Research Article

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The role of w-tilting modules in relative Gorenstein (co)homology

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Abstract: Let R be a ring, C be a left R -module and $S = \text{End}_R(C)$. When C is semidualizing, the Auslander class $\mathcal{A}_C(S)$ and the Bass class $\mathcal{B}_C(R)$ associated with C have been the subject of extensive investigations. It has been proved that these classes, also known as Foxby classes, are one of the central concepts of (relative) Gorenstein homological algebra. In this paper, we answer several natural questions which arise when we weaken the condition of C being semidualizing: if we let C be w-tilting (see Definition 2.1), we establish the conditions for the pair $(\mathcal{A}_C(S), \mathcal{A}_C(S)^{\perp})$ to be a perfect cotorsion theory and for the pair $({}^{\perp}\mathcal{B}_C(R), \mathcal{B}_C(R))$ to be a complete hereditary cotorsion theory. This tells us when the classes of Auslander and Bass are preenveloping and precovering, which generalizes a number of results disseminated in the literature. We investigate Gorenstein flat modules relative to a not necessarily semidualizing module C and we find conditions for the class of G_C -projective modules to be special precovering, the class of G_C -flat modules to be covering, the one of Gorenstein C -projective modules to be precovering and that of Gorenstein C -injective modules to be preenveloping. We also find how to recover Foxby classes from $\text{Add}_R(C)$ -resolutions of R .

Keywords: Gorenstein modules, Auslander class, Bass class, (weakly) Wakamatsu tilting module

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1 Introduction

Foxby classes (Auslander and Bass classes) have proven to be very useful when studying Gorenstein injective and projective dimensions: over Gorenstein rings, every module has finite Gorenstein projective and finite Gorenstein injective dimensions [1]. However, when the ring is no longer Gorenstein, modules of finite Gorenstein injective or projective dimensions constitute two new full subcategories. These were identified in the case of a commutative local Cohen-Macaulay ring with a dualizing module, at least for finitely generated modules. Foxby proved that the Auslander and Bass classes behave in a duality and that finitely generated modules in the Auslander class are precisely those of finite Gorenstein projective dimension, while finitely generated modules in the Bass class are those of finite Gorenstein injective dimension [2].

This result has been much appreciated and later on many authors have studied and extended the situation given by Foxby (see [3–5] or [6] for instance).

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On the other hand, the existence of Gorenstein projective precovers and Gorenstein injective preenvelopes is an active subject of study nowadays, not only in the category of modules but also in any category where the knowledge of homological properties is interesting, as for example the categories of sheaves over a topological space, the ones of complexes of modules, those of representations of quivers by modules, etc. (read for instance Enochs et al. in [7], Enochs et al. in [8], Gillespie in [9] or Krause in [10]). At the same time, generalizations of these concepts by means of a semidualizing or even a weakly Wakamatsu tilting (cotilting), module (see Definition 2.1) have been given (G_C -projective, G_C -injective, Gorenstein C -projective and Gorenstein C -injective modules), with the corresponding study of the dimensions relative to them [11–14]. However, little is yet known about the existence of precovers or preenvelopes relative to these new classes of modules: it is known that if C is weakly Wakamatsu tilting (cotilting), then every module of finite G_C -projective (injective) dimension has a special G_C -precover (preenvelope) [11, Corollary 3.6 and Theorem 4.11]. It is then clear at this point how important it would be to know when the existence of G_C -projective, $C - GP(R)$ and even G_C -flat precovers, and G_C -injective and $C - GI(R)$ preenvelopes, can be guaranteed, or, from another point of view, when all modules have finite G_C -projective (injective) dimension.

In this paper, we will address that problem, and we will do so by a systematic use of the duality between the Auslander and Bass classes. We will study when the pairs of classes $(\mathcal{A}_C(S), \mathcal{A}_C(S)^\perp)$ and $({}^\perp\mathcal{B}_C(R), \mathcal{B}_C(R))$ are complete cotorsion theories (in fact, $(\mathcal{A}_C(S), \mathcal{A}_C(S)^\perp)$ is not only complete but also perfect) in Theorem 3.16 and Corollary 3.12 (two of the main results of the paper), respectively.

Moreover, we will see (Theorem 3.10 and Corollary 4.8) that, with the extra assumption that C is self-small, the classes $\mathcal{A}_C(S)$ and $\mathcal{B}_C(R)$ can both be defined as the right orthogonal classes of some given modules. This is a big step in understanding their structure.

And finally we will be able to prove the last two main results of the paper: Corollary 5.7, which says that, under certain circumstances, the class of all G_C -flat modules is covering in $R\text{-Mod}$, and Corollary 5.14, which states that $G_C P(R)$ is special precovering in $R\text{-Mod}$.

The paper is organized as follows. Section 2 is devoted to setting the conditions assumed in all this work, to introducing the definitions of all the concepts related to this problem, and to stating and, sometimes proving, the basic results needed throughout the paper.

In Section 3, we study the problems addressed above regarding the pairs of classes $(\mathcal{A}_C(S), \mathcal{A}_C(S)^\perp)$ and $({}^\perp\mathcal{B}_C(R), \mathcal{B}_C(R))$. We give conditions for the first pair to be a perfect cotorsion theory (Theorem 3.16) and conditions for the second pair to be a complete cotorsion theory (Corollary 3.12). In particular, we find conditions for $\mathcal{A}_C(S)$ and $\mathcal{B}_C(R)$ to be precovering and preenveloping. We also study when the classes $\text{Add}_R(C)$ and $\text{Prod}_S(\text{Hom}_R(C, E))$ (${}_R E$ is an injective cogenerator) are covering (Corollary 3.5) and enveloping (Corollary 3.4), respectively, with precovers (preenvelopes) of modules in $\mathcal{B}_C(R)$ ($\mathcal{A}_C(S)$) being epimorphisms (monomorphisms) as can be checked in Propositions 3.8 and 3.9, respectively. This fact will lead us to prove that the Bass class $\mathcal{B}_C(R)$ can be recovered from a right $\text{Add}_R(C)$ -resolution of R , as a direct sum of C and the right orthogonal class of all cokernels in such resolution (Theorem 3.10). The dual of this result relative to the Auslander class will need a little more work and will be proved in Section 4.

In Section 4, we study when the class $C - GP(R)$ of Gorenstein C -projective modules and the class $C - GI(R)$ of Gorenstein C -injective modules are precovering (Proposition 4.7) and preenveloping (Corollary 4.4), respectively. Using this, we get, as applications, how to recover the Auslander class from a right $\text{Add}_R(C)$ -resolution of R on one side (Corollary 4.8), and on the other, what is the common part of the Bass class and its left 1-orthogonal class, and the common part of the Auslander class and its first right orthogonal class (Proposition 4.9).

Finally, in Section 5 we study the existence of G_C -projective precovers. We find the conditions for the class of G_C -flat modules to be covering (Corollary 5.7) and to get the inclusion of the class of G_C -projective modules in the one of G_C -flat modules (Theorem 5.9). From this fact we deduce the existence of G_C -projective precovers (indeed special G_C -projective precovers) in Corollary 5.14.

2 Preliminaries

Throughout this paper R will be an associative (not necessarily commutative) ring with identity, and all modules will, unless otherwise specified, be unital left R -modules. When it is necessary to use right R -modules, they will be denoted by M_R , while in these cases left R -modules will be denoted by ${}_R M$. Following this terminology, ${}_R M_S$ will mean that M is an (R, S) -bimodule. The category of all left (right if necessary) R -modules will be denoted by $R\text{-Mod}$ ($\text{Mod-}R$). For an R -module C we use $\text{Add}_R(C)$ to denote the class of all R -modules which are isomorphic to direct summands of direct sums of copies of C , and $\text{Prod}_R(C)$ will denote the class of all R -modules which are isomorphic to direct summands of direct products of copies of C . We will denote by $\text{Gen}_R(C)$ the class of left R -modules which are generated by C , that is, modules isomorphic to quotients of modules in $\text{Add}_R(C)$ and by $\text{Cogen}_R(C)$ the class of R -modules which are cogenerated by C , that is, they are isomorphic to submodules of modules in $\text{Prod}_R(C)$.

By $\sigma_R[C]$ we mean the full subcategory of $R\text{-Mod}$ consisting of all modules subgenerated by C , that is, modules isomorphic to submodules of quotients of direct sums of arbitrary copies of C . Dually, $\pi_R[C]$ will be the class of left R -modules which are isomorphic to quotients of submodules of direct products of copies of C (see [15]). We then see that $\text{Gen}_R(C) \subseteq \sigma_R[C] \cap \pi_R[C]$ and that $\text{Cogen}_R(C) \subseteq \pi_R[C]$.

From now on, and unless otherwise specified, C will be a left R -module and S its endomorphism ring ($S = \text{End}_R(C)$).

The character module of any module M will be represented by M^+ , and $\mathcal{P}(R)$, $\mathcal{I}(R)$ and $\mathcal{F}(R)$ will be the classes of all projective, injective and flat R -modules, respectively. As usual, the projective and injective dimensions of M will be denoted as $pd(M)$ and $id(M)$, respectively.

If $\mathcal{A}$ is a class of modules, a left $\mathcal{A}$ -resolution of a left R -module M is a $\text{Hom}(\mathcal{A}, -)$ -exact complex (so not necessarily exact) with components in $\mathcal{A}$, that is, a complex

$$\cdots \rightarrow A_1 \rightarrow A_0 \rightarrow M \rightarrow 0$$

such that $A_i \in \mathcal{A} \forall i$ and such that the resulting complex after applying $\text{Hom}_R(A, -)$ for any $A \in \mathcal{A}$ is exact. Right $\mathcal{A}$ -resolutions (or $\mathcal{A}$ -coresolutions) of ${}_R M$ are defined dually.

Given two classes of modules $\mathcal{C}$ and $\mathcal{D}$, a complete $(\mathcal{C}, \mathcal{D})$ -resolution of M is an exact, $\text{Hom}(\mathcal{C}, -)$ -exact and $\text{Hom}(-, \mathcal{D})$ -exact sequence of modules

$$\cdots \rightarrow C_1 \rightarrow C_0 \rightarrow D_0 \rightarrow D_1 \rightarrow \cdots$$

with $C_i \in \mathcal{C}$, $D_i \in \mathcal{D}$ and $M = \text{Im}(C_0 \rightarrow D_0)$. A complete $\mathcal{C}$ -resolution will mean a complete $(\mathcal{C}, \mathcal{C})$ -resolution.

A degreewise finite projective resolution is a projective resolution (that is, an exact left $\mathcal{P}(R)$ -resolution) in which all the components are finitely generated.

For any class of left R -modules $\mathcal{C}$ we use the following standard notation:

$$\begin{aligned} \mathcal{C}^{\perp_1} &= \{M \in R\text{-Mod} ; \text{Ext}_R^1(C, M) = 0, \forall C \in \mathcal{C}\}, \\ \mathcal{C}^{\perp} &= \{M \in R\text{-Mod} ; \text{Ext}_R^i(C, M) = 0, \forall C \in \mathcal{C}, \forall i \geq 1\}. \end{aligned}$$

Similarly, ${}^{\perp_1}\mathcal{C}$ and ${}^{\perp}\mathcal{C}$ are defined.

Recall that if $\mathcal{F}$ is any class of modules, an $\mathcal{F}$ -precover of a module M is a morphism $\varphi : F \rightarrow M$ with F in the class $\mathcal{F}$, such that for any other module F' in the class $\mathcal{F}$, the map $\text{Hom}(F', \varphi) : \text{Hom}(F', F) \rightarrow \text{Hom}(F', M)$ is surjective. The morphism φ is said to be a special $\mathcal{F}$ -precover provided that φ is surjective and $\ker(\varphi)$ holds in the class $\mathcal{F}^{\perp_1}$. And φ is an $\mathcal{F}$ -cover if $\text{Hom}(F', \varphi)(f) = \varphi \Rightarrow f$ is an automorphism. Preenvelopes, special preenvelopes and envelopes are defined dually.

A class of modules $\mathcal{F}$ is said to be precovering if every module has an $\mathcal{F}$ -precover. Analogously, special precovering, covering, preenveloping, special preenveloping and enveloping classes are defined.

A cotorsion theory is defined as a pair of classes $(\mathcal{C}, \mathcal{D})$ such that $\mathcal{C}^{\perp_1} = \mathcal{D}$ and ${}^{\perp_1}\mathcal{D} = \mathcal{C}$. The cotorsion theory is as follows:

- Cogenerated by a set if there is a set X such that $X^{\perp_1} = \mathcal{D}$.
- Complete if any left R -module has a special $\mathcal{D}$ -preenvelope (or equivalently, any left R -module has a special $\mathcal{C}$ -precover).
- Perfect if any left R -module has a $\mathcal{C}$ -cover and a $\mathcal{D}$ -envelope.
- Hereditary if $\mathcal{C}^{\perp_1} = \mathcal{C}^{\perp}$ [16, Definition 1.2.10].

Associated with every bimodule ${}_R C_S$ there are two classes: the Auslander ($\mathcal{A}_C(S)$) and the Bass ($\mathcal{B}_C(R)$) classes (also called Foxby classes), defined as follows:

$\mathcal{A}_C(S)$ is the class of all left S -modules M satisfying:

- A1. $\text{Tor}_{\geq 1}^S(C, M) = 0$.
- A2. $\text{Ext}_R^{\geq 1}(C, C \otimes_S M) = 0$.
- A3. The canonical map $\mu_M : M \rightarrow \text{Hom}_R(C, C \otimes_S M)$ is an isomorphism of S -modules.

$\mathcal{B}_C(R)$ consists of all left R -modules N satisfying:

- B1. $\text{Ext}_R^{\geq 1}(C, N) = 0$.
- B2. $\text{Tor}_{\geq 1}^S(C, \text{Hom}_R(C, N)) = 0$.
- B3. The canonical map $\nu_N : C \otimes_S \text{Hom}_R(C, N) \rightarrow N$ is an isomorphism of R -modules.

So we see that the functors $\text{Hom}_R(C, -)$ and $C \otimes_S -$ establish an equivalence between the subcategories $\mathcal{A}_C(S)$ and $\mathcal{B}_C(R)$.

Of course the classes $\mathcal{A}_C(R)$ and $\mathcal{B}_C(S)$ in $\text{Mod-}R$ and in $\text{Mod-}S$, respectively, can also be considered and they are in a situation of equivalence as well:

$\mathcal{A}_C(R)$ is the class of all right R -modules M satisfying:

$\text{Tor}_{\geq 1}^R(M, C) = 0$, $\text{Ext}_S^{\geq 1}(C, M \otimes_R C) = 0$ and the canonical map

$$\mu_M : M \rightarrow \text{Hom}_S(C, M \otimes_R C)$$

is an isomorphism of R -modules.

$\mathcal{B}_C(S)$ consists of all right S -modules N satisfying:

$\text{Ext}_S^{\geq 1}(C, N) = 0$, $\text{Tor}_{\geq 1}^R(\text{Hom}_S(C, N), C) = 0$ and the canonical map

$$\nu_N : \text{Hom}_S(C, N) \otimes_R C \rightarrow N$$

is an isomorphism of S -modules.

We now recall some concepts that will be needed throughout the paper.

Definition 2.1. [11] A left R -module C is said to be weakly Wakamatsu tilting (w-tilting for short) if it has the following two properties:

- (1) $\text{Ext}_R^{\geq 1}(C, C^{(I)}) = 0$ for every set I (that is, $C^{(I)} \in \text{Add}_R(C)^\perp$ for every set I).
- (2) R has an exact right $\text{Add}_R(C)$ -resolution $0 \rightarrow R \rightarrow C_0 \rightarrow C_1 \rightarrow C_2 \rightarrow \dots$

If C satisfies (1) but perhaps not (2), then C will be said to be Σ -self-orthogonal. If $\text{Ext}_R^{\geq 1}(C, C) = 0$, then C is self-orthogonal.

Dually, a left R -module U is said to be weakly cotilting (w-cotilting for short) if it satisfies the following two conditions:

- (1) $\text{Ext}_R^{\geq 1}(U^X, U) = 0$ for every set X .
- (2) There is an injective cogenerator D in $R\text{-Mod}$ with an exact left $\text{Prod}_R(U)$ -resolution $\dots \rightarrow A_1 \rightarrow A_0 \rightarrow D \rightarrow 0$.

If U satisfies (1) but possibly not (2), then U will be said to be $[\perp]$ -self-orthogonal.

We will say that a bimodule ${}_R C_S$ is a weakly Wakamatsu (co)tilting bimodule if ${}_R C$ and C_S are weakly Wakamatsu (co)tilting modules.

Definition 2.2. [3, Definition 2.1] Let C be a class of left R -modules and M be any left R -module. M is said to be Hom-faithful relative to C if:

$$C \in C \quad \text{and} \quad \text{Hom}_R(M, C) = 0 \Rightarrow C = 0.$$

When M is Hom-faithful relative to $R\text{-Mod}$ then we just say that M is Hom-faithful, and M is self-Hom-faithful if it is Hom-faithful relative to $\sigma[M]$.

On the other hand, M is said to be Hom-cofaithful relative to C if:

$$C \in C \quad \text{and} \quad \text{Hom}_R(C, M) = 0 \Rightarrow C = 0.$$

Again, M is Hom-cofaithful if M is Hom-cofaithful relative to $R\text{-Mod}$ and M is self-Hom-cofaithful if it is Hom-cofaithful relative to $\pi[M]$.

A right S -module N is said to be $\otimes_S$ -faithful relative to a class $\mathcal{D}$ of left S -modules if:

$$D \in \mathcal{D} \quad \text{and} \quad N \otimes_S D = 0 \Rightarrow D = 0.$$

N is $\otimes_S$ -faithful if N is $\otimes_S$ -faithful relative to $S\text{-Mod}$.

A module M is self-small if $\text{Hom}_R(M, M^{(I)}) \cong \text{Hom}_R(M, M)^{(I)}$ for every set I .

In what follows, E will be a fixed injective cogenerator of $R\text{-Mod}$ and $C^\vee = \text{Hom}_R(C, E)$.

Proposition 2.3. *For any bimodule ${}_R C_S$ we have that $\mathcal{A}_C(S) \subseteq \text{Cogen}_S(C^\vee)$ and that $\mathcal{B}_C(R) \subseteq \text{Gen}_R(C)$.*

Proof. Take any $M \in \mathcal{A}_C(S)$. Since E is an injective cogenerator in $R\text{-Mod}$, we know that there is a monomorphism $0 \rightarrow C \otimes_S M \rightarrow E^I$ for some index set I . Apply $\text{Hom}_R(C, -)$ to get a monomorphism of left S -modules

$$0 \rightarrow \text{Hom}_R(C, C \otimes_S M) \rightarrow \text{Hom}_R(C, E^I) \cong (C^\vee)^I.$$

But we know that $M \cong \text{Hom}_R(C, C \otimes_S M)$ by the properties of $\mathcal{A}_C(S)$ so we get $M \in \text{Cogen}_S(C^\vee)$.

On the other hand, suppose that N holds in $\mathcal{B}_C(R)$, take a free S -presentation of $\text{Hom}_R(C, N)$, $S^{(I)} \rightarrow \text{Hom}_R(C, N) \rightarrow 0$, and apply $C \otimes_S -$. We get an epimorphism

$$C^{(I)} \cong C \otimes_S S^{(I)} \rightarrow C \otimes_S \text{Hom}_R(C, N) \rightarrow 0,$$

and since $N \cong C \otimes_S \text{Hom}_R(C, N)$ we have that $N \in \text{Gen}_R(C)$. □

3 Cotorsion theories via Foxby classes

In this section, we study the properties of the different cotorsion theories arising from the Auslander and Bass classes. Namely, we will see under what conditions the pairs of classes $(\mathcal{A}_C(S), \mathcal{A}_C(S)^{\perp_1})$ and $({}^{\perp_1}\mathcal{B}_C(R), \mathcal{B}_C(R))$ are complete cotorsion theories (Theorem 3.16 and Corollary 3.12, respectively).

Moreover, we will prove that when C is w-tilting and self-small, the Bass class $\mathcal{B}_C(R)$ can be seen as the right orthogonal class of certain modules constructed from a right $\text{Add}_R(C)$ -resolution of R (Theorem 3.10).

We start by showing that the classes $\text{Add}_R(C)$ and $\text{Prod}_R(C^\vee)$ can be thought of, under our standard conditions on the module C , as other classes from which several interesting properties can be easily deduced. This identification will be very useful throughout the paper.

It is known (see [17]) that when R is commutative and C is semidualizing, the classes $\text{Add}_R(C)$ and $C \otimes_S \mathcal{P}(S)$ coincide. We start by extending this result to the noncommutative case and with significantly lower restrictions on C .

Proposition 3.1. *If ${}_R C$ is self-small, then $\text{Add}_R(C) = C \otimes_S \mathcal{P}(S)$.*

Proof. The class $C \otimes_S \mathcal{P}(S)$ clearly lies inside $\text{Add}_R(C)$.

Let then N be any module of $\text{Add}_R(C)$. We know there is a module T such that $N \oplus T = C^{(I)}$, so

$$\text{Hom}_R(C, N) \oplus \text{Hom}_R(C, T) \cong \text{Hom}_R(C, C^{(I)}) \cong \text{Hom}_R(C, C)^{(I)} = S^{(I)}.$$

This implies both that $\text{Hom}_R(C, N) \in \mathcal{P}(S)$ and that

$$(C \otimes_S \text{Hom}_R(C, N)) \oplus (C \otimes_S \text{Hom}_R(C, T)) \cong C^{(I)},$$

so we have two commutative diagrams

$$\begin{array}{ccccc} 0 & \longrightarrow & C \otimes_S \text{Hom}_R(C, N) & \longrightarrow & C \otimes_S \text{Hom}_R(C, C^{(I)}) \\ & & \downarrow \nu_N & & \downarrow \cong \\ 0 & \longrightarrow & N & \longrightarrow & C^{(I)} \end{array}$$

and

$$\begin{array}{ccccc} C \otimes_S \text{Hom}_R(C, C^{(I)}) & \longrightarrow & C \otimes_S \text{Hom}_R(C, N) & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \nu_N & & \\ C^{(I)} & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

which show that $N \cong C \otimes_S \text{Hom}_R(C, N)$. $\square$

As for the classes $\text{Hom}_R(C, I(R))$ and $\text{Prod}_S(C^\vee)$ we have the following.

Proposition 3.2. *Let S be any ring, C be a self-orthogonal right S -module admitting a degreewise finite projective resolution and $R = \text{End}_S(C)$. Then, $\text{Prod}_S(C^\vee) = \text{Hom}_R(C, I(R))$.*

Proof. The inclusion $\text{Hom}_R(C, I(R)) \subseteq \text{Prod}_S(C^\vee)$ clearly holds without any assumption on C .

On the other hand, C_S admitting a degreewise finite projective resolution makes $\mathcal{A}_C(S)$ closed under direct products, and by [3, Theorem 2.5] we have $I(R) \subseteq \mathcal{B}_C(R)$. Hence, $C^\vee \in \mathcal{A}_C(S)$ and so $\text{Prod}_S(C^\vee) \subseteq \mathcal{A}_C(S)$.

If we now let $L \in \text{Prod}_S(C^\vee)$, then $L \oplus K = (C^\vee)^J \cong \text{Hom}_R(C, E^J)$ for some $K \in S\text{-Mod}$, and then we have that

$$(C \otimes_S L) \oplus (C \otimes_S K) \cong C \otimes_S \text{Hom}_R(C, E^J) \cong E^J$$

($I(R) \subseteq \mathcal{B}_C(R)$). Hence, $\text{Hom}_R(C, C \otimes_S L) \oplus \text{Hom}_R(C, C \otimes_S K) \cong \text{Hom}_R(C, E^J)$ and therefore $L \in \text{Hom}_R(C, I(R))$ since $L \cong \text{Hom}_R(C, C \otimes_S L)$. $\square$

There exist in the literature a number of results providing information about when the image of a (pre)cover or a (pre)envelope by means of a functor in an adjoint situation is again a (pre)cover or a (pre)envelope. And surely there are solutions to the question of whether or not the “image” of a (pre)covering or (pre)enveloping class is (pre)covering or (pre)enveloping. However, we have not found them and so we give below two results that may shed some light on this problem and that will be useful later in the paper.

Theorem 3.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be two abelian categories and $F : \mathcal{C} \rightarrow \mathcal{D}$ be a left adjoint functor of $G : \mathcal{D} \rightarrow \mathcal{C}$. Then, the following statements hold:*

- (1) *If $\mathcal{Y}$ is a (pre)enveloping subcategory of $\mathcal{D}$, then $G(\mathcal{Y})$ is a (pre)enveloping subcategory of $\mathcal{C}$.*
- (2) *If $\mathcal{X}$ is a (pre)covering subcategory of $\mathcal{C}$, then $F(\mathcal{X})$ is a (pre)covering subcategory of $\mathcal{D}$.*

Proof. We only prove 1. since 2. is just a dualization.

Assume that $\mathcal{Y}$ is preenveloping. Given any object A of $\mathcal{C}$ let $\varphi : F(A) \rightarrow Y$ be a $\mathcal{Y}$ -preenvelope. Let us prove that if $\mu_A : A \rightarrow GF(A)$ is the unit of the adjunction in A , then $G(\varphi)\mu_A : A \rightarrow G(Y)$ is a $G(\mathcal{Y})$ -preenvelope.

Take any morphism $f : A \rightarrow G(Y')$ for any object Y' of $\mathcal{Y}$, call $v : FG \rightarrow 1_{\mathcal{C}}$ the co-unit of (F, G) and let $\eta : \text{Hom}_{\mathcal{C}}(-, G(-)) \rightarrow \text{Hom}_{\mathcal{D}}(F(-), -)$ be the natural equivalence provided by the adjunction. We have the situation

$$\begin{array}{ccccc}
 F(A) & \xrightarrow{F(\eta_{A,Y}^{-1}(\varphi))} & FG(Y) & \xrightarrow{\nu_Y} & Y \\
 \downarrow F(f) & & & & \\
 FG(Y) & & & & \\
 \downarrow \nu_Y & & & & \\
 Y & & & &
 \end{array}$$

where $\eta_{A,Y}^{-1}(\varphi) = G(\varphi)\mu_A$ by the properties of the unit of the adjunction (F, G) .

Now, using once more the properties of the unit and the co-unit of the adjunction we have

$$\nu_Y F(\eta_{A,Y}^{-1}(\varphi)) = \nu_Y F(G(\varphi)\mu_A) = \eta_{A,Y}(G(\varphi)\mu_A) = \eta_{A,Y}(\eta_{A,Y}^{-1}(\varphi)) = \varphi$$

and since φ is a $\mathcal{Y}$ -preenvelope, there is $\omega : Y \rightarrow Y'$ with $\omega\varphi = \nu_Y F(f)$. Applying now G we obtain $G(\omega\varphi) = G(\omega)G(\varphi) = G(\nu_Y F(f))$. But again, using the properties of μ_A and ν_Y we get

$$G(\nu_Y F(f))\mu_A = \eta_{A,Y'}^{-1}(\nu_Y F(f)) = \eta_{A,Y'}^{-1}(\eta_{A,Y'}(f)) = f$$

so we see that $G(\omega) : GY \rightarrow GY'$ is such that $G(\omega)G(\varphi)\mu_A = f$, that is, $G(\varphi)\mu_A$ is a $G(\mathcal{Y})$ -preenvelope.

If $\mathcal{Y}$ is now enveloping and $\varphi : F(A) \rightarrow Y$ is a $\mathcal{Y}$ -envelope, we already know that $G(\varphi)\mu_A : A \rightarrow GY$ is a $G(\mathcal{Y})$ -preenvelope, so the diagram

$$\begin{array}{ccc}
 A & \xrightarrow{G(\varphi)\mu_A} & G(Y) \\
 \downarrow G(\varphi)\mu_A & \nearrow \alpha & \\
 G(Y) & &
 \end{array}$$

can be completed commutatively by some α . Then we have $\alpha G(\varphi)\mu_A = G(\varphi)\mu_A = \eta_{A,Y}^{-1}(\varphi)$.

On the other hand, since $\eta_{A,Y}(\alpha G(\varphi)\mu_A) : F(A) \rightarrow Y$ and φ is a $\mathcal{Y}$ -preenvelope, there exists $\beta : Y \rightarrow Y$ such that $\beta\varphi = \eta_{A,Y}(\alpha G(\varphi)\mu_A)$. Therefore, we have

$$\beta\varphi = \eta_{A,Y}(\alpha G(\varphi)\mu_A) = \eta_{A,Y}\eta_{A,Y}^{-1}(\varphi) = \varphi.$$

But φ is a $\mathcal{Y}$ -envelope so the latter implies that β must be an automorphism and then $G(\beta) = \alpha$ must be an automorphism too. $\square$

We know (see [18]) that the classes $\text{Add}_R(C)$ and $\text{Prod}_S(C^\vee)$ are always precovering and preenveloping respectively. Now, we can immediately find conditions for $\text{Prod}_S(C^\vee)$ to be enveloping and for $\text{Add}_R(C)$ to be covering.

Corollary 3.4. *Let S be any ring, C be a self-orthogonal right S -module admitting a degreewise finite projective resolution and $R = \text{End}_S(C)$. Then, $\text{Prod}_S(C^\vee)$ is enveloping.*

Proof. By Proposition 3.2, we know that $\text{Prod}_S(C^\vee) = \text{Hom}_R(C, I(R))$, so apply Theorem 3.3 to the class $I(S)$ with the adjunction $(C \otimes_S -, \text{Hom}_R(C, -))$. $\square$

Following the same type of arguments we also get the following.

Corollary 3.5. *Assume that C is Σ -self-orthogonal as a left R -module and that S is left perfect. Then, $\text{Add}_R(C)$ is covering in $R\text{-Mod}$.*

We now turn to study conditions which guarantee when a module is w-(co)tilting.

Recall that given a subcategory C of $R\text{-Mod}$, a module Q in C is said to be Ext-projective in C if $Q \in {}^\perp C$. The module Q is a generator of C if for any $M \in C$ there is an exact sequence $0 \rightarrow M' \rightarrow Q' \rightarrow M \rightarrow 0$ with $M' \in C$ and $Q' \in \text{Add}_R(Q)$. Ext-injective modules and cogenerator modules can be defined dually.

The next result follows somewhat similar arguments to those of [19, Proposition 3.6].

Proposition 3.6. *Let C be a preenveloping class of left R -modules containing the class of injective modules and with an Ext-projective generator Q of C such that $\text{Add}_R(Q) \subseteq C$. Then Q is w-tilting.*

Proof. Since Q is Ext-projective in C and $\text{Add}(Q) \subseteq C$ we immediately have that Q is Σ -self-orthogonal, so we only have to prove that R has an exact right $\text{Add}(Q)$ -resolution.

Start by choosing a C -preenvelope $g_0 : R \rightarrow T_0$ of R . Since C contains the injective modules, the injective envelope $E(R)$ of R holds in C and then the diagram

$$\begin{array}{ccc} R & \xrightarrow{g_0} & T_0 \\ \downarrow & \nearrow & \\ E(R) & & \end{array}$$

can be completed commutatively. This shows that every C -preenvelope (in particular g_0) is monic.

Now, Q is a generator in C so there is an exact sequence

$$0 \rightarrow Y_0 \rightarrow Q_0 \xrightarrow{\alpha} T_0 \rightarrow 0$$

with $Q_0 \in \text{Add}_R(Q)$ and $Y_0 \in C$. Then, any $f_0 \in \text{Hom}_R(R, Q_0)$ with $\alpha f_0 = g_0$ is a monic C -preenvelope since for any morphism $h : R \rightarrow X$ with $X \in C$ we have the situation

$$\begin{array}{ccccc} & & g_0 & & \\ & \nearrow & & \searrow & \\ R & \xrightarrow{f_0} & Q_0 & \xrightarrow{\alpha} & T_0 \\ \downarrow h & & & & \\ & & X & & \end{array}$$

But g_0 is a C -preenvelope so there is an $h_0 : T_0 \rightarrow X$ completing the diagram commutatively and then $h_0 \alpha : Q_0 \rightarrow X$ shows that f_0 is a preenvelope.

Thus, the exact sequence

$$0 \rightarrow R \xrightarrow{f_0} Q_0 \rightarrow K_0 \rightarrow 0$$

is also $\text{Hom}_R(-, C)$ -exact, and since $\text{Ext}_R^{\geq 1}(Q_0, C) = 0$ (Q is Ext-projective in C and so Q_0 is too) we get that $\text{Ext}_R^{\geq 1}(K_0, C) = 0$. This means that we can substitute R by K_0 and apply again the same argument to K_0 , and then to K_1 and so on, getting at the end an exact right $\text{Add}_R(Q)$ -resolution

$$0 \rightarrow R \rightarrow Q_0 \rightarrow Q_1 \rightarrow \dots$$

□

Dually, the following result can be easily proved.

Proposition 3.7. *Let C be a precovering class in $R\text{-Mod}$ containing the class of projective modules and with an Ext-injective cogenerator D of C such that $\text{Prod}_R(D) \subseteq C$. Then, D is w-cotilting.*

We now give results which find sufficient conditions on C for $\text{Add}_R(C)$ to be precovering in $\mathcal{B}_C(R)$ and $\text{Prod}_S(C^\vee)$ to be preenveloping in $A_C(S)$.

Proposition 3.8. *Let C be a self-small and Σ -self-orthogonal left R -module. Any module in $\mathcal{B}_C(R)$ has an epic $\text{Add}_R(C)$ -precover with kernel in $\mathcal{B}_C(R)$.*

Proof. By [11, Proposition 5.4] we have that $\mathcal{P}(S) \subseteq \mathcal{A}_C(S)$ so we can apply [20, Theorem 2.14] to get, for any $M \in \mathcal{B}_C(R)$, an epic $C \otimes_S \mathcal{P}(S)$ -precover $\gamma : L \rightarrow M$. But Proposition 3.1 says that $\text{Add}_R(C) = C \otimes_S \mathcal{P}(S)$ so we have indeed found an epic $\text{Add}_R(C)$ -precover $\gamma : L \rightarrow M$ of M .

Now, ${}_R C$ being self-small and Σ -self-orthogonal means that $\text{Add}_R(C) \subseteq \mathcal{B}_C(R)$, so indeed $L \in \mathcal{B}_C(R)$. And $L, M \in \mathcal{B}_C(R)$ implies $\text{Ext}_R^{\geq 1}(C, L) = \text{Ext}_R^{\geq 1}(C, M) = 0$, so applying $\text{Hom}_R(C, -)$ to the exact sequence

$$0 \rightarrow \ker \gamma \rightarrow L \xrightarrow{\gamma} M \rightarrow 0$$

we immediately get that $\text{Ext}_R^{\geq 1}(C, \ker \gamma) = 0$.

Similarly, the fact that M is in $\mathcal{B}_C(R)$ implies $\text{Tor}_{\geq 1}^S(C, \text{Hom}_R(C, M)) = 0$, so applying $C \otimes_S -$ to the exact sequence

$$0 \rightarrow \text{Hom}_R(C, \ker \gamma) \rightarrow \text{Hom}_R(C, L) \rightarrow \text{Hom}_R(C, M) \rightarrow 0$$

we get that $\text{Tor}_{\geq 1}^S(C, \text{Hom}_R(C, \ker \gamma)) \cong \text{Tor}_{\geq 1}^S(C, \text{Hom}_R(C, L)) = 0$ ($L \in \mathcal{B}_C(R)$).

Finally, the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & C \otimes_S \text{Hom}_R(C, \ker \gamma) & \longrightarrow & C \otimes_S \text{Hom}_R(C, L) & \longrightarrow & C \otimes_S \text{Hom}_R(C, M) \longrightarrow 0 \\ & & & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & \ker \gamma & \longrightarrow & L & \longrightarrow & M \longrightarrow 0 \end{array}$$

shows that $\ker \gamma \cong C \otimes_S \text{Hom}_R(C, \ker \gamma)$ canonically. $\square$

Proposition 3.9. Suppose C is self-orthogonal and has a degreewise finite projective resolution as a right S -module, and $\text{End}_S(C) \cong R$ canonically. Then, any $N \in \mathcal{A}_C(S)$ has a monic $\text{Prod}_S(C^\vee)$ -preenvelope $0 \rightarrow N \xrightarrow{\alpha} T$ with $\text{coker } \alpha \in \mathcal{A}_C(S)$.

Proof. By Proposition 3.2 we have that $\text{Prod}_S(C^\vee) = \text{Hom}_R(C, I(R))$, and by [3, Theorem 2.5] that $I(R) \subseteq \mathcal{B}_C(R)$, so finally we get $\text{Prod}_S(C^\vee) = \text{Hom}_R(C, I(R)) \subseteq \mathcal{A}_C(S)$. Then, applying [20, Theorem 2.11] we find a monic $\text{Prod}_S(C^\vee)$ -preenvelope $0 \rightarrow N \xrightarrow{\alpha} T$.

Now, applying $\text{Hom}_S(-, C^\vee)$ to the exact sequence

$$0 \rightarrow N \rightarrow T \rightarrow A \rightarrow 0$$

we immediately get that $\text{Ext}_S^{\geq 1}(A, C^\vee) = 0$ since $N, T \in \mathcal{A}_C(S)$. Thus, by [1, Theorem 3.2.1],

$$\text{Tor}_{\geq 1}^S(C, A)^\vee \cong \text{Ext}_S^{\geq 1}(A, C^\vee) = 0$$

and then $\text{Tor}_{\geq 1}^S(C, A) = 0$, that is, $A = \text{coker } \alpha$ satisfies A1.

But then the sequence

$$0 \rightarrow C \otimes_S N \rightarrow C \otimes_S T \rightarrow C \otimes_S A \rightarrow 0$$

is also exact, and applying $\text{Hom}_R(C, -)$ we get another short exact sequence which associated long exact sequence gives $\text{Ext}_R^{\geq 1}(C, C \otimes_S A) = 0$ (so A satisfies A2 as well), and a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_R(C, C \otimes_S N) & \longrightarrow & \text{Hom}_R(C, C \otimes_S T) & \longrightarrow & \text{Hom}_R(C, C \otimes_S A) \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & N & \longrightarrow & T & \longrightarrow & A \longrightarrow 0 \end{array}$$

which shows that $A \cong \text{Hom}_R(C, C \otimes_S A)$ canonically, that is, A satisfies A3. $\square$

We will now see that if ${}_R C$ is w -tilting, then the class $\mathcal{B}_C(R)$ can be recovered from C itself and from the cokernels of any of the exact right $\text{Add}_R(C)$ -resolutions of R , at least when ${}_R C$ is self-small.

Theorem 3.10. Let C be a self-small and w -tilting left R -module and

$$0 \rightarrow R \xrightarrow{t_0} C_0 \xrightarrow{t_1} C_1 \xrightarrow{t_2} \dots$$

be an exact right $\text{Add}_R(C)$ -resolution of R . Then, $\mathcal{B}_C(R) = (C \oplus (\oplus_{i=0}^\infty \text{coker } t_i))^\perp$.

Proof. Call $K_i = \text{coker } t_i$, $i \geq 0$, $K = C \oplus (\oplus_{i=0}^{\infty} K_i)$ and choose any $M \in \mathcal{B}_C(R)$. By Proposition 3.8, we know that M has an epic $\text{Add}_R(C)$ -precover $\gamma : L \rightarrow M$ with $\ker \gamma \in \mathcal{B}_C(R)$, so the exact sequence

$$0 \rightarrow \ker \gamma \rightarrow L \xrightarrow{\gamma} M \rightarrow 0$$

is $\text{Hom}_R(C, -)$ -exact.

Applying both $\text{Hom}_R(-, M)$ and $\text{Hom}_R(-, L)$ to the exact sequence

$$0 \rightarrow R \xrightarrow{t_0} C_0 \rightarrow K_0 \rightarrow 0$$

we get a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_R(K_0, L) & \longrightarrow & \text{Hom}_R(C_0, L) & \xrightarrow{\text{Hom}(t_0, L)} & \text{Hom}_R(R, L) \longrightarrow 0 \\ & & \downarrow \text{Hom}(K_0, \gamma) & & \downarrow \text{Hom}(C_0, \gamma) & & \downarrow \text{Hom}(R, \gamma) \\ 0 & \longrightarrow & \text{Hom}_R(K_0, M) & \longrightarrow & \text{Hom}_R(C_0, M) & \xrightarrow{\text{Hom}(t_0, M)} & \text{Hom}_R(R, M) \longrightarrow \text{Ext}^1(K_0, M) \end{array}$$

with $\text{Ext}_R^{\geq 1}(C_0, M) = 0$ (and so $\text{Ext}_R^{\geq 2}(K_0, M) = 0$) since $M \in \mathcal{B}_C(R)$.

Now, the commutativity of the diagram shows that $\text{Hom}_R(t_0, M)$ is an epimorphism so we see that $\text{Ext}_R^1(K_0, M) = 0$ and so that $\text{Ext}_R^{\geq 1}(K_0, H) = 0$ for all H in $\mathcal{B}_C(R)$.

Now, for any $j \geq 0$ we have that $\text{Ext}_R^{\geq 1}(C_{j+1}, \ker \gamma) = 0$, so the resulting long exact sequence after applying $\text{Hom}_R(-, \ker \gamma)$ to the exact sequence

$$0 \rightarrow K_j \rightarrow C_{j+1} \rightarrow K_{j+1} \rightarrow 0$$

shows that $\text{Ext}_R^i(K_j, \ker \gamma) \cong \text{Ext}_R^{i+1}(K_{j+1}, \ker \gamma) \forall i \geq 1$.

On the other hand, applying $\text{Hom}_R(K_j, -)$ to the exact sequence

$$0 \rightarrow \ker \gamma \rightarrow L \xrightarrow{\gamma} M \rightarrow 0,$$

and noting that $\text{Ext}_R^{\geq 1}(K_j, L) = 0$ ($0 \rightarrow R \rightarrow C_0 \rightarrow C_1 \rightarrow \dots$ is a right $\text{Add}_R(C)$ -resolution) we immediately see that $\text{Ext}_R^i(K_j, M) \cong \text{Ext}_R^{i+1}(K_j, \ker \gamma)$ for every $i \geq 1$ and every $j \geq 0$.

Therefore, we get

$$\text{Ext}_R^i(K_j, \ker \gamma) \cong \text{Ext}_R^i(K_{j+1}, M) \forall i \geq 1, \forall j \geq 0.$$

But we know that $\text{Ext}_R^{\geq 1}(K_0, \ker \gamma) = 0$ so we have that $\text{Ext}_R^{\geq 1}(K_1, M) = 0$ and then, by induction, we see that $\text{Ext}_R^{\geq 1}(K_j, M) = 0 \forall j \geq 0$ and so that $\text{Ext}_R^{\geq 1}(K, M) = 0$.

Conversely, $\mathcal{P}(S) \subseteq \mathcal{A}_C(S)$ [11, Proposition 5.4] and any $M \in K^\perp$ is in $C^\perp$, so by [20, Theorem 2.14] it suffices to find an exact left $C \otimes_S \mathcal{P}(S)$ -resolution of M , which, by Proposition 3.1, is equivalent to finding an exact left $\text{Add}_R(C)$ -resolution of M .

Let $R^{(X)} \rightarrow M$ be an epimorphism, consider the morphism $\oplus t_0 : R^{(X)} \rightarrow C_0^{(X)}$ and construct the pushout

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & D & \longrightarrow & R^{(X)} & \longrightarrow & M \longrightarrow 0 \\ & & \parallel & & \downarrow \oplus t_0 & & \downarrow \\ 0 & \longrightarrow & D & \longrightarrow & C_0^{(X)} & \longrightarrow & T \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & K_0^{(X)} & = & K_0^{(X)} \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

Since $\text{Ext}_R^1(K_0^{(X)}, M) \cong \text{Ext}_R^1(K_0, M)^X = 0$, the right-hand column splits and hence there is an epimorphism $C_0^{(X)} \rightarrow M$ and this means that every $\text{Add}_R(C)$ -precover $C' \rightarrow M$ of M (which exists by [18, Corollary 3.7]) is epic.

Now, applying $\text{Hom}_R(K_i, -)$ to the exact sequence

$$0 \rightarrow B \rightarrow C' \rightarrow M \rightarrow 0$$

we get that $0 = \text{Ext}_R^j(K_i, M) \cong \text{Ext}_R^{j+1}(K_i, B) \forall j \geq 1, \forall i \geq 0$, and applying $\text{Hom}_R(C, -)$ to the same sequence we obtain that $B \in C^\perp$. But then, the long exact sequence associated with

$$0 \rightarrow K_i \rightarrow C_{i+1} \rightarrow K_{i+1} \rightarrow 0$$

after applying $\text{Hom}_R(-, B)$ shows that $\text{Ext}_R^j(K_i, B) \cong \text{Ext}_R^{j+1}(K_{i+1}, B) = 0 \forall j \geq 1, i \geq 0$, that is, $B \in K^\perp$. Repeating this procedure we find an exact left $\text{Add}_R(C)$ -resolution of M . $\square$

Now let us show that when C satisfies the conditions of Theorem 3.10, the class $\mathcal{B}_C(R)$ is the right-hand class of a cotorsion theory which is in fact complete and hereditary. We recall first what a (co)resolving class of modules is.

Definition 3.11. A class of modules C is said to be resolving provided that it contains the class of projective modules and is closed under extensions and kernels of epimorphisms, that is, if

$$0 \rightarrow K \rightarrow M \rightarrow N \rightarrow 0$$

is a short exact sequence and $K \in C$, then $M \in C \Leftrightarrow N \in C$.

Dually, C is coresolving if it contains the class of injective modules and is closed under extensions and cokernels of monomorphisms (if $K \in C$, then $M \in C \Leftrightarrow N \in C$).

Corollary 3.12. Let ${}_R C$ be self-small and w -tilting. Then, $({}^\perp \mathcal{B}_C(R), \mathcal{B}_C(R))$ is a complete and hereditary cotorsion theory. In particular, $\mathcal{B}_C(R)$ is special preenveloping and ${}^\perp \mathcal{B}_C(R)$ is special precovering.

Proof. Call $K = C \oplus (\oplus_{i=0}^\infty \text{coker } t_i)$ the module of Theorem 3.10, choose any projective representation of K ,

$$\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow K \rightarrow 0,$$

call $L_i = \ker(P_{i-1} \rightarrow P_{i-2})$ (where $P_{-1} = K$) and consider the module $X = K \oplus (\oplus L_i)$. We claim that $\mathcal{B}_C(R) = X^\perp$.

For any left R -module A we have

$$\text{Ext}_R^1(X, A) \cong \text{Ext}_R^1(K, A) \times \left(\prod \text{Ext}_R^1(L_i, A) \right),$$

and of course $\text{Ext}_R^1(L_i, A) \cong \text{Ext}_R^{i+1}(K, A)$, so we get

$$\text{Ext}_R^1(X, A) \cong \prod_{i \geq 1} \text{Ext}_R^i(K, A).$$

Therefore, we see that $X^\perp = K^\perp = \mathcal{B}_C(R)$ ($K^\perp = \mathcal{B}_C(R)$ by Theorem 3.10), that is, the pair $({}^\perp \mathcal{B}_C(R), \mathcal{B}_C(R))$ is cogenerated by the set X . This clearly implies that it is indeed a cotorsion theory since

$$({}^\perp \mathcal{B}_C(R))^\perp = ({}^\perp (X^\perp))^\perp = X^\perp = \mathcal{B}_C(R).$$

Now, it is well known that when a cotorsion theory is cogenerated by a set, then it is complete (see for instance [1, Theorem 7.4.1]), and $\mathcal{B}_C(R)$ is closed under extensions and cokernels of monomorphisms [11, Proposition 5.6] so it is indeed a coresolving class, and applying [16, Theorem 1.2.10] we get that $({}^\perp \mathcal{B}_C(R), \mathcal{B}_C(R))$ is also hereditary. $\square$

We now recall the definition of duality pairs.

Definition 3.13. [21, Definition 2.1] A left duality pair over R is a pair $(\mathcal{M}, C)$, where $\mathcal{M}$ is a class of left R -modules and C is a class of right R -modules, subject to the following conditions:

- (1) For a left R -module M , one has $M \in \mathcal{M}$ if and only if $M^+ \in C$.
- (2) C is closed under direct summands and finite direct sums.

Analogously, right duality pairs with $\mathcal{M}$ in $\text{Mod-}R$ and C is $R\text{-Mod}$ are defined.

A left duality pair $(\mathcal{M}, C)$ is called (co)product-closed if the class $\mathcal{M}$ is closed under (co)products in the category of all left R -modules.

A left duality pair $(\mathcal{M}, C)$ is called perfect if it is coproduct-closed, $\mathcal{M}$ is closed under extensions, and R belongs to $\mathcal{M}$.

We now want to check when $\mathcal{A}_C(R)$, $\mathcal{A}_C(S)$, $\mathcal{B}_C(R)$ and $\mathcal{B}_C(S)$ are covering, and indeed, which of these pairs behave as a perfect cotorsion theory situation.

We start our way with the next result, whose proof follows the arguments of [6, Proposition 7.2].

Proposition 3.14. Assume that ${}_RC$ has a degreewise finite projective resolution. The following statements hold:

- (a) $M \in \mathcal{A}_C(S) \Leftrightarrow M^+ \in \mathcal{B}_C(S)$.
- (b) $M \in \mathcal{B}_C(R) \Leftrightarrow M^+ \in \mathcal{A}_C(R)$.

Now, let S be any ring, C be a right S -module admitting a degreewise finite projective resolution and $R = \text{End}_S(C)$. The following statements hold:

- (c) $M \in \mathcal{A}_C(R) \Leftrightarrow M^+ \in \mathcal{B}_C(R)$.
- (d) $M \in \mathcal{B}_C(S) \Leftrightarrow M^+ \in \mathcal{A}_C(S)$.

Corollary 3.15. Let ${}_RC$ have a degreewise finite projective resolution. The following statements hold:

- (a) $(\mathcal{A}_C(S), \mathcal{B}_C(S))$ is a coproduct-closed left duality pair over S and $\mathcal{A}_C(S)$ is closed under extensions. If, in addition, ${}_RC$ is Σ -self-orthogonal, then the left duality pair is perfect.
- (b) $(\mathcal{B}_C(R), \mathcal{A}_C(R))$ is a coproduct-closed left duality pair over R .

If, in addition, C_S has a degreewise finite projective resolution, then the pairs $(\mathcal{A}_C(S), \mathcal{B}_C(S))$ and $(\mathcal{B}_C(R), \mathcal{A}_C(R))$ are also product-closed.

On the other hand, if C_S has a degreewise finite projective resolution, then we have:

- (c) $(\mathcal{A}_C(R), \mathcal{B}_C(R))$ is a coproduct-closed right duality pair over R and $\mathcal{A}_C(R)$ is closed under extensions. Furthermore, if C_S is Σ -self-orthogonal, then the pair is a perfect right duality pair.
- (d) $(\mathcal{B}_C(S), \mathcal{A}_C(S))$ is a coproduct-closed right duality pair over S .

If, in addition, ${}_RC$ has a degreewise finite projective resolution, then the pairs $(\mathcal{A}_C(R), \mathcal{B}_C(R))$ and $(\mathcal{B}_C(S), \mathcal{A}_C(S))$ are also product-closed.

Proof. Items (a) and (b): by Proposition 3.14 we see that the two pairs are left duality pairs. Of course both are coproduct-closed as well (and product closed when C_S has a degreewise finite projective resolution), and by [11, Proposition 5.4] $\mathcal{A}_C(S)$ is closed under extensions. If ${}_RC$ is Σ -self-orthogonal, then $\mathcal{A}_C(S)$ contains all projective S -modules again by [11, Proposition 5.4].

Items (c) and (d) are proved dually. □

With the tools developed so far we can answer several of the questions raised in Section 1. This answer generalizes [21, Theorem 3.2] and improves [3, Theorem 4.4].

Theorem 3.16.

- (i) Let ${}_RC$ has a degreewise finite projective resolution. The following assertions hold.
 - (a) The classes $\mathcal{A}_C(S)$ and $\mathcal{B}_C(R)$ are both covering.
 - (b) If, furthermore, ${}_RC$ is Σ -self-orthogonal, then $(\mathcal{A}_C(S), \mathcal{A}_C(S)^{\perp_1})$ is a perfect cotorsion theory.
- (ii) Assume now that C_S has a degreewise finite projective resolution. The following statements are true:
 - (a) The classes $\mathcal{A}_C(R)$ and $\mathcal{B}_C(S)$ are covering.
 - (b) If, in addition, C_S is Σ -self-orthogonal, then $(\mathcal{A}_C(R), \mathcal{A}_C(R)^{\perp_1})$ is a perfect cotorsion theory.

(iii) If both ${}_R C$ and C_S have degreewise finite projective resolutions, then the classes $\mathcal{A}_C(S)$, $\mathcal{B}_C(R)$, $\mathcal{A}_C(R)$ and $\mathcal{B}_C(S)$ are all preenveloping.

Proof.

- (i) By Corollary 3.15 $(\mathcal{A}_C(S), \mathcal{B}_C(S))$ and $(\mathcal{B}_C(R), \mathcal{A}_C(R))$ are coproduct-closed left duality pairs, so by [21, Theorem 3.1, (b)] both $\mathcal{A}_C(S)$ and $\mathcal{B}_C(R)$ are covering.
 Now, if ${}_R C$ is Σ -self-orthogonal, then $(\mathcal{A}_C(S), \mathcal{B}_C(S))$ is a perfect duality pair [11, Proposition 5.4] so, again by [21, Theorem 3.1], $(\mathcal{A}_C(S), \mathcal{A}_C(S)^\perp)$ is a perfect cotorsion theory.
 (ii) Dual to (i).
 (iii) If both ${}_R C$ and C_S have degreewise finite projective resolutions, then all the classes $\mathcal{A}_C(S)$, $\mathcal{B}_C(R)$, $\mathcal{A}_C(R)$ and $\mathcal{B}_C(S)$ are closed under products and then [21, Theorem 3.1, (a)] says that all of them are preenveloping. $\square$

Corollary 3.17. Suppose that ${}_R C$ has a degreewise finite projective resolution. Then, the pair $(\mathcal{A}_C(S), \mathcal{A}_C(S)^\perp)$ is a hereditary cotorsion theory if and only if ${}_R C$ is Σ -self-orthogonal.

Proof. If $(\mathcal{A}_C(S), \mathcal{A}_C(S)^\perp)$ is a cotorsion theory, then $\mathcal{A}_C(S) = {}^\perp(\mathcal{A}_C(S)^\perp)$ and so $\mathcal{P}(S) \subseteq \mathcal{A}_C(S)$. Then, by [11, Proposition 5.4, 1.] we get that ${}_R C$ is Σ -self-orthogonal.

Conversely, if ${}_R C$ is Σ -self-orthogonal, then $(\mathcal{A}_C(S), \mathcal{A}_C(S)^\perp)$ is a cotorsion theory by Theorem 3.16 and so again $\mathcal{P}(S) \subseteq \mathcal{A}_C(S)$, and if $A \in \mathcal{A}_C(S)$ is arbitrary and

$$0 \rightarrow K \rightarrow P \rightarrow A \rightarrow 0$$

is exact with P a projective left S -module, then $K \in \mathcal{A}_C(S)$ [11, Proposition 5.4, 4.]. Therefore, for any $M \in \mathcal{A}_C(S)^\perp$, the exactness of the sequence

$$\cdots \rightarrow \text{Ext}_S^{i+1}(A, M) \rightarrow \text{Ext}_S^{i+1}(P, M) \rightarrow \text{Ext}_S^{i+1}(K, M) \rightarrow \cdots$$

shows, using induction, that $\text{Ext}_S^{\geq 1}(A, M) = 0$ and so that $M \in \mathcal{A}_C(S)^\perp$. $\square$

Using Theorem 3.16 and Proposition 3.6 we can see what conditions we need on C and $\mathcal{B}_C(R)$ to ensure that ${}_R C$ is w-tilting.

Corollary 3.18. Suppose that ${}_R C$ is Σ -self-orthogonal, that C_S is self-orthogonal, that $R \cong \text{End}_S(C)$ canonically and that both ${}_R C$ and C_S have degreewise finite projective resolutions. Then, $\mathcal{B}_C(R)$ is a coresolving, preenveloping and covering class, ${}_R C$ is an Ext-projective generator in $\mathcal{B}_C(R)$ and $\text{Add}_R(C) \subseteq \mathcal{B}_C(R)$. As a consequence, ${}_R C$ is w-tilting.

Proof. The class $\mathcal{B}_C(R)$ is preenveloping and covering (Theorem 3.16), coresolving [3, Theorem 2.5], $\text{Add}_R(C) \subseteq \mathcal{B}_C(R)$ [11, Proposition 5.6] and of course ${}_R C$ is Ext-projective in $\mathcal{B}_C(R)$ by definition.

Now, any $B \in \mathcal{B}_C(R)$ has an epic $\text{Add}_R(C)$ -precover with kernel in $\mathcal{B}_C(R)$ (Proposition 3.8), so C is a generator in $\mathcal{B}_C(R)$.

Finally, applying Proposition 3.6 we get that ${}_R C$ is w-tilting. $\square$

In a similar way we have the following result.

Corollary 3.19. Suppose that ${}_R C$ is Σ -self-orthogonal and Hom-faithful, that C_S is self-orthogonal, that $R \cong \text{End}_S(C)$ canonically and that both ${}_R C$ and C_S have degreewise finite projective resolutions. Then, $\mathcal{A}_C(S)$ is a resolving, covering and preenveloping class, $C^\vee$ is an Ext-injective cogenerator in $\mathcal{A}_C(S)$ and $\text{Prod}_S(C^\vee) \subseteq \mathcal{A}_C(S)$. As a consequence, $C^\vee$ is w-cotilting.

Proof. Theorem 3.16 shows that $\mathcal{A}_C(S)$ is both covering and preenveloping, $\mathcal{A}_C(S)$ is resolving by [11, Proposition 5.4], $C^\vee$ is a cogenerator of $\mathcal{A}_C(S)$ by Proposition 3.9, and it is Ext-injective by [1, Theorem 3.2.1], and $\text{Prod}_S(C^\vee) \subseteq \mathcal{A}_C(S)$ by [3, Theorem 2.5]. Therefore, $C^\vee$ is w-cotilting by Proposition 3.7. $\square$

4 Gorenstein C -injective preenvelopes and Gorenstein C -projective precovers. Applications

In this section, we address another interesting question: when are the classes of syzygies of complete $\text{Add}_R(C)$ -resolutions and complete $\text{Prod}_S(C^\vee)$ -resolutions, that is the classes $C\text{-GP}(R)$ and $C\text{-GI}(S)$ of Gorenstein C -projective modules and Gorenstein C -injective modules, precovering and preenveloping respectively? The answer to this question will be given in Proposition 4.7 and Corollary 4.4.

As applications we will see that, dually to Theorem 3.10, the Auslander class can also be recovered from a right $\text{Add}_R(C)$ resolution of R when C is w-tilting (Corollary 4.8), and we will also be able to compute the common parts of the Bass class with its first left-orthogonal class, and of the Auslander class with its first right-orthogonal class, obtaining that these intersections are precisely the classes of $\mathcal{P}_C$ -projective and $\mathcal{I}_C$ -injective modules, respectively (Proposition 4.9).

We start by defining all these C -relative concepts.

Definition 4.1. A left R -module is said to be $\mathcal{P}_C$ -projective if it is isomorphic to some $C \otimes_S P$ (P projective in $S\text{-Mod}$).

A module M is G_C -projective if it has a complete $(\mathcal{P}(R), C \otimes_S \mathcal{P}(S))$ -complete resolution. The class of all G_C -projective modules will be denoted by $G_C P(R)$.

M is said to have G_C -projective dimension ($\mathcal{P}_C$ -projective dimension) less than or equal to n , $G_C\text{-pd}(M) = n$ ($\mathcal{P}_C\text{-pd}(M) = n$), if there is an exact sequence

$$0 \rightarrow G_n \rightarrow \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$$

with $G_i \in G_C P(R)$ ($G_i \in \mathcal{P}_C(R)$) for every $i \in \{0, \dots, n\}$. If n is the least nonnegative integer for which such a sequence exists, then $G_C\text{-pd}(M) = n$ ($\mathcal{P}_C\text{-pd}(M) = n$), and if there is no such n , then $G_C\text{-pd}(M) = \infty$ ($\mathcal{P}_C\text{-pd}(M) = \infty$).

A right R -module M is said to be $\mathcal{I}_C$ -injective if it is isomorphic to $\text{Hom}_S(C, E)$ for some injective right S -module E , and it is G_C -injective if it admits a complete $(\text{Hom}_S(C, \mathcal{I}(S)), \mathcal{I}(R))$ -resolution (where $\mathcal{I}(R)$ denotes the class of injective right R -modules). We use $G_C \mathcal{I}(R)$ to denote the class of all G_C -injective R -modules.

The concepts of $\mathcal{I}_C$ -injective and of G_C -injective dimensions can be defined in a natural way now.

We denote by $C\text{-GP}(R)$ the class of all left R -modules with a complete $\text{Add}_R(C)$ -resolution, and by $C\text{-GI}(R)$ that of all left S -modules having a complete $\text{Prod}_S(C^\vee)$ -resolution. We call these modules Gorenstein C -projective and Gorenstein C -injective, respectively.

It is worth noting that when ${}_R C$ is self-small, the class of $\mathcal{P}_C$ -projective modules is nothing but $\text{Add}_R(C)$ (Proposition 3.1), so G_C -projective modules are zero syzygies of exact and $\text{Hom}_R(-, \text{Add}_R(C))$ -exact complexes

$$\cdots \rightarrow P^1 \rightarrow P^0 \rightarrow A_0 \rightarrow A_1 \rightarrow \cdots$$

with all P^i projective and all A_i in $\text{Add}_R(C)$.

Similarly, if C_S is self-orthogonal with a degreewise finite projective resolution and $R = \text{End}_S(C)$, then $\mathcal{I}_C$ -injective and G_C -injective right modules can be defined in terms of $\text{Prod}_S(C^\vee)$.

We start with the case of complete $\text{Prod}_S(C^\vee)$ -resolutions. We will need the following facts.

Lemma 4.2. Let $F : C \rightarrow \mathcal{D}$ be a left adjoint functor of a functor $G : \mathcal{D} \rightarrow C$, and μ, ν be the unit and the co-unit of the adjunction, respectively. If $A, B \in \text{Ob}(\mathcal{D})$ are two objects such that ν_A and $\mu_{G(B)}$ are both isomorphisms, then, for any morphism $f : G(A) \rightarrow G(B)$, it holds that $G(\nu_B F(f) \nu_A^{-1}) = f$.

Proof. We know that $G(\nu_A) \mu_{G(A)} = 1_{G(A)}$, so ν_A being an isomorphism implies that $\mu_{G(A)}$ is also an isomorphism and $G(\nu_A) = \mu_{G(A)}^{-1}$. Similarly, $G(\nu_B) = \mu_{G(B)}^{-1}$.

Thus, $G(\nu_B F(f) \nu_A^{-1}) = \mu_{G(B)}^{-1} G F(f) \mu_{G(A)}$.

Now, the naturality of μ says that the diagram

$$\begin{array}{ccc} G(A) & \xrightarrow{f} & G(B) \\ \mu_{G(A)} \downarrow & & \downarrow \mu_{G(B)} \\ GF(A) & \xrightarrow{GF(f)} & GF(B) \end{array}$$

is commutative, that is, $\mu_{G(B)}^{-1} GF(f) \mu_{G(A)} = f$ and we are done. $\square$

Proposition 4.3. *Let ${}_R C$ be Hom-faithful and suppose that C_S is $\otimes_S$ -faithful, has a degreewise finite projective resolution and $C^\vee \in \mathcal{A}_C(S)$. Then*

$$C - GI(S) = \text{Hom}_R(C, GI(R) \cap \mathcal{B}_C(R)).$$

Proof. We start by noting that by [11, Theorem 5.9] we have that $R \cong \text{End}_S(C)$ and that C_S is self-orthogonal. Then, by Proposition 3.2, $\text{Prod}_S(C^\vee) = \text{Hom}_R(C, I(R))$. Therefore, if $M \in C - GI(S)$ we can find a complete $\text{Hom}_R(C, I(R))$ -resolution of M

$$\mathbf{X}_\bullet : \cdots \rightarrow \text{Hom}_R(C, E_{-1}) \rightarrow \text{Hom}_R(C, E_0) \rightarrow \text{Hom}_R(C, E_1) \rightarrow \cdots.$$

Call $K_i = \ker(X_i \rightarrow X_{i+1})$ (so $M = K_1$). It is clear that $K_i \in C - GI(S) \forall i$ so, again by [11, Theorem 5.9], $K_i \in \mathcal{A}_C(S) \forall i$ and then $C \otimes_S K_i \in \mathcal{B}_C(R) \forall i$. In particular, $C \otimes_S M \in \mathcal{B}_C(R)$. But since $M \in \mathcal{A}_C(S)$ we also have that $M \cong \text{Hom}_R(C, C \otimes_S M)$, so if we prove that $C \otimes_S M \in GI(R)$, we will have the left to right inclusion.

Applying $C \otimes_S -$ to the resolution $\mathbf{X}_\bullet$ we get the complex

$$C \otimes_S \mathbf{X}_\bullet : \cdots \rightarrow E_{-1} \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots$$

since $I(R) \subseteq \mathcal{B}_C(R)$ [11, Theorem 5.9].

Now, for every i we have that the sequence

$$0 \rightarrow C \otimes_S K_i \rightarrow E_i \rightarrow C \otimes_S K_{i+1} \rightarrow 0$$

is exact since $K_{i+1} \in \mathcal{A}_C(S)$ (so $\text{Tor}_{\geq 1}^S(C, K_{i+1}) = 0$), and this means that $C \otimes_S \mathbf{X}_\bullet$ is exact.

We then see that to have $C \otimes_S M \in GI(R)$ we only need to check that $C \otimes_S \mathbf{X}_\bullet$ is $\text{Hom}_R(I(R), -)$ -exact.

Choose any injective left R -module E and any morphism $f : E \rightarrow C \otimes_S K_{i+1}$. We want to complete the diagram

$$\begin{array}{ccccccc} & & & & E & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & C \otimes_S K_i & \longrightarrow & E_i & \longrightarrow & C \otimes_S K_{i+1} \longrightarrow 0 \end{array}$$

If we apply $\text{Hom}_R(C, -)$ we get the following diagram with our original sequence

$$\begin{array}{ccccccc} & & & & \text{Hom}_R(C, E) & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & K_i & \longrightarrow & \text{Hom}_R(C, E_i) & \longrightarrow & K_{i+1} \longrightarrow 0 \end{array}$$

since $K_i \in \mathcal{A}_C(S) \forall i$.

But $\mathbf{X}_\bullet$ is $\text{Hom}_S(\text{Hom}_R(C, I(R)), -)$ -exact, so this last diagram can be completed commutatively. Finally, apply $C \otimes_S -$ to it to get the desired completion of our first diagram.

Conversely, let $N \in GI(R) \cap \mathcal{B}_C(R)$, choose a complete injective resolution of N

$$\mathbf{E}_\bullet : \quad \cdots \longrightarrow E_{-1} \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \cdots$$

$$\begin{array}{ccc} & & \nearrow \\ & N & \\ & & \searrow \end{array}$$

and call $L_i = \ker(E_i \rightarrow E_{i+1})$. Then, $L_i \in GI(R) \forall i$, but also $L_i \in \mathcal{B}_C(R) \forall i$ by [11, Proposition 5.6], so applying $\text{Hom}_R(C, -)$ to the complex above we get another exact complex ($L_i \in \mathcal{B}_C(R) \forall i$)

$$\text{Hom}(C, \mathbf{E}_\bullet) : \cdots \text{Hom}_R(C, E_0) \longrightarrow \text{Hom}_R(C, E_1) \longrightarrow \cdots$$

$$\begin{array}{ccc} & & \nearrow \\ & \text{Hom}_R(C, N) & \\ & & \searrow \end{array}$$

It only remains to prove that $\text{Hom}_R(C, \mathbf{E}_\bullet)$ is $\text{Hom}_S(\text{Hom}_R(C, I(R)), -)$ -exact and $\text{Hom}_S(-, \text{Hom}_R(C, I(R)))$ -exact.

Let then E be any left R -module and $f : \text{Hom}_R(C, E) \rightarrow \text{Hom}_R(C, L_{i+1})$ be any morphism. We want to prove that the diagram

$$\begin{array}{ccccccc} & & & & \text{Hom}_R(C, E) & & \\ & & & & \downarrow f & & \\ 0 & \longrightarrow & \text{Hom}_R(C, L_i) & \longrightarrow & \text{Hom}_R(C, E_i) & \xrightarrow{\text{Hom}(C, \delta_i)} & \text{Hom}_R(C, L_{i+1}) \longrightarrow 0 \end{array}$$

can be completed commutatively.

Applying $C \otimes_S -$ to the diagram we get

$$\begin{array}{ccccccc} & & & & E & & \\ & & & & \downarrow \nu_{L_{i+1}}(1_C \otimes f)\nu_E^{-1} & & \\ 0 & \longrightarrow & L_i & \longrightarrow & E_i & \xrightarrow{\delta_i} & L_{i+1} \longrightarrow 0 \end{array}$$

since all L_i and all E_i are in $\mathcal{B}_C(R)$, where

$$\nu_E : C \otimes_S \text{Hom}_R(C, E) \rightarrow E$$

is the natural homomorphism given in the definition of the Bass class, that is, the co-unit of the adjunction $(C \otimes_S -, \text{Hom}_R(C, -))$.

But $\mathbf{E}_\bullet$ is $\text{Hom}_R(I(R), -)$ -exact so this last diagram can be completed commutatively by some $g : E \rightarrow E_i$. Applying again $\text{Hom}_R(C, -)$ we get $\text{Hom}_R(C, g) : \text{Hom}_R(C, E) \rightarrow \text{Hom}_R(C, E_i)$ such that

$$\text{Hom}_R(C, \delta_i)\text{Hom}_R(C, g) = \text{Hom}_R(C, \nu_{L_{i+1}}(1_C \otimes f)\nu_E^{-1}),$$

and $\text{Hom}_R(C, \nu_{L_{i+1}}(1_C \otimes f)\nu_E^{-1}) = f$ by Lemma 4.2, so we are done.

The $\text{Hom}_S(-, \text{Hom}_R(C, I(R)))$ -exactness of $\text{Hom}_R(C, \mathbf{E}_\bullet)$ is completely dual. $\square$

Corollary 4.4. Suppose that ${}_R C$ is Hom-faithful and that C_S is $\otimes$ -faithful, has a degreewise finite projective resolution and $C^\vee \in \mathcal{A}_C(S)$. If $\text{id}_R(C) < \infty$, then $C - GI(S) = \text{Hom}_R(C, GI(R))$.

If, in addition, R is left noetherian, then $C - GI(S)$ is preenveloping.

Proof. By [11, Theorem 5.9] we have $I(R) \subseteq \mathcal{B}_C(R)$, so [3, Proposition 3.2] says that $GI(R) \subseteq \mathcal{B}_C(R)$. Then, an application of Proposition 4.3 gives the first part.

Now, if R is left noetherian, then $GI(R)$ is preenveloping in $R\text{-Mod}$, so by Theorem 3.3 $\text{Hom}_R(C, GI(R))$ is preenveloping in $S\text{-Mod}$. $\square$

We now give the dual results concerning complete $\text{Add}_R(C)$ -resolutions. We omit their proofs since they are dual to those of Proposition 4.3 and Corollary 4.4, respectively.

Proposition 4.5. *Let ${}_R C$ be Hom-faithful, self-small and Σ -self-orthogonal. Then,*

$$C - GP(R) = C \otimes_S (GP(S) \cap \mathcal{A}_C(S)).$$

Corollary 4.6. *Let ${}_R C$ be self-small, Σ -self-orthogonal, Hom-faithful and suppose that E is an injective cogenerator in $R\text{-Mod}$ such that $pd_S(\text{Hom}_R(C, E)) < \infty$. Then, $C - GP(R) = C \otimes_S GP(S)$.*

As a consequence of [17, Proposition 2.12], we have that the class $C - GP(R)$ is precovering in the subcategory of all modules having finite $C - GP(R)$ -dimension. Now we can give conditions for $C - GP(R)$ to be precovering in the whole category $R\text{-Mod}$.

Proposition 4.7. *Suppose that both $id_R(C)$ and $id_S(C)$ are finite, that ${}_R C$ is self-Hom-faithful and Σ -self-orthogonal with a degreewise finite projective resolution, and that S is right noetherian and left n -perfect. Then, $C - GP(R)$ is precovering in $R\text{-Mod}$.*

Proof. The class $\mathcal{B}_C(R)$ and that of all left R -modules having a finite left $C - GP(R)$ -resolution $\overline{(C - GP(R))}$ coincide by [3, Theorem 3.4]. Then, this class $\overline{(C - GP(R))}$ is covering in $R\text{-Mod}$ by Theorem 3.16.

Let then M be any R -module and $\varphi : \overline{G} \rightarrow M$ be a $\overline{(C - GP(R))}$ -precover. By definition, $\overline{G}$ being in $\overline{(C - GP(R))}$ implies that $\overline{G}$ has $G - CP(R)$ -precover $\psi : G \rightarrow \overline{G}$. It is immediate to verify that $\varphi\psi : G \rightarrow M$ is a $G - CP(R)$ -precover of M . $\square$

In Theorem 3.10 we showed how, under certain circumstances, the Bass class $\mathcal{B}_C(R)$ can be recovered from C and the cokernels of an exact right $\text{Add}_R(C)$ -resolution of R . We now show that, when C satisfies some additional conditions, the Auslander class $\mathcal{A}_C(R)$ can also be recovered, in a different way, from the same modules.

Corollary 4.8. *Suppose that ${}_R C$ is self-small and w -tilting (so let*

$$0 \rightarrow R \xrightarrow{t_0} C_0 \xrightarrow{t_1} C_1 \xrightarrow{t_2} \dots$$

be an exact right $\text{Add}_R(C)$ -resolution of R and call $K = (C \oplus (\oplus_{i=0}^{\infty} \text{coker } t_i))^{\perp}$, that $R = \text{End}_S(C)$ and that C_S has a degreewise finite projective resolution. Then,

$$\mathcal{A}_C(R) = \{M \mid \text{Tor}_{i \geq 1}^R(M, K) = 0\}^{\perp} (K^+).$$

Proof. Using Proposition 3.14 we get that $M \in \mathcal{A}_C(R) \Leftrightarrow M^+ \in \mathcal{B}_C(R)$, and by Theorem 3.10 we know that $\mathcal{B}_C(R) = K^{\perp}$, so we have

$$M \in \mathcal{A}_C(R) \Leftrightarrow \text{Ext}_R^i(K, M^+) = 0 \quad \forall i \geq 1.$$

But $\text{Ext}_R^i(K, M^+) \cong \text{Tor}_i^R(M, K)^+$ (see for instance [1, Theorem 3.2.1]) so indeed $M \in \mathcal{A}_C(R) \Leftrightarrow \text{Tor}_i^R(M, K)^+ = 0 \Leftrightarrow \text{Tor}_i^R(M, K) = 0 \quad \forall i \geq 1$.

But at the same time, we have $\text{Ext}_R^i(M, K^+) \cong \text{Tor}_i^R(M, K)^+$, so again

$$\text{Ext}_R^i(M, K^+) = 0 \quad \forall i \geq 1 \Leftrightarrow \text{Tor}_i^R(M, K) = 0 \quad \forall i \geq 1$$

and we finally get $M \in \mathcal{A}_C(R) \Leftrightarrow \text{Ext}_R^i(M, K^+) = 0 \quad \forall i \geq 1$. $\square$

We finish this section by proving that, in some circumstances, the common part of the two components of the pairs $(\mathcal{A}_C(S), \mathcal{A}_C(S)^{\perp_1})$ and $({}^{\perp_1}\mathcal{B}_C(R), \mathcal{B}_C(R))$ is precisely the class of $\mathcal{I}_C$ -injective left S -modules and the class of $\mathcal{P}_C$ -projective left R -modules, respectively.

Proposition 4.9. *The following statements hold.*

- (1) ${}_R C$ is self-small and Σ -self-orthogonal if and only if $\mathcal{B}_C(R) \cap {}^{\perp_1} \mathcal{B}_C(R) = C \otimes_S \mathcal{P}(S)$.
- (2) Suppose that ${}_R C$ is Hom-faithful and that C_S has a degreewise finite projective resolution. Then, $R \cong \text{End}_S(C)$ (canonically) and C_S is self-orthogonal if and only if $\mathcal{A}_C(S) \cap \mathcal{A}_C(S)^{\perp_1} = \text{Hom}_R(C, I(R))$.

Proof. (1) $\Rightarrow$ (2) On one hand, we know by Proposition 3.1 that $\text{Add}_R(C) = C \otimes_S \mathcal{P}(S)$ and by [11, Proposition 5.6] that $\text{Add}_R(C) \subseteq \mathcal{B}_C(R)$. On the other hand, for any projective left S -module P and any left R -module $B \in \mathcal{B}_C(R)$ we have that $\text{Ext}_R^1(C \otimes_S P, B)$ is a direct summand of $\text{Ext}_R^1(C \otimes_S S^{(I)}, B)$ for some index set I . But

$$\text{Ext}_R^1(C \otimes_S S^{(I)}, B) \cong \text{Ext}_R^1(C^{(I)}, B) \cong \text{Ext}_R^1(C, B)^I = 0$$

so $\text{Ext}_R^1(C \otimes_S P, B) = 0$.

We then have $C \otimes_S \mathcal{P}(S) \subseteq \mathcal{B}_C(R) \cap {}^{\perp_1} \mathcal{B}_C(R)$.

Conversely, choose any $M \in \mathcal{B}_C(R) \cap {}^{\perp_1} \mathcal{B}_C(R)$. By Proposition 3.8 we know that M has an epic $\text{Add}_R(C)$ -precover $0 \rightarrow T \rightarrow L \rightarrow M \rightarrow 0$ with kernel $T \in \mathcal{B}_C(R)$. But then $\text{Ext}_R^1(M, T) = 0$ and so the sequence splits, which proves that $M \in \text{Add}_R(C) = C \otimes_S \mathcal{P}(S)$.

($\Leftarrow$) For any set I we have $C^{(I)} \cong C \otimes_S S^{(I)} \in C \otimes_S \mathcal{P}(S) = \mathcal{B}_C(R) \cap {}^{\perp_1} \mathcal{B}_C(R)$, so $C^{(I)} \in \mathcal{B}_C(R)$. Then, $\text{Add}_R(C) \subseteq \mathcal{B}_C(R)$ and then ${}_R C$ is Σ -self-orthogonal and self-small by [11, Proposition 5.6].

(2) $\Rightarrow$ (1) By [11, Theorem 5.9] we know that $I(R) \subseteq \mathcal{B}_C(R)$ so $\text{Hom}_R(C, I(R)) \subseteq \mathcal{A}_C(S)$.

On the other hand, for any injective $E \in I(R)$ and any $A \in \mathcal{A}_C(S)$ we have

$$\text{Ext}_S^1(A, \text{Hom}_R(C, E)) \cong \text{Hom}_R(\text{Tor}_1^S(C, A), E)$$

by [1, Theorem 3.2.1]. But $\text{Tor}_1^S(C, A) = 0$ since $A \in \mathcal{A}_C(S)$, so $\text{Hom}_R(C, E) \in \mathcal{A}_C(S)^{\perp_1}$ and then $\text{Hom}_R(C, I(R)) \subseteq \mathcal{A}_C(S) \cap \mathcal{A}_C(S)^{\perp_1}$.

Conversely, if $M \in \mathcal{A}_C(S) \cap \mathcal{A}_C(S)^{\perp_1}$, [20, Theorem 2.11] guarantees the existence of a monic $\text{Hom}_R(C, I(R))$ -preenvelope $0 \rightarrow M \rightarrow X$. Now, since $\text{Hom}_R(C, I(R)) \subseteq \mathcal{A}_C(S)$, we have that $M, X \in \mathcal{A}_C(S)$, so by [11, Proposition 5.4, 2] the cokernel X/M is also in $\mathcal{A}_C(S)$. But $M \in \mathcal{A}_C(S)^{\perp_1}$, so applying $\text{Hom}_S(X/M, -)$ to the preenvelope we see that the sequence

$$0 \rightarrow M \rightarrow X \rightarrow X/M \rightarrow 0$$

splits, and since $\text{Hom}_R(C, I(R)) = \text{Prod}_S(C^\vee)$, we see that $M \in \text{Hom}_R(C, I(R))$.

($\Leftarrow$) $\text{Hom}_R(C, I(R)) \subseteq \mathcal{A}_C(S) \Rightarrow C^\vee \in \mathcal{A}_C(S)$, so the result follows by [11, Theorem 5.9]. $\square$

5 G_C -flat and G_C -projective covers and dimensions

The information existing in the literature concerning the existence of relative Gorenstein flat precovers is very limited. In this section, we want to address two natural problems which arise when studying this class of modules: when do relative Gorenstein flat (pre)covers exist? and, when is the class of relative Gorenstein projective modules included in the class of relative Gorenstein flats? And, related to this last problem, can we compare relative Gorenstein projective and flat dimensions? Answers to the first two questions will be given in Corollary 5.7 and Theorem 5.9, respectively, and a partial answer to the third question will be given in Theorem 5.13.

To better know how the class of relative Gorenstein flat modules look like, we will study its relations with the class of relative Gorenstein injective modules in Proposition 5.6.

As a consequence of the results to be obtained in this section, we will also give sufficient conditions for the class of relative projective modules to be special precovering in $R\text{-Mod}$ (Corollary 5.14).

We start by recalling the concept of relative Gorenstein flat module.

Definition 5.1. A left R -module M is said to be G_C -flat if there exists an exact and $(\text{Hom}_S(C, I(S)) \otimes_R -)$ -exact sequence of left R -modules

$$\mathbf{X} = \cdots \rightarrow F^1 \rightarrow F^0 \rightarrow C \otimes_S F_0 \rightarrow C \otimes_S F_1 \rightarrow \cdots,$$

where the F^i 's are all flat left R -modules, the F_i 's are all flat left S -modules and $M = \text{Im}(F^0 \rightarrow C \otimes_S F_0)$.

The class of all G_C -flat left R -modules will be denoted by $G_C F(R)$.

It is immediate to see that the class $G_C F(R)$ is closed under direct sums.

Proposition 5.2. Suppose that ${}_R C$ is Σ -self-orthogonal and has a degreewise finite projective resolution. Then, $\mathcal{F}(S) \subseteq \mathcal{A}_C(S)$ and so we have $C \otimes_S \mathcal{F}(S) \subseteq \mathcal{B}_C(R)$ and $\text{Hom}_R(C, C \otimes_S \mathcal{F}(S)) \subseteq \mathcal{F}(S)$.

Proof. By [11, Proposition 5.4] we know that $\mathcal{P}(S) \subseteq \mathcal{A}_C(S)$, and flat modules are direct limits of projective modules, so ${}_R C$ having a degreewise finite projective resolution (which means that both $\text{Hom}_R(C, -)$ and $\text{Ext}_R^{\geq 1}(C, -)$ preserve direct limits) proves that $\mathcal{F}(S) \subseteq \mathcal{A}_C(S)$. But then $C \otimes_S \mathcal{F}(S) \subseteq \mathcal{B}_C(R)$ and so $\text{Hom}_R(C, C \otimes_S \mathcal{F}(S)) \subseteq \mathcal{A}_C(S)$. Therefore, for any $F \in \mathcal{F}(S)$ we have

$$F \cong \text{Hom}_R(C, C \otimes_S F)$$

so $\text{Hom}_R(C, C \otimes_S F)$ is a flat left S -module. □

From Proposition 5.2, we can give conditions for the class $C \otimes_S \mathcal{F}(S)$ to be closed under direct products and pure submodules, which, as a consequence, implies that $C \otimes_S \mathcal{F}(S)$ is preenveloping.

Proposition 5.3. Let ${}_R C$ be Hom-faithful, Σ -self-orthogonal and have a degreewise finite projective resolution. Then, $C \otimes_S \mathcal{F}(S)$ is closed under pure submodules.

If S is right coherent and C_S is finitely presented, then $C \otimes_S \mathcal{F}(S)$ is closed under direct products.

As a consequence, when all conditions above are satisfied, $C \otimes_S \mathcal{F}(S)$ is preenveloping in $R\text{-Mod}$.

Proof. To prove that $C \otimes_S \mathcal{F}(S)$ is closed under pure submodules one can follow [6, Lemma 5.2] and use Proposition 5.2 when necessary (since [6, Lemma 5.2] is proved when C is semidualizing).

The statement about $C \otimes_S \mathcal{F}(S)$ being closed under products is clear.

Finally, we know that if a Kaplansky class is closed under direct products, then it is preenveloping [1, Theorem 11.9.4] so we only have to prove that $C \otimes_S \mathcal{F}(S)$ is Kaplansky.

For let $F \in \mathcal{F}(S)$ be any flat left S -module and $x \in C \otimes_S F$ be any element. Then, we can write x as $x = \sum_{i=1}^n c_i \otimes x_i$, and by [1, Lemma 5.3.12] we know that there is a cardinal number α (dependent only on $|R|$) such that for any i there exists a pure submodule $A_i \leq F$ containing x_i such that $|A_i| < \alpha$. But then $A = \sum_{i=1}^n A_i \leq F$ is a pure submodule of F so it is indeed a flat module, and of course $|A| < \alpha$. If we choose any cardinal number $\aleph > \max\{|C|, \alpha\}$ we get that $|C \otimes_S A| \leq |C \times A| < \aleph$. Therefore, we have $x \in C \otimes_S A \in C \otimes_S \mathcal{F}(S)$ and $|C \otimes_S A| < \aleph$.

Now, $A \leq F$ being pure means that the sequence

$$0 \rightarrow C \otimes_S A \rightarrow C \otimes_S F \rightarrow C \otimes_S \frac{F}{A} \rightarrow 0$$

is exact, so since F/A is flat we immediately see that $\frac{C \otimes_S F}{C \otimes_S A}$ holds in $C \otimes_S \mathcal{F}(S)$ and so that $C \otimes_S \mathcal{F}(S)$ is Kaplansky. □

It is a very interesting question to know whether or not the class of I_C -injective right R -modules, that is, the class $\text{Hom}_S(C, I(S))$ is embedded in the class of all G_C -injective modules. It is known that if C is semidualizing then the answer is yes, but what happens in more general cases? We will need an answer to this question in the proof of our Proposition 5.5.

Proposition 5.4. *Suppose that ${}_R C$ is self-orthogonal and that it has a degreewise finite projective resolution. Then, the class $\text{Hom}_S(C, \mathcal{I}(S))$ is inside the class $G_C I(R)$ of all G_C -injective right R -modules.*

Proof. For any injective right S -module I , let

$$0 \rightarrow \text{Hom}_S(C, I) \rightarrow E_0 \rightarrow E_1 \rightarrow E_2 \rightarrow \dots$$

be an injective resolution of the left R -module $\text{Hom}_S(C, I)$. Let us prove that the sequence is $\text{Hom}_R(\text{Hom}_S(C, \mathcal{I}(S)), -)$ -exact.

Call $K_i = \ker(E_i \rightarrow E_{i+1})$ for every $i \geq 1$, choose any injective right S -module I' and apply the functor $\text{Hom}_R(\text{Hom}_S(C, I'), -)$ to the exact sequence

$$0 \rightarrow \text{Hom}_S(C, I) \rightarrow E_0 \rightarrow K_1 \rightarrow 0.$$

We get that $\text{Ext}_R^i(\text{Hom}_S(C, I'), K_1) \cong \text{Ext}_R^{i+1}(\text{Hom}_S(C, I'), \text{Hom}_S(C, I))$ for every i .

But

$$\text{Ext}_R^i(\text{Hom}_S(C, I'), \text{Hom}_S(C, I)) \cong \text{Hom}_S(\text{Tor}_i^R(\text{Hom}_S(C, I'), C), I)$$

by [1, Theorem 3.2.1], and

$$\text{Hom}_S(\text{Tor}_i^R(\text{Hom}_S(C, I'), C), I) \cong \text{Hom}_S(\text{Hom}_S(\text{Ext}_R^i(C, C), I'), I)$$

by [1, Theorem 3.2.13], so we get $\text{Ext}_R^i(\text{Hom}_S(C, I'), K_1) = 0 \forall i$ since C is self-orthogonal.

Now, $\text{Ext}_R^i(\text{Hom}_S(C, I'), K_j) \cong \text{Ext}_R^{i+j}(\text{Hom}_S(C, I'), \text{Hom}_S(C, I)) = 0$ so our original injective resolution is $\text{Hom}_R(\text{Hom}_S(C, \mathcal{I}(S)), -)$ -exact. $\square$

Now we give a slight modification of the dual of Holm's Corollary 2.11 in [22] for Gorenstein projective modules, extended by White in [14, Corollary 3.8] to the case of G_C -projective modules for a semidualizing module C .

Proposition 5.5. *Suppose that ${}_R C$ is self-orthogonal and has a degreewise finite projective resolution. If $0 \rightarrow M \rightarrow \text{Hom}_S(C, I) \rightarrow G \rightarrow 0$ is a short exact sequence of right R -modules where I is an injective right S -module, G is G_C -injective and $\text{Ext}_R^1(A, M) = 0$ for every $\mathcal{I}_C$ -injective right R -module A , then M is G_C -injective as well.*

Proof. Since G is G_C -injective, there is a $\text{Hom}_R(\text{Hom}_S(C, \mathcal{I}(S)), -)$ -exact short exact sequence

$$0 \rightarrow K \rightarrow \text{Hom}_S(C, I_0) \rightarrow G \rightarrow 0$$

for some injective right S -module I_0 . We claim that the right R -module K is G_C -injective.

Of course, K has an exact left $\text{Hom}_S(C, \mathcal{I}(S))$ -resolution since G is G_C -injective. Let then

$$0 \rightarrow K \rightarrow E_0 \rightarrow E_1 \rightarrow \dots$$

be an injective resolution of K . If this resolution is $\text{Hom}_R(\text{Hom}_S(C, \mathcal{I}(S)), -)$ -exact, then pasting together both resolutions of K we get that K is G_C -injective.

Call then $K_i = \ker(E_i \rightarrow E_{i+1}) \forall i \geq 1$ and choose any $I' \in \mathcal{I}(S)$.

Applying $\text{Hom}_R(\text{Hom}_S(C, I'), -)$ to the exact sequence

$$0 \rightarrow K_j \rightarrow E_j \rightarrow K_{j+1} \rightarrow 0$$

we get that

$$\text{Ext}_R^i(\text{Hom}_S(C, I'), K_{j+1}) \cong \text{Ext}_R^{i+1}(\text{Hom}_S(C, I'), K_j),$$

so we actually have

$$\text{Ext}_R^i(\text{Hom}_S(C, I'), K_j) \cong \text{Ext}_R^{i+j}(\text{Hom}_S(C, I'), K) \forall i, j \geq 1.$$

But, as in the proof of Proposition 5.4, $\text{Ext}_R^i(\text{Hom}_S(C, I'), \text{Hom}_S(C, I_0)) = 0$, so from the exact sequence

$$0 \rightarrow K \rightarrow \text{Hom}_S(C, I_0) \rightarrow G \rightarrow 0$$

we see that $\text{Ext}_R^{i+1}(\text{Hom}_S(C, I'), K) \cong \text{Ext}_R^i(\text{Hom}_S(C, I'), G)$, and of course, since G is G_C -injective, $\text{Ext}_R^i(\text{Hom}_S(C, I'), G) = 0$, so we get $\text{Ext}_R^{i \geq 2}(\text{Hom}_S(C, I'), K) = 0$.

As for $\text{Ext}_R^1(\text{Hom}_S(C, I'), K) = 0$, this follows from the $\text{Hom}_R(\text{Hom}_S(C, I(S)), -)$ -exactness of

$$0 \rightarrow K \rightarrow \text{Hom}_S(C, I_0) \rightarrow G \rightarrow 0.$$

Therefore, K is G_C -injective.

Now, constructing the pullback of $\text{Hom}_S(C, I) \rightarrow G$ and $\text{Hom}_S(C, I_0) \rightarrow G$ we get a commutative diagram with exact rows

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & M & \xlongequal{\quad} & M & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & \text{Hom}_S(C, I) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & K & \longrightarrow & \text{Hom}_S(C, I_0) & \longrightarrow & G \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

Since K is G_C -injective we have $\text{Ext}_R^1(\text{Hom}_S(C, I), K) = 0$ and so we get $P \cong K \oplus \text{Hom}_S(C, I)$, that is, P is G_C -injective (see Proposition 5.4). But we know by the hypotheses $\text{Ext}_R^1(\text{Hom}_S(C, I_0), M) = 0$ so $P \cong M \oplus \text{Hom}_S(C, I_0)$.

Finally, G_C -injective modules are closed under direct summands as a consequence of [23, Proposition 4.8]: though the authors assume that their classes $\mathcal{V}$ and $\mathcal{W}$ are closed under finite direct sums and direct summands throughout the paper, their arguments are similar to and based on those of [24, Theorem 3.6], and the only necessary assumption is that the classes are closed under finite direct sums. But both $\text{Hom}_S(C, I(S))$ and $I(R)$ are closed under finite direct sums, so $G_C(R)$ -injective modules are closed under direct summands.

Therefore, M is G_C -injective. □

The following result has been proved when C is semidualizing [25, Proposition 2.5]. Here we give the following extension whose proof follows that of [25] with the only modification of using Propositions 5.3, 5.4 and 5.5, which are adaptations to our context of the analogous cases when C is semidualizing.

Proposition 5.6. *Let M be any left R -module. The following statements hold:*

- (1) *If M is G_C -flat, then M^+ is a G_C -injective right R -module.*
- (2) *Suppose now that S is right coherent, that ${}_R C$ is Σ -self-orthogonal, Hom-faithful and that it has a degreewise finite projective resolution, and finally that C_S is finitely presented. If $M^+ \in G_C I(R)$, then $M \in G_C F(R)$.*

Corollary 5.7. *Suppose that C_S finitely presented, that ${}_R C$ is Σ -self-orthogonal and Hom-faithful, and that it has a degreewise finite projective resolution. If S is right coherent, then the class $G_C F(R)$ is covering in $R\text{-Mod}$.*

Proof. Since $G_C I(R)$ is closed under products and direct summands, Proposition 5.6 shows that the pair $(G_C F(R), G_C I(R))$ is a left duality pair, which, in addition, is coproduct closed, so [21, Theorem 3.1] guarantees the existence of a $G_C F(R)$ -cover for every left R -module. □

Lemma 5.8. Let ${}_R C$ be Σ -self-orthogonal and $X^\bullet$ be a complete $(\mathcal{P}(R), C \otimes_S \mathcal{P}(S))$ -resolution of certain module. If

$$0 \rightarrow C \otimes_S Q_n \rightarrow C \otimes_S Q_{n-1} \rightarrow \cdots \rightarrow C \otimes_S Q_0 \rightarrow L \rightarrow 0$$

is an exact complex and all Q_i are projective left S -modules, then the sequence of complexes of abelian groups

$$0 \rightarrow \operatorname{Hom}_R(X^\bullet, C \otimes_S Q_n) \rightarrow \cdots \rightarrow \operatorname{Hom}_R(X^\bullet, C \otimes_S Q_0) \rightarrow \operatorname{Hom}_R(X^\bullet, L) \rightarrow 0$$

is also exact.

Proof. We have to prove that for any integer m , the sequence

$$0 \rightarrow \operatorname{Hom}_R(X^m, C \otimes_S Q_n) \rightarrow \cdots \rightarrow \operatorname{Hom}_R(X^m, C \otimes_S Q_0) \rightarrow \operatorname{Hom}_R(X^m, L) \rightarrow 0$$

is exact.

If $m \leq 0$, then X^m is projective and so the corresponding sequence is exact.

On the other hand, if $m \geq 1$, then $X^m = C \otimes_S K$ for some projective left S -module K . Then, K is a direct summand of $S^{(J)}$ for some set J , and likewise, Q_n is a direct summand of some $S^{(I)}$. Therefore, $\operatorname{Ext}_R^i(C \otimes_S K, C \otimes_S Q_n)$ is (isomorphic to) a direct summand of $\operatorname{Ext}_R^i(C \otimes_S S^{(J)}, C \otimes_S S^{(I)}) \cong \operatorname{Ext}_R^i(C, C^{(I)})^J$.

But ${}_R C$ is Σ -self-orthogonal so $\operatorname{Ext}_R^i(C, C^{(I)})^J = 0$ and then $\operatorname{Ext}_R^{\geq 1}(C \otimes_S K, C \otimes_S Q_n) = 0$.

The same argument shows that $\operatorname{Ext}_R^{\geq 1}(C \otimes_S K, C \otimes_S Q_i) = 0$ for every i , so making use of the long exact sequence associated with

$$0 \rightarrow C \otimes_S Q_n \rightarrow C \otimes_S Q_{n-1} \rightarrow A_{n-1} \rightarrow 0$$

we see that $\operatorname{Ext}_R^{\geq 1}(X^m, A_{n-1}) = 0$.

An inductive process gives now that indeed $\operatorname{Ext}_R^{\geq 1}(X^m, A_i) = 0$ for every i so the sequences

$$0 \rightarrow \operatorname{Hom}_R(X^m, C \otimes_S Q_n) \rightarrow \cdots \rightarrow \operatorname{Hom}_R(X^m, C \otimes_S Q_0) \rightarrow \operatorname{Hom}_R(X^m, L) \rightarrow 0$$

are all exact. $\square$

Theorem 5.9. Suppose that S is right coherent, that C_S is finitely presented and that ${}_R C$ is Σ -self-orthogonal. If $\mathcal{P}_C\text{-pd}(C \otimes_S F) < \infty$ for every flat and cotorsion left S -module F , then $G_C P(R) \subseteq G_C F(R)$.

Proof. Let M be any G_C -projective left R -module and $X^\bullet$ a complete $(\mathcal{P}(R), C \otimes_S \mathcal{P}(S))$ -resolution of M .

If we prove that the complex $\operatorname{Hom}_S(C, I) \otimes_R X^\bullet$ is exact for any injective right S -module I , then we will have that M is G_C -flat.

Of course $\operatorname{Hom}_S(C, I) \otimes_R X^\bullet$ is exact if and only if $(\operatorname{Hom}_S(C, I) \otimes_R X^\bullet)^+$ is exact, and

$$(\operatorname{Hom}_S(C, I) \otimes_R X^\bullet)^+ \cong \operatorname{Hom}_R(X^\bullet, \operatorname{Hom}_S(C, I)^+) \cong \operatorname{Hom}_R(X^\bullet, C \otimes_S I^+).$$

Now, I^+ is cotorsion, and since S is right coherent, I^+ is also flat, so, by the hypotheses, $\mathcal{P}_C\text{-pd}(C \otimes_S I^+) < \infty$. This means that there is an exact sequence

$$0 \rightarrow C \otimes_S Q_n \rightarrow C \otimes_S Q_{n-1} \rightarrow \cdots \rightarrow C \otimes_S Q_0 \rightarrow C \otimes_S I^+ \rightarrow 0$$

with all Q_i projective left S -modules.

Then, by Lemma 5.8 we get that the sequence of complexes

$$0 \rightarrow \operatorname{Hom}_R(X^\bullet, C \otimes_S Q_n) \rightarrow \cdots \rightarrow \operatorname{Hom}_R(X^\bullet, C \otimes_S Q_0) \rightarrow \operatorname{Hom}_R(X^\bullet, C \otimes_S I^+) \rightarrow 0$$

is exact. Since each $\operatorname{Hom}_R(X^\bullet, C \otimes_S Q_i)$ is an exact complex we get by induction that $\operatorname{Hom}_R(X^\bullet, C \otimes_S I^+)$ is exact too. $\square$

Our next goal will be to find conditions for every G_C -flat module to have finite G_C -projective dimension. We will need the following result, inspired by the proof of [26, Proposition 3.1].

Recall that a module M is said to have flat cotorsion dimension less than or equal to a nonnegative integer number n whenever there exists an exact sequence

$$0 \rightarrow M \rightarrow F_0 \rightarrow F_1 \rightarrow \cdots \rightarrow F_n \rightarrow 0$$

with all F_i flat and cotorsion modules.

The following result gives an interesting property of flat modules over n -perfect rings. We believe it is surely known but we have not found its proof so we state and prove it here.

Proposition 5.10. *The flat cotorsion dimension of any flat left R -module over a left n -perfect ring R is always less than or equal to n .*

Proof. Let F be any flat left R -module and

$$0 \rightarrow F \rightarrow C_0 \rightarrow C_1 \rightarrow \cdots$$

be a cotorsion resolution of F . If we call $K_i = \ker(C_i \rightarrow C_{i+1})$ for every $i \geq 1$, then we know that K_i is flat by Wakamatsu's Lemma (see for instance [27, Lemma 2.1.2]), and this implies that every C_i is flat and cotorsion. Therefore, we see that we only need to prove that K_n is cotorsion. But for every flat module $\bar{F}$ we have $\text{Ext}_R^{\geq 1}(\bar{F}, C_i) = 0 \ \forall i$, so

$$\text{Ext}_R^i(\bar{F}, K_j) \cong \text{Ext}_R^{i+j}(\bar{F}, F) \ \forall i, j \geq 1.$$

Now, since R is left n -perfect we have $\text{Ext}_R^i(\bar{F}, F) = 0 \ \forall i \geq n+1$ so

$$\text{Ext}_R^1(\bar{F}, K_n) \cong \text{Ext}_R^{n+1}(\bar{F}, F) = 0$$

and K_n is cotorsion. □

Lemma 5.11. *Suppose that S is left n -perfect, that C_S is finitely presented and that ${}_R C$ is Σ -self-orthogonal. Then, given any exact complex $\mathbf{X}$ in $R\text{-Mod}$ with $\mathbf{X}^i$ G_C -projective for every i , we have: if $\text{Hom}_S(C, I) \otimes_R \mathbf{X}$ is exact for any injective right S -module I , then $\text{Hom}_R(\mathbf{X}, C \otimes_S P)$ is exact for every projective left S -module P .*

Proof. In first place, we note that the natural morphism of left S -modules $F \rightarrow F^{++}$ is a pure injection, so if F is flat and cotorsion then F is a direct summand of F^{++} .

Now, C_S is finitely presented so there is a natural isomorphism

$$(\text{Hom}_S(C, F^+) \otimes_R \mathbf{X})^+ \cong \text{Hom}_R(\mathbf{X}, C \otimes_S F^{++}).$$

Thus, since F^+ is an injective right S -module, the complex $(\text{Hom}_S(C, F^+) \otimes_R \mathbf{X})^+$ (and so the complex $\text{Hom}_R(\mathbf{X}, C \otimes_S F^{++})$) is exact by the hypotheses, and therefore its direct summand $\text{Hom}_R(\mathbf{X}, C \otimes_S F)$ is exact. Hence, $\text{Hom}_R(\mathbf{X}, C \otimes_S F)$ is an exact complex for every flat and cotorsion left S -module F .

Now, since S left n -perfect, by Proposition 5.10 we have that if P is any projective left S -module, then we can find an exact sequence

$$0 \rightarrow P \rightarrow F_0 \rightarrow F_1 \rightarrow \cdots \rightarrow F_n \rightarrow 0$$

in which every F_i is a flat and cotorsion left S -module. Then, the sequence of left R -modules

$$0 \rightarrow C \otimes_S P \rightarrow C \otimes_S F_0 \rightarrow \cdots \rightarrow C \otimes_S F_n \rightarrow 0$$

is exact and, if we prove that

$$0 \rightarrow \text{Hom}_R(\mathbf{X}, C \otimes_S P) \rightarrow \text{Hom}_R(\mathbf{X}, C \otimes_S F_0) \rightarrow \cdots \rightarrow \text{Hom}_R(\mathbf{X}, C \otimes_S F_n) \rightarrow 0$$

is exact, we will have that the complex $\text{Hom}_R(\mathbf{X}, C \otimes_S P)$ is exact since we have already proved that all the complexes $\text{Hom}_R(\mathbf{X}, C \otimes_S F_i)$ are exact.

But each ${}_S F_i$ is flat so it has finite projective dimension and then each $C \otimes_S F_i$ has finite $\mathcal{P}_C$ -projective dimension. Thus, $\text{Ext}_R^i(\mathbf{X}^j, C \otimes_S F_i) = 0 \ \forall i \geq 1 \ \forall j$ since every $\mathbf{X}^j$ is G_C -projective (see [28, Corollary 3.5]). □

We now give an extension of the concept of n -perfect ring, adapted to this new setting of C -relativity.

Definition 5.12. Given a nonnegative integer number n , a ring R will be said to be left (C, n) -perfect if every flat left R -module has $\mathcal{P}_C$ -projective dimension less than or equal to n .

Theorem 5.13. Suppose that R is left (C, n) -perfect, that S is left n -perfect and left coherent and that ${}_R C$ is weakly Wakamatsu tilting and has a degreewise finite projective resolution. Then, every G_C -flat left R -module has finite G_C -projective dimension.

Proof. Let M be a G_C -flat left R -module. Then, there is a $(\text{Hom}_S(C, I(S)) \otimes_R -)$ -exact complex

$$\mathbf{X} : \cdots \rightarrow \mathbf{X}^{-1} \rightarrow \mathbf{X}^0 \rightarrow \mathbf{X}^1 \rightarrow \cdots$$

in which $M = \text{Im}(\mathbf{X}^0 \rightarrow \mathbf{X}^1)$, $\mathbf{X}^i$ is a flat left R -module for every $i \leq 0$ and $\mathbf{X}^i \cong C \otimes_S F^i$ where F^i is a flat left S -module for every $i > 0$.

Therefore, $\mathcal{P}_C\text{-pd}(\mathbf{X}^i) \leq n \ \forall i \leq 0$ since R is left (C, n) -perfect, and $\text{pd}(F^i) \leq n \ \forall i \geq 1$ since S left n -perfect. Then, $\mathcal{P}_C\text{-pd}(C \otimes_S F^i) \leq n \ \forall i \geq 1$ and then, by [28, Theorem 3.6], we get that $G_C\text{-pd}(\mathbf{X}^i) \leq n \ \forall i$.

Now, for any $i \leq 0$ let

$$0 \rightarrow K^i \rightarrow P^i \rightarrow \mathbf{X}^i \rightarrow 0$$

be a short exact sequence in $R\text{-Mod}$ with P^i projective, and if $i \geq 1$ consider an exact sequence

$$0 \rightarrow K^i \rightarrow P^i \rightarrow F^i \rightarrow 0$$

in $S\text{-Mod}$ with P^i projective. Then, the sequences of R -modules

$$0 \rightarrow C \otimes_S K^i \rightarrow C \otimes_S P^i \rightarrow C \otimes_S F^i = \mathbf{X}^i \rightarrow 0$$

are also exact.

For any $i \leq -1$ we have a commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & P^i & \xlongequal{\quad} & P^i & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathbf{X}^{i-1} & \longrightarrow & \mathbf{X}^i & \longrightarrow & \mathbf{X}^{i+1} & \longrightarrow & \mathbf{X}^{i+2} & \longrightarrow & \cdots \\ & & & & \downarrow & & & & & & \\ & & & & 0 & & & & & & \end{array}$$

with exact rows and columns. Similarly, if $i \geq 1$ we have

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & C \otimes_S P^i & \xlongequal{\quad} & C \otimes_S P^i & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathbf{X}^{i-1} & \longrightarrow & \mathbf{X}^i & \longrightarrow & \mathbf{X}^{i+1} & \longrightarrow & \mathbf{X}^{i+2} & \longrightarrow & \cdots \\ & & & & \downarrow & & & & & & \\ & & & & 0 & & & & & & \end{array}$$

and if $i = 0$ we have

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & P^0 & \xlongequal{\quad} & P^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathbf{X}^{-1} & \longrightarrow & \mathbf{X}^0 & \longrightarrow & \mathbf{X}^1 & \longrightarrow & \mathbf{X}^2 & \longrightarrow & \cdots \\ & & & & \downarrow & & & & & & \\ & & & & 0 & & & & & & \end{array}$$

Therefore, if for any module N we call $\overline{N}[i]$ the complex

$$\cdots \longrightarrow 0 \longrightarrow N \rightrightarrows N \longrightarrow 0 \longrightarrow \cdots$$

with N at the i th and $(i+1)$ th positions, we get an exact sequence of complexes

$$0 \rightarrow \mathbf{W} \rightarrow \mathbf{P}_0 = (\oplus_{i \leq 0} \overline{P^i}[i]) \oplus (\oplus_{i \geq 1} \overline{C \otimes_S P^i}[i]) \rightarrow \mathbf{X} \rightarrow 0$$

in which every $\mathbf{W}^i$ is obtained in the following way: for $i \leq 0$ as the pullback

$$\begin{array}{ccccccc} 0 & \longrightarrow & K^i & \longrightarrow & \mathbf{W}^i & \longrightarrow & P^{i-1} \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & K^i & \longrightarrow & P^i & \longrightarrow & \mathbf{X}^i \longrightarrow 0 \end{array}$$

For $i \geq 2$ as the pullback

$$\begin{array}{ccccccc} 0 & \longrightarrow & C \otimes_S K^i & \longrightarrow & \mathbf{W}^i & \longrightarrow & C \otimes_S P^{i-1} \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C \otimes_S K^i & \longrightarrow & C \otimes_S P^i & \longrightarrow & C \otimes_S P^i \longrightarrow 0 \end{array}$$

For $i = 1$ as the pullback

$$\begin{array}{ccccccc} 0 & \longrightarrow & C \otimes_S K^1 & \longrightarrow & \mathbf{W}^1 & \longrightarrow & P^0 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C \otimes_S K^1 & \longrightarrow & C \otimes_S P^1 & \longrightarrow & C \otimes_S P^1 \longrightarrow 0 \end{array}$$

But $\mathbf{P}_0$ and $\mathbf{X}$ are exact complexes so $\mathbf{W}$ is exact too. Then, since $\mathbf{P}_0$ and $\mathbf{X}$ are $\text{Hom}_S(C, I(S))_{\otimes_R}$ -exact, so is $\mathbf{W}$.

Moreover, we see from the pullbacks above that

$$\mathbf{W}^i \cong \begin{cases} K^i \oplus P^{i-1} & \text{if } i \leq 0, \\ (C \otimes_S K^1) \oplus P^0 & \text{if } i = 1. \end{cases}$$

Now, for any $i > 1$ we know that $C \otimes_S P^{i-1}$ is a direct summand of $C \otimes_S S^{(I)} \cong C^{(I)}$, so $\text{Ext}_R^1(C \otimes_S P^{i-1}, C \otimes_S K^i)$ is a direct summand of

$$\text{Ext}_R^1(C^{(I)}, C \otimes_S K^i) \cong \text{Ext}_R^1(C, C \otimes_S K^i)^I.$$

Since ${}_R C$ has a degreewise finite projective resolution by [1, Theorem 3.2.15] we have

$$\text{Ext}_R^1(C, C \otimes_S K^i) \cong \text{Ext}_R^1(C, C) \otimes_S K^i = 0$$

so we see that

$$\mathbf{W}^i \cong (C \otimes_S K^i) \oplus (C \otimes_S P^{i-1}) \quad \forall i > 1.$$

Therefore, the complex $\mathbf{W}$ is

$$\cdots \rightarrow K^{-1} \oplus P^{-2} \rightarrow K^0 \oplus P^{-1} \rightarrow (C \otimes_S K^1) \oplus P^0 \rightarrow (C \otimes_S K^2) \oplus (C \otimes_S P^1) \rightarrow \cdots$$

Moreover, for every $i \leq 0$, $\mathbf{P}_0^i$ is a projective left R -module so it is G_C -projective; for any $i \geq 2$, $\mathbf{P}_0^i = (C \otimes_S P^{i-1}) \oplus (C \otimes_S P^i)$ and $G_C\text{-pd}(C \otimes_S P^i) \leq \mathcal{P}_C\text{-pd}(C \otimes_S P^i) = 0$; and finally $G_C\text{-pd}(P^0 \oplus (C \otimes_S P^1)) = 0$, so in the exact sequence

$$0 \rightarrow \mathbf{W} \rightarrow \mathbf{P}_0 \rightarrow \mathbf{X} \rightarrow 0$$

we have

$$\sup\{G_C\text{-pd}(\mathbf{P}_0^i), G_C\text{-pd}(\mathbf{X}^i) - 1\} = G_C\text{-pd}(\mathbf{X}^i) - 1 \leq n - 1$$

(unless $\mathbf{X}^i$ is G_C -projective, in which case $\mathbf{W}^i$ will also be G_C -projective by [11, Proposition 2.9]).

Hence, by [11, Proposition 3.11] we get that $G_C\text{-pd}(\mathbf{W}^i) \leq n - 1 \forall i$.

This means that we can apply the same argument to the complex $\mathbf{W}$ and we can start an inductive procedure to get an exact sequence of exact and $(\text{Hom}_S(C, I(S)) \otimes_R -)$ -exact complexes

$$0 \rightarrow \mathbf{W}_t \rightarrow \mathbf{P}_{t-1} \rightarrow \cdots \rightarrow \mathbf{P}_0 \rightarrow \mathbf{X} \rightarrow 0$$

with $t \leq n$ and $G_C\text{-pd}(\mathbf{W}_t^i) = 0 \forall i$.

Now, $\text{Add}_R(C) = C \otimes_S \mathcal{P}(S)$ since ${}_R C$ is self-small (it has a degreewise finite projective resolution). Then, by [11, Proposition 2.16] we see that $G_C P^2(R) = G_C P(R)$, and since every $\mathbf{W}_t^i$ has G_C -projective dimension zero, we get that $Z^0(\mathbf{W}_t)$ is a G_C -projective left R -module. But we have already seen that the left R -modules $Z^0(\mathbf{P}_i)$ are all G_C -projective, so the exact sequence

$$0 \rightarrow Z^0(\mathbf{W}_t) \rightarrow Z^0(\mathbf{P}_{t-1}) \rightarrow \cdots \rightarrow Z^0(\mathbf{P}_0) \rightarrow Z^0(\mathbf{X}) = M \rightarrow 0$$

shows that $G_C\text{-pd}(M) \leq n$. □

We now give conditions for the class $G_C P(R)$ to be special precovering.

Corollary 5.14. *Let R be a left (C, n) -perfect ring, S be left n -perfect and right coherent, ${}_R C$ be a Hom-faithful weakly Wakamatsu tilting module with a degreewise finite projective resolution and C_S be finitely presented. Then, $G_C P(R)$ is special precovering in $R\text{-Mod}$.*

Proof. By Theorems 5.13 and 5.7, we have the inclusions

$$G_C P(R) \subseteq G_C F(R) \subseteq \overline{G_C P(R)},$$

where $\overline{G_C P(R)}$ denotes the class of left R -modules of finite G_C -projective dimension.

Now, given any left R -module M , by Corollary 5.7 we know that we have a $G_C F(R)$ -cover $\varphi : F \rightarrow M$, and then, by [11, Theorem 3.5], there is a special $G_C P(R)$ -precover $\psi : P \rightarrow F$. We claim that $\psi\varphi : P \rightarrow M$ is a special $G_C P(R)$ -precover of M .

Since φ is a cover and ψ is a special precover, we get that $\ker(\varphi)$ holds in $G_C F(R)^{\perp_1}$ and $\ker(\psi)$ holds in $G_C P(R)^{\perp_1}$. But $G_C F(R)^{\perp_1} \subseteq G_C P(R)^{\perp_1}$ so indeed both kernels are in $G_C P(R)^{\perp_1}$. Then, from the left-hand column of the pullback diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & \ker(\psi) & \xlongequal{\quad} & \ker(\psi) & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & D & \longrightarrow & P & \xrightarrow{\varphi\psi} & M \longrightarrow 0 \\ & & \downarrow & & \downarrow \psi & & \parallel \\ 0 & \longrightarrow & \ker(\varphi) & \longrightarrow & F & \xrightarrow{\varphi} & M \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

we see that D is in $G_C P(R)^{\perp_1}$, and this immediately implies that $\varphi\psi$ is a special $G_C P(R)$ -precover. □

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