×10^{-4}	1.9934	2.01314×10^{-1}	0.9949	3.90295×10^{-3}	1.9187
$\frac{1}{80}$	$\frac{1}{80}$	2.560060×10^{-5}	1.9971	1.00716×10^{-1}	0.9992	1.00212×10^{-3}	1.9615

Table 2: Numerical results with $\alpha = 0.5$, Example 6.1

h	τ	$\ y - y_h\ _{0,\infty}$	Rate	$\ y - y_h\ _{1,\infty}$	Rate	$\ P_h y - y_h\ _{1,\infty}$	Rate
$\frac{1}{10}$	$\frac{1}{10}$	2.14883×10^{-3}	—	7.84300×10^{-1}	—	5.02080×10^{-2}	—
$\frac{1}{20}$	$\frac{1}{30}$	5.56062×10^{-4}	1.9502	4.01220×10^{-1}	0.9670	1.41640×10^{-2}	1.8257
$\frac{1}{40}$	$\frac{1}{90}$	1.35887×10^{-4}	2.0328	2.01314×10^{-1}	0.9949	3.76104×10^{-3}	1.9130
$\frac{1}{80}$	$\frac{1}{270}$	3.32722×10^{-5}	2.0300	1.00716×10^{-1}	0.9992	9.68909×10^{-4}	1.9567

Table 3: Numerical results with $\alpha = 0.95$, Example 6.1

h	τ	$\ y - y_h\ _{0,\infty}$	Rate	$\ y - y_h\ _{1,\infty}$	Rate	$\ P_h y - y_h\ _{1,\infty}$	Rate
$\frac{1}{10}$	$\frac{1}{10}$	4.06759×10^{-3}	—	7.84402×10^{-1}	—	4.31645×10^{-2}	—
$\frac{1}{20}$	$\frac{1}{40}$	1.05515×10^{-3}	1.9467	4.01235×10^{-1}	0.9671	1.21829×10^{-2}	1.8250
$\frac{1}{40}$	$\frac{1}{160}$	2.56966×10^{-4}	2.0378	2.01316×10^{-1}	0.9950	3.25106×10^{-3}	1.9059
$\frac{1}{80}$	$\frac{1}{640}$	6.24374×10^{-5}	2.0411	1.00717×10^{-1}	0.9992	8.42640×10^{-4}	1.9479

Example 6.2. This is a 2D example. The data are as follows:

$$\begin{aligned}\Omega &= (0, 1) \times (0, 1), T = 1.0, A(x) = E, \\ y(t, x) &= t^3 \sin(2\pi x_1) \sin(2\pi x_2), \\ f(t, x) &= \left(\frac{6}{t^\alpha \Gamma(4 - \alpha)} + 8\pi^2 \right) y(t, x).\end{aligned}$$

This example is solved by Algorithm 6.1. In Tables 4–6, the errors $\|y - y_h\|_{0,\infty}$, $\|y - y_h\|_{1,\infty}$ and $\|P_h y - y_h\|_{1,\infty}$ based on different α values and a sequence of uniformly refined meshes' size h and time step size τ are shown. It is easy to see that $\|y - y_h\|_{0,\infty}$ and $\|P_h y - y_h\|_{1,\infty}$ are the second-order convergent while $\|y - y_h\|_{1,\infty}$ is the first-order convergent. We plot the profile of the numerical solution y_h with $t = 0.5$ and $\alpha = 0.5$, where $h = \frac{1}{40}$ and $\tau = \frac{1}{90}$ in Figure 2.

Table 4: Numerical results with $\alpha = 0.05$, Example 6.2

h	τ	$\ y - y_h\ _{0,\infty}$	Rate	$\ y - y_h\ _{1,\infty}$	Rate	$\ P_h y - y_h\ _{1,\infty}$	Rate
$\frac{1}{10}$	$\frac{1}{10}$	5.45891×10^{-2}	—	1.35785	—	6.88961×10^{-3}	—
$\frac{1}{20}$	$\frac{1}{20}$	1.42607×10^{-2}	1.9366	6.93043×10^{-1}	0.9703	1.75476×10^{-3}	1.9731
$\frac{1}{40}$	$\frac{1}{40}$	3.60530×10^{-3}	1.9839	3.48335×10^{-1}	0.9925	4.41402×10^{-4}	1.9911
$\frac{1}{80}$	$\frac{1}{80}$	9.03851×10^{-4}	1.9960	1.74395×10^{-1}	0.9981	1.10657×10^{-4}	1.9960

Table 5: Numerical results with $\alpha = 0.5$, Example 6.2

h	τ	$\ y - y_h\ _{0,\infty}$	Rate	$\ y - y_h\ _{1,\infty}$	Rate	$\ P_h y - y_h\ _{1,\infty}$	Rate
$\frac{1}{10}$	$\frac{1}{10}$	5.39968×10^{-2}	—	1.35788	—	1.25360×10^{-2}	—
$\frac{1}{20}$	$\frac{1}{30}$	1.40961×10^{-2}	1.9376	6.93048×10^{-1}	0.9703	3.24284×10^{-3}	1.9507
$\frac{1}{40}$	$\frac{1}{90}$	3.56467×10^{-3}	1.9835	3.48335×10^{-1}	0.9925	8.03542×10^{-4}	2.0128
$\frac{1}{80}$	$\frac{1}{270}$	8.94056×10^{-4}	1.9953	1.74395×10^{-1}	0.9981	1.97545×10^{-4}	2.0242

Table 6: Numerical results with $\alpha = 0.95$, Example 6.2

h	τ	$\ y - y_h\ _{0,\infty}$	Rate	$\ y - y_h\ _{1,\infty}$	Rate	$\ P_h y - y_h\ _{1,\infty}$	Rate
$\frac{1}{10}$	$\frac{1}{10}$	5.25616×10^{-2}	—	1.35809	—	2.71477×10^{-2}	—
$\frac{1}{20}$	$\frac{1}{40}$	1.36714×10^{-2}	1.9428	6.93079×10^{-1}	0.9705	7.29615×10^{-3}	1.8956
$\frac{1}{40}$	$\frac{1}{160}$	3.45608×10^{-3}	1.9840	3.48339×10^{-1}	0.9925	1.82080×10^{-3}	2.0026
$\frac{1}{80}$	$\frac{1}{640}$	8.88086×10^{-4}	1.9604	1.74396×10^{-1}	0.9981	4.52293×10^{-4}	2.0092

Figure 2: The numerical solution y_h when $t = \alpha = 0.5$, $h = \frac{1}{40}$, $\tau = \frac{1}{90}$, Example 6.2.

Example 6.3. This is a 2D example. The data are as follows:

$$\begin{aligned}\Omega &= (0, 1) \times [0, 1], \quad T = 1.0, \\ A(x) &= \begin{cases} 4E, & x_1 + x_2 \leq 1, \\ E, & x_1 + x_2 > 1, \end{cases} \\ y(t, x) &= \begin{cases} t^2 \sin(\pi x_1) \sin(\pi x_2), & x_1 + x_2 \leq 1, \\ t^2 \sin(2\pi x_1) \sin(2\pi x_2), & x_1 + x_2 > 1, \end{cases} \\ f(t, x) &= \left(\frac{2}{t^\alpha \Gamma(3 - \alpha)} + 8\pi^2 \right) y(t, x). \end{aligned}$$

We take $\tau = 10^{-2}$ and solve this example by using Algorithms 6.1 and 6.2. Numerical results based on a sequence of uniformly refined meshes and adaptive meshes are listed in Tables 7 and 8, respectively. It is clear that the adaptive meshes generated via the error estimators η^n are able to save substantial computational work. We plot the profile of the numerical solution y_h at $t = 0.5$, where adaptive mesh nodes = 1,241 in Figure 3.

Table 7: Numerical results for Example 6.3 on uniform meshes

Uniform meshes	1	2	3	4
Nodes of y_h^n	121	441	1,681	6,561
$\ \nabla y - \nabla y_h\ _{0,\infty}$	3.85733×10^{-1}	3.72531×10^{-1}	3.65088×10^{-1}	3.60944×10^{-1}
$\ G_h y_h - \nabla y_h\ _{0,\infty}$	3.91964×10^{-1}	2.22143×10^{-1}	1.62862×10^{-1}	1.17960×10^{-1}

Table 8: Numerical results for Example 6.3 on adaptive meshes

Adaptive meshes	1	2	3	4
Nodes of y_h^n	121	383	817	1,241
$\ \nabla y - \nabla y_h\ _{0,\infty}$	3.85733×10^{-1}	3.70120×10^{-1}	3.67693×10^{-1}	3.58027×10^{-1}
$\ G_h y_h - \nabla y_h\ _{0,\infty}$	3.91964×10^{-1}	2.12143×10^{-1}	1.52864×10^{-1}	1.15966×10^{-1}

Figure 3: The numerical solution y_h at $t = 0.5$ on adaptive mesh (nodes = 1,241).

7 Conclusion

Although there has been extensive research on FEMs for FPDEs, mostly focused on convergence analysis [7–11]. While there is little work on *a posteriori* error estimates of FEM for FPDEs. Hence, our results on *a posteriori* error estimates and adaptive FEM for fractional evolution equations are new.

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