Research Article

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Some new results on the weaving of K -g-frames in Hilbert spaces

<https://doi.org/10.1515/math-2021-0103>

received November 6, 2020; accepted September 12, 2021

Abstract: In this paper, we provide some conditions for a K -woven pair of K -g-frames to be preserved under an operator and particularly, we report that applying two different operators to a K -woven pair of K -g-frames can leave them K -woven. Several new methods on the construction of K -woven pair of K -g-frames are also given. We end the paper with a new perturbation result on the weaving of K -g-frames, which shows that, under the perturbation condition involved in one known result on this topic, two K -g-frames can be K -woven in the whole space, not merely in the subspace $\text{Range}(K)$.

Keywords: K -g-frame, K -woven, perturbation, pseudo-inverse

MSC 2020: 42C15, 47B40

1 Introduction

Inspired by the double inequalities induced from the frame coefficients of a multi-wavelet frame for $L^2(\mathbb{R})$, Sun [1] introduced the notion of g-frames, which cover various generalizations of (classical) frames such as pseudo-frames, fusion frames, outer frames, and oblique frames. Even though many properties of frames have counterparts in g-frame setting, there are some essential differences between them. For example, an exact g-frame is not necessarily a g-Riesz basis, and not all g-Riesz frames contain a g-Riesz basis. This makes many scholars pay their attention to g-frames [2–5].

Later, K -frames, also known as frames for operators, were put forward by Găvruta in [6] to examine atomic decompositions for a linear bounded operator K , which, in many aspects, behave quite differently to frames, see [7–11]. After those works, Xiao et al. [12] showed us a more general type of frames named K -g-frames by combining g-frames with K -frames. Asgari and Rahimi in [13] called these frames “Generalized frames for operators.” The readers can check [14–16] for more details on K -g-frames.

Motivated by a question arising in distributed signal processing, Bemrose et al. [17,18] proposed the concept of weaving frames, which were further investigated in [19,20]. Now, weaving has become to be a research hotspot because of some potential applications. We refer to the papers [21–23] for more information on the weaving of fusion frames or g-frames, and the paper [24] for information on the weaving of K -frames.

Recently, the concept of weaving for more general frames, namely weaving K -g-frames, was introduced in [25], which provides a wider perspective on frame theory, and a deep study of them would help to explore the application approach of weaving theory of frames in pre-processing of signals and wireless sensor networks and some other fields. So it is natural for us to come up with the idea to examine this new framework, in order to further develop and enrich frame theory and its application scenarios.

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In [25], the authors studied the erasures and perturbations of weaving for K -g-frames. We found, however, that the main results they obtained are all based on the subspace $\text{Range}(K)$. The aim of this article is to present some new results on the weaving for K -g-frames, which are established in the whole space. More precisely, in Section 2 we examine conditions under which a K -woven pair of K -g-frames can be preserved when applying linear bounded operators to them. Section 3 is devoted to constructing K -woven pair of K -g-frames by using several new methods. In Section 4, we give a new perturbation result on the weaving of K -g-frames, which shows that the Paley-Wiener-type perturbation condition involved in Theorem 3 of [25] is valid for the whole space.

Throughout this paper, $\mathbb{I}$ is an index set (finite or countable), $\mathcal{H}$, $\mathcal{K}$, and $\mathcal{H}_i$'s ($i \in \mathbb{I}$) are complex separable Hilbert spaces. We denote by $B(\mathcal{H}, \mathcal{K})$ the collection of all linear bounded operators from $\mathcal{H}$ to $\mathcal{K}$, and $B(\mathcal{H}, \mathcal{H})$ is abbreviated to $B(\mathcal{H})$.

Definition 1.1. (see [12]) Suppose $K \in B(\mathcal{H})$ and $\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)$ for each $i \in \mathbb{I}$. We say that $\{\Lambda_i\}_{i \in \mathbb{I}}$ is a K -g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$ with lower and upper frame bounds C and D , respectively, if there are constants $C, D > 0$ such that

$$C\|K^*f\|^2 \leq \sum_{i \in \mathbb{I}} \|\Lambda_i f\|^2 \leq D\|f\|^2, \quad \forall f \in \mathcal{H}. \quad (1.1)$$

Recall that a K -g-frame $\{\Lambda_i\}_{i \in \mathbb{I}}$ is said to be C -tight, if

$$C\|K^*f\|^2 = \sum_{i \in \mathbb{I}} \|\Lambda_i f\|^2, \quad \forall f \in \mathcal{H}.$$

If only the inequality to the right-hand side in (1.1) holds, then $\{\Lambda_i\}_{i \in \mathbb{I}}$ is called a g-Bessel sequence with g-Bessel bound D , related to which there are two well-defined operators:

$$\begin{aligned} T : \ell^2(\{\mathcal{H}_i\}_{i \in \mathbb{I}}) &\rightarrow \mathcal{H}, \quad T\{g_i\}_{i \in \mathbb{I}} = \sum_{i \in \mathbb{I}} \Lambda_i^* g_i \quad (\text{synthesis operator}) \\ S : \mathcal{H} &\rightarrow \mathcal{H}, \quad Sf = \sum_{i \in \mathbb{I}} \Lambda_i^* \Lambda_i f \quad (\text{g-frame operator}) \end{aligned}$$

Definition 1.2. (see [25]) Let $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Gamma_i\}_{i \in \mathbb{I}}$ be K -g-frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$. If there are universal constants C and D such that for any partition $\{\sigma_1, \sigma_2\}$ of $\mathbb{I}$, $\{\Lambda_i\}_{i \in \sigma_1} \cup \{\Gamma_i\}_{i \in \sigma_2}$ is a K -g-frame for $\mathcal{H}$ with frame bounds C and D , then we call that $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Gamma_i\}_{i \in \mathbb{I}}$ are K -woven in $\mathcal{H}$, and each $\{\Lambda_i\}_{i \in \sigma_1} \cup \{\Gamma_i\}_{i \in \sigma_2}$ is called a weaving.

The K -g-frames $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Gamma_i\}_{i \in \mathbb{I}}$ are said to be C -tight K -woven, if for any partition $\{\sigma_1, \sigma_2\}$ of $\mathbb{I}$, $\{\Lambda_i\}_{i \in \sigma_1} \cup \{\Gamma_i\}_{i \in \sigma_2}$ is a C -tight K -g-frame.

Note that if $K = \text{Id}_{\mathcal{H}}$, the identical operator on $\mathcal{H}$, then the weaving of K -g-frames turns to be the weaving of g-frames.

Lemma 1.3. (see [26]) Suppose that $\Phi \in B(\mathcal{H}, \mathcal{K})$ has closed range, then there exists a unique operator $\Phi^\dagger \in B(\mathcal{K}, \mathcal{H})$, called the pseudo-inverse of Φ , satisfying

$$\begin{aligned} \Phi\Phi^\dagger &= P_{\text{Range}(\Phi)}, \quad \Phi^\dagger\Phi = P_{\text{Range}(\Phi^\dagger)}, \quad (\Phi\Phi^\dagger)^* = \Phi\Phi^\dagger, \quad (\Phi^\dagger\Phi)^* = \Phi^\dagger\Phi, \\ \text{Ker}\Phi^\dagger &= (\text{Range}(\Phi))^\perp, \quad \text{Range}(\Phi^\dagger) = (\text{Ker}\Phi)^\perp, \end{aligned}$$

where $P_{\text{Range}(\Phi)}$ denotes the orthogonal projection onto $\text{Range}(\Phi)$.

Lemma 1.4. (see [27]) Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces. Also let $P \in B(\mathcal{H}_1, \mathcal{H})$ and $Q \in B(\mathcal{H}_2, \mathcal{H})$. The following statements are equivalent.

- (1) $\text{Range}(P) \subset \text{Range}(Q)$.
- (2) There exists $\lambda > 0$ such that $PP^* \leq \lambda QQ^*$.
- (3) There exists $\Theta \in B(\mathcal{H}_1, \mathcal{H}_2)$ such that $P = Q\Theta$.

In the sequel, the notations D_Λ , T_Λ , and S_Λ are used to denote, respectively, the g-Bessel bound, the synthesis operator, and the g-frame operator of the g-Bessel sequence $\{\Lambda_i\}_{i \in \mathbb{I}}$ (the same goes for $\{\Gamma_i\}_{i \in \mathbb{I}}$ and $\{\Theta_i\}_{i \in \mathbb{I}}$) and if, furthermore, $\{\Lambda_i\}_{i \in \mathbb{I}}$ is a K -g-frame, then we use C_Λ to denote the lower frame bound (the same goes for $\{\Gamma_i\}_{i \in \mathbb{I}}$). As usual, the notation $\Phi^\dagger$ is reserved for the pseudo-inverse of Φ , if it exists.

2 Operator perturbation of the weaving of K -g-frames

The main purpose of this section is to investigate the stability of K -woven pair of K -g-frames under operator perturbation.

Theorem 2.1. Suppose that K -g-frames $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Gamma_i\}_{i \in \mathbb{I}}$ are K -woven in $\mathcal{H}$ with the universal frame bounds C and D . If $U \in B(\mathcal{H})$ has closed range with $UK = KU$ and $\text{Range}(K^*) \subset \text{Range}(U)$, then $\{\Lambda_i U^*\}_{i \in \mathbb{I}}$ and $\{\Gamma_i U^*\}_{i \in \mathbb{I}}$ are K -woven in $\mathcal{H}$ with the universal frame bounds $C\|U^+\|^{-2}$ and $D\|U\|^2$.

Proof. For each $f \in \mathcal{H}$ we have

$$\sum_{i \in \mathbb{I}} \|\Lambda_i U^* f\|^2 \leq D_\Lambda \|U^* f\|^2 \leq D_\Lambda \|U\|^2 \|f\|^2.$$

Since $UK = KU$, it follows that $K^* U^* = U^* K^*$. Noting that U has closed range and that $\text{Range}(K^*) \subset \text{Range}(U)$, by Lemma 1.3 we get

$$\|K^* f\|^2 = \|UU^\dagger K^* f\|^2 = \|(U^\dagger)^* K^* U^* f\|^2 \leq \|U^\dagger\|^2 \|K^* U^* f\|^2, \quad \forall f \in \mathcal{H},$$

which tells us that

$$\sum_{i \in \mathbb{I}} \|\Lambda_i U^* f\|^2 \geq C_\Lambda \|K^* U^* f\|^2 \geq C_\Lambda \|U^\dagger\|^{-2} \|K^* f\|^2.$$

Therefore, $\{\Lambda_i U^*\}_{i \in \mathbb{I}}$ is a K -g-frame for $\mathcal{H}$ with frame bounds $C_\Lambda \|U^\dagger\|^{-2}$ and $D_\Lambda \|U\|^2$. Similarly, we can show that $\{\Gamma_i U^*\}_{i \in \mathbb{I}}$ is a K -g-frame for $\mathcal{H}$ with frame bounds $C_\Gamma \|U^\dagger\|^{-2}$ and $D_\Gamma \|U\|^2$.

Since $\{\Lambda_i\}_{i \in \sigma_1} \cup \{\Gamma_i\}_{i \in \sigma_2}$ is a K -g-frame for $\mathcal{H}$ with frame bounds C and D for any partition $\{\sigma_1, \sigma_2\}$ of $\mathbb{I}$, it follows that $\{\Lambda_i U^*\}_{i \in \sigma_1} \cup \{\Gamma_i U^*\}_{i \in \sigma_2}$ is a K -g-frame for $\mathcal{H}$ with frame bounds $C\|U^\dagger\|^{-2}$ and $D\|U\|^2$. That is, $\{\Lambda_i U^*\}_{i \in \mathbb{I}}$ and $\{\Gamma_i U^*\}_{i \in \mathbb{I}}$ are K -woven in $\mathcal{H}$ with the universal frame bounds $C\|U^\dagger\|^{-2}$ and $D\|U\|^2$. $\square$

Corollary 2.2. Suppose that $K \in B(\mathcal{H})$ has a dense range, that $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Gamma_i\}_{i \in \mathbb{I}}$ are K -g-frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$, and that $U \in B(\mathcal{H})$ has closed range with $UK = KU$. If $\{\Lambda_i U^*\}_{i \in \mathbb{I}}$ and $\{\Gamma_i U^*\}_{i \in \mathbb{I}}$, and $\{\Lambda_i U\}_{i \in \mathbb{I}}$ and $\{\Gamma_i U\}_{i \in \mathbb{I}}$ are both K -woven in $\mathcal{H}$, then $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Gamma_i\}_{i \in \mathbb{I}}$ are K -woven in $\mathcal{H}$.

Proof. Let C and D be the universal frame bounds of $\{\Lambda_i U^*\}_{i \in \mathbb{I}}$ and $\{\Gamma_i U^*\}_{i \in \mathbb{I}}$. Then for any partition $\{\sigma_1, \sigma_2\}$ of $\mathbb{I}$,

$$C\|K^* f\|^2 \leq \sum_{i \in \sigma_1} \|\Lambda_i U^* f\|^2 + \sum_{i \in \sigma_2} \|\Gamma_i U^* f\|^2, \quad \forall f \in \mathcal{H}. \quad (2.1)$$

Since K has a dense range, $\overline{\text{Range}}(K) = \mathcal{H}$, and $\text{Ker}(K^*) = (\overline{\text{Range}}(K))^\perp = \{0\}$ as a consequence. This together with (2.1) gives $\text{Range}(U) = (\text{Ker}(U^*))^\perp \supset (\text{Ker}(K^*))^\perp = \mathcal{H}$, meaning that U is surjective. A similar discussion can show that U^* is also surjective. Hence, U is invertible on $\mathcal{H}$. Since $UK = KU$, $U^{-1}K = KU^{-1}$. Now the conclusion follows from Theorem 2.1, since $\{\Lambda_i\}_{i \in \mathbb{I}} = \{\Lambda_i U^*(U^{-1})^*\}_{i \in \mathbb{I}}$ and $\{\Gamma_i\}_{i \in \mathbb{I}} = \{\Gamma_i U^*(U^{-1})^*\}_{i \in \mathbb{I}}$, and $\text{Range}(K^*) \subset \mathcal{H} = \text{Range}(U^{-1})$. $\square$

Corollary 2.3. Let K -g-frames $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Gamma_i\}_{i \in \mathbb{I}}$ be K -woven in $\mathcal{H}$ and $U \in B(\mathcal{H})$ with $UK = KU$. If U is surjective, then $\{\Lambda_i U^*\}_{i \in \mathbb{I}}$ and $\{\Gamma_i U^*\}_{i \in \mathbb{I}}$ are K -woven in $\mathcal{H}$.

Proof. It is an immediate consequence of Theorem 2.1. $\square$

In fact, the inverse of Corollary 2.3 remains true if the K -woven is tight and K^* is surjective, as shown in the following result.

Theorem 2.4. Suppose that K -g-frames $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Gamma_i\}_{i \in \mathbb{I}}$ are C -tight K -woven in $\mathcal{H}$, that $U \in B(\mathcal{H})$ with $UK = KU$, and that $\text{Range}(K^*) = \mathcal{H}$. If $\{\Lambda_i U^*\}_{i \in \mathbb{I}}$ and $\{\Gamma_i U^*\}_{i \in \mathbb{I}}$ are K -woven in $\mathcal{H}$, then U is surjective.

Proof. Let C' and D' be the universal frame bounds of $\{\Lambda_i U^*\}_{i \in \mathbb{I}}$ and $\{\Gamma_i U^*\}_{i \in \mathbb{I}}$. Then for any partition $\{\sigma_1, \sigma_2\}$ of $\mathbb{I}$ and any $f \in \mathcal{H}$,

$$C' \|K^* f\|^2 \leq \sum_{i \in \sigma_1} \|\Lambda_i U^* f\|^2 + \sum_{i \in \sigma_2} \|\Gamma_i U^* f\|^2 \leq D' \|f\|^2. \quad (2.2)$$

We also have

$$C \|K^* g\|^2 = \sum_{i \in \sigma_1} \|\Lambda_i g\|^2 + \sum_{i \in \sigma_2} \|\Gamma_i g\|^2$$

for any $g \in \mathcal{H}$, and the condition $UK = KU$ yields $U^* K^* = K^* U^*$. Thus,

$$C \|U^* K^* f\|^2 = C \|K^* U^* f\|^2 = \sum_{i \in \sigma_1} \|\Lambda_i U^* f\|^2 + \sum_{i \in \sigma_2} \|\Gamma_i U^* f\|^2.$$

This fact along with (2.2) implies that

$$\|U^* K^* f\|^2 = C^{-1} \left(\sum_{i \in \sigma_1} \|\Lambda_i U^* f\|^2 + \sum_{i \in \sigma_2} \|\Gamma_i U^* f\|^2 \right) \geq C^{-1} C' \|K^* f\|^2,$$

from which we conclude that U^* is bounded below as K^* is surjective and, consequently, U is surjective. $\square$

With a similar discussion to Theorem 2.4 we can obtain the following result.

Proposition 2.5. Suppose that K -g-frames $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Gamma_i\}_{i \in \mathbb{I}}$ are 1-tight K -woven in $\mathcal{H}$, that $U \in B(\mathcal{H})$ with $UK = KU$, and that $\text{Range}(K^*) = \mathcal{H}$. If $\{\Lambda_i U^*\}_{i \in \mathbb{I}}$ and $\{\Gamma_i U^*\}_{i \in \mathbb{I}}$ are 1-tight K -woven in $\mathcal{H}$, then U is a co-isometry.

Furthermore, we report in the following result that two different operators are applied to a K -woven pair of K -g-frames can also leave them K -woven.

Theorem 2.6. Suppose that K -g-frames $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Gamma_i\}_{i \in \mathbb{I}}$ are K -woven in $\mathcal{H}$ with the universal frame bounds C and D , that $U_i \in B(\mathcal{H})$ are surjective with $U_i K = K U_i$ ($i = 1, 2$). If $\text{Ker}(K^*) \subset \text{Ker}(U_i^*)$ for $i = 1, 2$, and $\|U_1^\dagger\| \|U_2 - U_1\| \|\widetilde{K}^{-1}\| < \sqrt{C/D_\Gamma}$, then $\{\Lambda_i U_1^*\}_{i \in \mathbb{I}}$ and $\{\Gamma_i U_2^*\}_{i \in \mathbb{I}}$ are K -woven in $\mathcal{H}$ with the universal frame bounds $(\sqrt{C} \|U_1^\dagger\|^{-1} - \sqrt{D_\Gamma} \|U_2 - U_1\| \|\widetilde{K}^{-1}\|)^2$ and $D_\Lambda \|U_1\|^2 + D_\Gamma \|U_2\|^2$, where $\widetilde{K}^* : \text{Ker}^\perp(K^*) \rightarrow \text{Range}(K^*)$ is the restriction of K^* on $\text{Ker}^\perp(K^*)$.

Proof. The proof of Theorem 2.1 tells us that both $\{\Lambda_i U_1^*\}_{i \in \mathbb{I}}$ and $\{\Gamma_i U_2^*\}_{i \in \mathbb{I}}$ are K -g-frames for $\mathcal{H}$. Now for any partition $\{\sigma_1, \sigma_2\}$ of $\mathbb{I}$ and any $g \in \text{Ker}^\perp(K^*)$ we have

$$\begin{aligned} & \left(\sum_{i \in \sigma_1} \|\Lambda_i U_1^* g\|^2 + \sum_{i \in \sigma_2} \|\Gamma_i U_2^* g\|^2 \right)^{1/2} \\ &= \left(\sum_{i \in \sigma_1} \|\Lambda_i U_1^* g\|^2 + \sum_{i \in \sigma_2} \|\Gamma_i U_1^* g + \Gamma_i (U_2^* - U_1^*) g\|^2 \right)^{1/2} \\ &\geq \left(\sum_{i \in \sigma_1} \|\Lambda_i U_1^* g\|^2 + \sum_{i \in \sigma_2} \|\Gamma_i U_1^* g\|^2 \right)^{1/2} - \left(\sum_{i \in \sigma_2} \|\Gamma_i (U_2^* - U_1^*) g\|^2 \right)^{1/2} \end{aligned}$$

$$\begin{aligned}
&\geq \sqrt{C} \|K^* U_1^* g\| - \left(\sum_{i \in \mathbb{I}} \|\Gamma_i(U_2^* - U_1^*)g\|^2 \right)^{1/2} \\
&\geq \sqrt{C} \|K^* U_1^* g\| - \sqrt{D_\Gamma} \|U_2^* - U_1^*\| \|g\| \\
&= \sqrt{C} \|U_1^* K^* g\| - \sqrt{D_\Gamma} \|U_2 - U_1\| \|\widetilde{K}^{*-1} \widetilde{K}^* g\| \\
&\geq \sqrt{C} \|U_1^\dagger\|^{-1} \|K^* g\| - \sqrt{D_\Gamma} \|U_2 - U_1\| \|\widetilde{K}^{*-1}\| \|\widetilde{K}^* g\| \\
&= (\sqrt{C} \|U_1^\dagger\|^{-1} - \sqrt{D_\Gamma} \|U_2 - U_1\| \|\widetilde{K}^{*-1}\|) \|K^* g\|.
\end{aligned}$$

Since each $f \in \mathcal{H}$ can be represented as $f = f_1 + f_2$ with $f_1 \in \text{Ker}(K^*)$ and $f_2 \in \text{Ker}^\perp(K^*)$, and $\text{Ker}(K^*) \subset \text{Ker}(U_i^*)$ ($i = 1, 2$), it follows that

$$\begin{aligned}
&\sum_{i \in \sigma_1} \|\Lambda_i U_1^* f\|^2 + \sum_{i \in \sigma_2} \|\Gamma_i U_2^* f\|^2 \\
&= \sum_{i \in \sigma_1} \|\Lambda_i U_1^* (f_1 + f_2)\|^2 + \sum_{i \in \sigma_2} \|\Gamma_i U_2^* (f_1 + f_2)\|^2 \\
&= \sum_{i \in \sigma_1} \|\Lambda_i U_1^* f_2\|^2 + \sum_{i \in \sigma_2} \|\Gamma_i U_2^* f_2\|^2 \\
&\geq (\sqrt{C} \|U_1^\dagger\|^{-1} - \sqrt{D_\Gamma} \|U_2 - U_1\| \|\widetilde{K}^{*-1}\|)^2 \|K^* f_2\|^2 \\
&= (\sqrt{C} \|U_1^\dagger\|^{-1} - \sqrt{D_\Gamma} \|U_2 - U_1\| \|\widetilde{K}^{*-1}\|)^2 \|K^* (f_1 + f_2)\|^2 \\
&= (\sqrt{C} \|U_1^\dagger\|^{-1} - \sqrt{D_\Gamma} \|U_2 - U_1\| \|\widetilde{K}^{*-1}\|)^2 \|K^* f\|^2.
\end{aligned}$$

The required universal upper frame bound follows from the following calculation:

$$\sum_{i \in \sigma_1} \|\Lambda_i U_1^* f\|^2 + \sum_{i \in \sigma_2} \|\Gamma_i U_2^* f\|^2 \leq \sum_{i \in \mathbb{I}} \|\Lambda_i U_1^* f\|^2 + \sum_{i \in \mathbb{I}} \|\Gamma_i U_2^* f\|^2 \leq (D_\Lambda \|U_1\|^2 + D_\Gamma \|U_2\|^2) \|f\|^2, \quad \forall f \in \mathcal{H}. \quad \square$$

At the end of this section we discuss conditions under which the sum of a K -woven pair of K -g-frames and their perturbation under an operator can be still K -woven.

Theorem 2.7. Suppose that K -g-frames $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Gamma_i\}_{i \in \mathbb{I}}$ are K -woven in $\mathcal{H}$ with the universal frame bounds C and D and that $T \in B(\mathcal{H})$ is a positive operator. For any $\sigma \subset \mathbb{I}$, let S^σ be the g -frame operator of $\{\Lambda_i\}_{i \in \sigma} \cup \{\Gamma_i\}_{i \in \sigma^c}$. If $TS^\sigma = S^\sigma T$, then $\{\Lambda_i + \Lambda_i T^*\}_{i \in \mathbb{I}}$ and $\{\Gamma_i + \Gamma_i T^*\}_{i \in \mathbb{I}}$ are K -woven in $\mathcal{H}$ with the universal frame bounds C and $D\|\text{Id}_{\mathcal{H}} + T\|^2$.

Proof. For any partition $\{\sigma_1, \sigma_2\}$ of $\mathbb{I}$ and any $f \in \mathcal{H}$ we have

$$\begin{aligned}
&\sum_{i \in \sigma_1} \|(\Lambda_i + \Lambda_i T^*)f\|^2 + \sum_{i \in \sigma_2} \|(\Gamma_i + \Gamma_i T^*)f\|^2 \\
&= \sum_{i \in \sigma_1} \|\Lambda_i(\text{Id}_{\mathcal{H}} + T^*)f\|^2 + \sum_{i \in \sigma_2} \|\Gamma_i(\text{Id}_{\mathcal{H}} + T^*)f\|^2 \\
&\leq D\|(\text{Id}_{\mathcal{H}} + T)^*f\|^2 \leq D\|\text{Id}_{\mathcal{H}} + T\|^2 \|f\|^2.
\end{aligned}$$

Since T is a positive operator with $TS^{\sigma_1} = S^{\sigma_1}T$, we can get $TS^{\sigma_1} \geq 0$ and $S^{\sigma_1}T^* \geq 0$. If, now, we denote by S the g -frame operator of $\{\Lambda_i + \Lambda_i T^*\}_{i \in \sigma_1} \cup \{\Gamma_i + \Gamma_i T^*\}_{i \in \sigma_2}$, then

$$\begin{aligned}
Sf &= \sum_{i \in \sigma_1} (\Lambda_i + \Lambda_i T^*)^* (\Lambda_i + \Lambda_i T^*) f + \sum_{i \in \sigma_2} (\Gamma_i + \Gamma_i T^*)^* (\Gamma_i + \Gamma_i T^*) f \\
&= \left(\sum_{i \in \sigma_1} \Lambda_i^* \Lambda_i f + \sum_{i \in \sigma_2} \Gamma_i^* \Gamma_i f \right) + T \left(\sum_{i \in \sigma_1} \Lambda_i^* \Lambda_i f + \sum_{i \in \sigma_2} \Gamma_i^* \Gamma_i f \right) \\
&\quad + \left(\sum_{i \in \sigma_1} \Lambda_i^* \Lambda_i T^* f + \sum_{i \in \sigma_2} \Gamma_i^* \Gamma_i T^* f \right) + T \left(\sum_{i \in \sigma_1} \Lambda_i^* \Lambda_i T^* f + \sum_{i \in \sigma_2} \Gamma_i^* \Gamma_i T^* f \right) \\
&= S^\sigma f + TS^\sigma f + S^{\sigma_1} T^* f + TS^{\sigma_1} T^* f \geq S^\sigma f
\end{aligned}$$

for any $f \in \mathcal{H}$. Therefore,

$$\sum_{i \in \sigma_1} \|(\Lambda_i + \Lambda_i T^*)f\|^2 + \sum_{i \in \sigma_2} \|(\Gamma_i + \Gamma_i T^*)f\|^2 = \langle Sf, f \rangle \geq \langle S^0 f, f \rangle \geq C \|K^* f\|^2,$$

and we have the result. $\square$

Theorem 2.8. Suppose that K -g-frames $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Gamma_i\}_{i \in \mathbb{I}}$ are K -woven in $\mathcal{H}$ with the universal frame bounds C and D and that $T \in B(\mathcal{H})$ is a positive operator. If TKK^* is positive, then $\{\Lambda_i + \Lambda_i T^*\}_{i \in \mathbb{I}}$ and $\{\Gamma_i + \Gamma_i T^*\}_{i \in \mathbb{I}}$ are K -woven in $\mathcal{H}$ with the universal frame bounds C and $D\|\text{Id}_{\mathcal{H}} + T\|^2$.

Proof. For each $f \in \mathcal{H}$ we have

$$\sum_{i \in \mathbb{I}} \|(\Lambda_i + \Lambda_i T^*)f\|^2 = \sum_{i \in \mathbb{I}} \|\Lambda_i(\text{Id}_{\mathcal{H}} + T^*)f\|^2 \leq D_{\Lambda} \|(\text{Id}_{\mathcal{H}} + T)^* f\|^2 \leq D_{\Lambda} \|\text{Id}_{\mathcal{H}} + T\|^2 \|f\|^2. \quad (2.3)$$

Now, noting that TKK^* is a positive operator, we obtain

$$\begin{aligned} \sum_{i \in \mathbb{I}} \|(\Lambda_i + \Lambda_i T^*)f\|^2 &\geq C_{\Lambda} \|K^*(\text{Id}_{\mathcal{H}} + T^*)f\|^2 \\ &= C_{\Lambda} (\|K^* f\|^2 + 2\text{Re}\langle K^* f, K^* T^* f \rangle + \|K^* T^* f\|^2) \\ &= C_{\Lambda} (\|K^* f\|^2 + 2\text{Re}\langle TKK^* f, f \rangle + \|K^* T^* f\|^2) \\ &\geq C_{\Lambda} (\|K^* f\|^2 + \|K^* T^* f\|^2) \\ &\geq C_{\Lambda} \|K^* f\|^2. \end{aligned}$$

This together with (2.3) means that $\{\Lambda_i + \Lambda_i T^*\}_{i \in \mathbb{I}}$ is a K -g-frame for $\mathcal{H}$ with frame bounds C_{Λ} and $D_{\Lambda} \|\text{Id}_{\mathcal{H}} + T\|^2$. Similarly, we know that $\{\Gamma_i + \Gamma_i T^*\}_{i \in \mathbb{I}}$ is also a K -g-frame for $\mathcal{H}$ with frame bounds C_{Γ} and $D_{\Gamma} \|\text{Id}_{\mathcal{H}} + T\|^2$.

Since $\{\Lambda_i\}_{i \in \sigma_1} \cup \{\Gamma_i\}_{i \in \sigma_2}$ is a K -g-frame for $\mathcal{H}$ with frame bounds C and D for any partition $\{\sigma_1, \sigma_2\}$ of $\mathbb{I}$, following above discussion we conclude that $\{\Lambda_i + \Lambda_i T^*\}_{i \in \sigma_1} \cup \{\Gamma_i + \Gamma_i T^*\}_{i \in \sigma_2}$ is a K -g-frame for $\mathcal{H}$ with frame bounds C and $D\|\text{Id}_{\mathcal{H}} + T\|^2$. That is, $\{\Lambda_i + \Lambda_i T^*\}_{i \in \mathbb{I}}$ and $\{\Gamma_i + \Gamma_i T^*\}_{i \in \mathbb{I}}$ are K -woven in $\mathcal{H}$ with the universal frame bounds C and $D\|\text{Id}_{\mathcal{H}} + T\|^2$. $\square$

3 Constructions of the weaving of K -g-frames

In this section, we provide several new methods to construct K -woven pair of K -g-frames. We begin with a necessary and sufficient condition for two K -g-frames to be K -woven.

Theorem 3.1. Let $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Gamma_i\}_{i \in \mathbb{I}}$ be two K -g-frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$. Then $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Gamma_i\}_{i \in \mathbb{I}}$ are K -woven in $\mathcal{H}$ if and only if there exists a g -Bessel sequence $\{\Theta_i\}_{i \in \mathbb{I}}$ for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$ such that for any partition $\{\sigma_1, \sigma_2\}$ of $\mathbb{I}$ and any $f \in \mathcal{H}$,

$$Kf = \sum_{i \in \sigma_1} \Lambda_i^* \Theta_i f + \sum_{i \in \sigma_2} \Gamma_i^* \Theta_i f.$$

Proof. “ $\Leftarrow$.” It is sufficient to show the universal lower frame bound condition. For any $f, g \in \mathcal{H}$,

$$\begin{aligned} \langle Kg, f \rangle &= \sum_{i \in \sigma_1} \langle \Lambda_i^* \Theta_i g, f \rangle + \sum_{i \in \sigma_2} \langle \Gamma_i^* \Theta_i g, f \rangle \\ &= \left\langle g, \sum_{i \in \sigma_1} \Theta_i^* \Lambda_i f \right\rangle + \left\langle g, \sum_{i \in \sigma_2} \Theta_i^* \Gamma_i f \right\rangle \\ &= \left\langle g, \sum_{i \in \sigma_1} \Theta_i^* \Lambda_i f + \sum_{i \in \sigma_2} \Theta_i^* \Gamma_i f \right\rangle, \end{aligned}$$

implying that $K^*f = \sum_{i \in \sigma_1} \Theta_i^* \Lambda_i f + \sum_{i \in \sigma_2} \Theta_i^* \Gamma_i f$, and thus

$$\begin{aligned} \|K^*f\|^2 &= \sup_{\|h\|=1} |\langle K^*f, h \rangle|^2 \\ &= \sup_{\|h\|=1} \left| \sum_{i \in \sigma_1} \langle \Lambda_i f, \Theta_i h \rangle + \sum_{i \in \sigma_2} \langle \Gamma_i f, \Theta_i h \rangle \right|^2 \\ &\leq 2 \sup_{\|h\|=1} \left(\left| \sum_{i \in \sigma_1} \langle \Lambda_i f, \Theta_i h \rangle \right|^2 + \left| \sum_{i \in \sigma_2} \langle \Gamma_i f, \Theta_i h \rangle \right|^2 \right) \\ &\leq 2D_\Theta \left(\sum_{i \in \sigma_1} \|\Lambda_i f\|^2 + \sum_{i \in \sigma_2} \|\Gamma_i f\|^2 \right). \end{aligned}$$

Therefore,

$$\frac{1}{2D_\Theta} \|K^*f\|^2 \leq \sum_{i \in \sigma_1} \|\Lambda_i f\|^2 + \sum_{i \in \sigma_2} \|\Gamma_i f\|^2,$$

as desired.

“ $\Rightarrow$.” Let C be the universal lower frame bound of $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Gamma_i\}_{i \in \mathbb{I}}$. For any partition $\{\sigma_1, \sigma_2\}$ of $\mathbb{I}$, let T be the synthesis operator of the g-Bessel sequence $\{\Lambda_i\}_{i \in \sigma_1} \cup \{\Gamma_i\}_{i \in \sigma_2}$, and P_n be the projection on $\ell^2(\{\mathcal{H}_i\}_{i \in \mathbb{I}})$ that maps each element to its n th component, i.e., $p_n\{g_i\}_{i \in \mathbb{I}} = \{u_i\}_{i \in \mathbb{I}}$, where

$$u_i = \begin{cases} g_n & \text{if } i = n, \\ 0 & \text{if } i \neq n, \end{cases}$$

for each $\{g_i\}_{i \in \mathbb{I}} \in \ell^2(\{\mathcal{H}_i\}_{i \in \mathbb{I}})$. Since

$$C\langle KK^*f, f \rangle \leq \sum_{i \in \sigma_1} \|\Lambda_i f\|^2 + \sum_{i \in \sigma_2} \|\Gamma_i f\|^2 = \langle TT^*f, f \rangle, \quad \forall f \in \mathcal{H},$$

we know that $KK^* \leq \frac{1}{C} TT^*$. By Lemma 1.4, there exists $U \in B(\mathcal{H}, \ell^2(\{\mathcal{H}_i\}_{i \in \mathbb{I}}))$ such that $K = TU$. Taking $\Theta_i = P_i U$ for $i \in \mathbb{I}$, then

$$\sum_{i \in \mathbb{I}} \|\Theta_i f\|^2 = \sum_{i \in \mathbb{I}} \|P_i Uf\|^2 = \sum_{i \in \mathbb{I}} \|(Uf)_i\|^2 = \|Uf\|^2 \leq \|U\|^2 \|f\|^2$$

for any $f \in \mathcal{H}$, showing that $\{\Theta_i\}_{i \in \mathbb{I}}$ is a g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$. Now

$$\begin{aligned} Kf &= TUf \\ &= \sum_{i \in \sigma_1} \Lambda_i^*(Uf)_i + \sum_{i \in \sigma_2} \Gamma_i^*(Uf)_i \\ &= \sum_{i \in \sigma_1} \Lambda_i^*(P_i Uf) + \sum_{i \in \sigma_2} \Gamma_i^*(P_i Uf) \\ &= \sum_{i \in \sigma_1} \Lambda_i^* \Theta_i f + \sum_{i \in \sigma_2} \Gamma_i^* \Theta_i f. \end{aligned} \quad \square$$

Corollary 3.2. Suppose that $K \in B(\mathcal{H})$ has closed range and that $\Lambda_i, \Gamma_i \in B(\mathcal{H}, \mathcal{H}_i)$ for each $i \in \mathbb{I}$. If $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Gamma_i\}_{i \in \mathbb{I}}$ are a K -woven pair of K -g-frames in $\mathcal{H}$, then they are a woven pair of g-frames in $\text{Range}(K)$.

Proof. Let $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Gamma_i\}_{i \in \mathbb{I}}$ be a K -woven pair of K -g-frames in $\mathcal{H}$. Then, by Theorem 3.1 there is a g-Bessel sequence $\{\Theta_i\}_{i \in \mathbb{I}}$ for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$ such that for any partition $\{\sigma_1, \sigma_2\}$ of $\mathbb{I}$ and any $f \in \mathcal{H}$ we have

$$Kf = \sum_{i \in \sigma_1} \Lambda_i^* \Theta_i f + \sum_{i \in \sigma_2} \Gamma_i^* \Theta_i f.$$

For each $h \in \text{Range}(K)$, we see from Lemma 1.3 that

$$h = KK^+h = \sum_{i \in \sigma_1} \Lambda_i^*(\Theta_i K^+ P_{\text{Range}(K)})h + \sum_{i \in \sigma_2} \Gamma_i^*(\Theta_i K^+ P_{\text{Range}(K)})h.$$

Taking $\Phi_i = \Theta_i K^+ P_{\text{Range}(K)}$ for any $i \in \mathbb{I}$, then it is easy to check that $\{\Phi_i\}_{i \in \mathbb{I}}$ is a g-Bessel sequence for $\mathcal{H}$ with bound $D_\Theta \|K^+\|^2$. Now for any $g, h \in \text{Range}(K)$,

$$\begin{aligned} \langle g, h \rangle &= \left\langle \sum_{i \in \sigma_1} \Lambda_i^* \Phi_i g + \sum_{i \in \sigma_2} \Gamma_i^* \Phi_i g, h \right\rangle \\ &= \sum_{i \in \sigma_1} \langle \Lambda_i^* \Phi_i g, h \rangle + \sum_{i \in \sigma_2} \langle \Gamma_i^* \Phi_i g, h \rangle \\ &= \sum_{i \in \sigma_1} \langle g, \Phi_i^* \Lambda_i h \rangle + \sum_{i \in \sigma_2} \langle g, \Phi_i^* \Gamma_i h \rangle \\ &= \left\langle g, \sum_{i \in \sigma_1} \Phi_i^* \Lambda_i h + \sum_{i \in \sigma_2} \Phi_i^* \Gamma_i h \right\rangle. \end{aligned}$$

It follows that $h = \sum_{i \in \sigma_1} \Phi_i^* \Lambda_i h + \sum_{i \in \sigma_2} \Phi_i^* \Gamma_i h$. Hence,

$$\begin{aligned} \|h\|^2 &= \sup_{\|g\|=1} |\langle h, g \rangle|^2 \\ &= \sup_{\|g\|=1} \left| \sum_{i \in \sigma_1} \langle \Lambda_i h, \Phi_i g \rangle + \sum_{i \in \sigma_2} \langle \Gamma_i h, \Phi_i g \rangle \right|^2 \\ &\leq 2 \sup_{\|g\|=1} \left(\left| \sum_{i \in \sigma_1} \langle \Lambda_i h, \Phi_i g \rangle \right|^2 + \left| \sum_{i \in \sigma_2} \langle \Gamma_i h, \Phi_i g \rangle \right|^2 \right) \\ &\leq 2D_\Theta \|K^+\|^2 \left(\sum_{i \in \sigma_1} \|\Lambda_i h\|^2 + \sum_{i \in \sigma_2} \|\Gamma_i h\|^2 \right). \end{aligned}$$

The universal upper frame bound condition is trivial, and the proof is complete. $\square$

Corollary 3.3. Let $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Gamma_i\}_{i \in \mathbb{I}}$ be two K -g-frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$. Then $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Gamma_i\}_{i \in \mathbb{I}}$ are K -woven in $\mathcal{H}$ if and only if there exists $M > 0$ such that for each $f \in \mathcal{H}$, there is $\{c_i^f\}_{i \in \mathbb{I}} \in \ell^2(\{\mathcal{H}_i\}_{i \in \mathbb{I}})$ satisfying

- (a) $Kf = \sum_{i \in \sigma_1} \Lambda_i^* c_i^f + \sum_{i \in \sigma_2} \Gamma_i^* c_i^f$ for any partition $\{\sigma_1, \sigma_2\}$ of $\mathbb{I}$, and
 (b) $\sum_{i \in \mathbb{I}} \|c_i^f\|^2 \leq M \|f\|^2$.

Proof. “ $\Rightarrow$.” By Theorem 3.1 there is a g-Bessel sequence $\{\Theta_i\}_{i \in \mathbb{I}}$ for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$ such that $Kf = \sum_{i \in \sigma_1} \Lambda_i^* \Theta_i f + \sum_{i \in \sigma_2} \Gamma_i^* \Theta_i f$ for any partition $\{\sigma_1, \sigma_2\}$ of $\mathbb{I}$ and any $f \in \mathcal{H}$. Taking $M = D_\Theta$ and $c_i^f = \Theta_i f$ for any $i \in \mathbb{I}$, then

$$Kf = \sum_{i \in \sigma_1} \Lambda_i^* c_i^f + \sum_{i \in \sigma_2} \Gamma_i^* c_i^f \quad \text{and} \quad \sum_{i \in \mathbb{I}} \|c_i^f\|^2 = \sum_{i \in \mathbb{I}} \|\Theta_i f\|^2 \leq D_\Theta \|f\|^2 = M \|f\|^2.$$

“ $\Leftarrow$.” The proof is similar to the “ $\Leftarrow$ ” part of Theorem 3.1, we omit the details. $\square$

In next two theorems, we show that the weaving of K -g-frames can be checked in subspaces.

Theorem 3.4. Suppose that K -g-frames $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Gamma_i\}_{i \in \mathbb{I}}$ are K -woven in $\mathcal{M}$, a closed subspace of $\mathcal{H}$, with the universal frame bounds C and D .

- (1) If $\text{Range}(K) \subset \mathcal{M}$, then $\{\Lambda_i P_{\mathcal{M}}\}_{i \in \mathbb{I}}$ and $\{\Gamma_i P_{\mathcal{M}}\}_{i \in \mathbb{I}}$ are K -woven in $\mathcal{H}$ with the same universal frame bounds.
 (2) If $\mathcal{M}$ is dense in $\mathcal{H}$, then $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Gamma_i\}_{i \in \mathbb{I}}$ are K -woven in $\mathcal{H}$ with the same universal frame bounds.

Proof.

(1) For any $f \in \mathcal{H}$, $P_{\mathcal{M}}f \in \mathcal{M}$ and thus,

$$\sum_{i \in \mathbb{I}} \|\Lambda_i P_{\mathcal{M}} f\|^2 \leq D_{\Lambda} \|P_{\mathcal{M}} f\|^2 \leq D_{\Lambda} \|f\|^2.$$

Noting now that each $f \in \mathcal{H}$ has a decomposition $f = f_1 + f_2$ with $f_1 \in \mathcal{M}$ and $f_2 \in \mathcal{M}^{\perp}$, and that $\langle K^* f_2, g \rangle = \langle f_2, Kg \rangle = 0$ for any $g \in \mathcal{H}$ leading to $K^* f_2 = 0$, we obtain

$$\sum_{i \in \mathbb{I}} \|\Lambda_i P_{\mathcal{M}} f\|^2 \geq C_{\Lambda} \|K^* P_{\mathcal{M}} f\|^2 = C_{\Lambda} \|K^* P_{\mathcal{M}}(f_1 + f_2)\|^2 = C_{\Lambda} \|K^* f_1\|^2 = C_{\Lambda} \|K^*(f_1 + f_2)\|^2 = C_{\Lambda} \|K^* f\|^2.$$

Altogether we know that $\{\Lambda_i P_{\mathcal{M}}\}_{i \in \mathbb{I}}$ is a K -g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$ with frame bounds C_{Λ} and D_{Λ} and, similarly, we can show that $\{\Gamma_i P_{\mathcal{M}}\}_{i \in \mathbb{I}}$ is a K -g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$ with frame bounds C_{Γ} and D_{Γ} .

Since $\{\Lambda_i\}_{i \in \sigma_1} \cup \{\Gamma_i\}_{i \in \sigma_2}$ is a K -g-frame for $\mathcal{M}$ with respect to $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$ with frame bounds C and D for any partition $\{\sigma_1, \sigma_2\}$ of $\mathbb{I}$, from above discussion we know that $\{\Lambda_i P_{\mathcal{M}}\}_{i \in \sigma_1} \cup \{\Gamma_i P_{\mathcal{M}}\}_{i \in \sigma_2}$ is a K -g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$ with frame bounds C and D . That is, $\{\Lambda_i P_{\mathcal{M}}\}_{i \in \mathbb{I}}$ and $\{\Gamma_i P_{\mathcal{M}}\}_{i \in \mathbb{I}}$ are K -woven in $\mathcal{H}$ with the universal frame bounds C and D .

(2) For any $g \in \mathcal{M}$ we have

$$C_{\Lambda} \|K^* g\|^2 \leq \sum_{i \in \mathbb{I}} \|\Lambda_i g\|^2 \leq D_{\Lambda} \|g\|^2. \quad (3.1)$$

We prove first that $\{\Lambda_i\}_{i \in \mathbb{I}}$ is a g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$. Assume on the contrary that there exists $f \in \mathcal{H}$ such that

$$\sum_{i \in \mathbb{I}} \|\Lambda_i f\|^2 > D_{\Lambda} \|f\|^2.$$

Then there is a finite set $\mathbb{J} \subset \mathbb{I}$ so that

$$\sum_{i \in \mathbb{J}} \|\Lambda_i f\|^2 > D_{\Lambda} \|f\|^2.$$

Since $\mathcal{M}$ is dense in $\mathcal{H}$, there exists $h \in \mathcal{M}$ satisfying

$$\sum_{i \in \mathbb{J}} \|\Lambda_i h\|^2 > D_{\Lambda} \|h\|^2,$$

a contradiction to (3.1). Next we show that $\{\Lambda_i\}_{i \in \mathbb{I}}$ satisfies the opposite inequality of a K -g-frame. Let $f \in \mathcal{H}$, then for any $\varepsilon > 0$ there is $g \in \mathcal{M}$ such that $\|f - g\| < \varepsilon$. Clearly, from (3.1) we can get that $C_{\Lambda} \|K^* f\|^2 \leq D_{\Lambda} \|f\|^2$. Now

$$\begin{aligned} \left(\sum_{i \in \mathbb{I}} \|\Lambda_i f\|^2 \right)^{1/2} &= \|\{\Lambda_i f\}_{i \in \mathbb{I}}\| \geq \|\{\Lambda_i g\}_{i \in \mathbb{I}}\| - \|\{\Lambda_i(f - g)\}_{i \in \mathbb{I}}\| \\ &\geq \sqrt{C_{\Lambda}} \|K^* g\| - \sqrt{D_{\Lambda}} \|f - g\| \\ &= \sqrt{D_{\Lambda}} \|f - g\| + \sqrt{C_{\Lambda}} \|K^* g\| - 2\sqrt{D_{\Lambda}} \|f - g\| \\ &\geq \sqrt{C_{\Lambda}} \|K^*(f - g)\| + \sqrt{C_{\Lambda}} \|K^* g\| - 2\sqrt{D_{\Lambda}} \varepsilon \\ &\geq \sqrt{C_{\Lambda}} \|K^*(f - g) + K^* g\| - 2\sqrt{D_{\Lambda}} \varepsilon \\ &= \sqrt{C_{\Lambda}} \|K^* f\| - 2\sqrt{D_{\Lambda}} \varepsilon. \end{aligned}$$

Therefore,

$$\sum_{i \in \mathbb{I}} \|\Lambda_i f\|^2 \geq C_{\Lambda} \|K^* f\|^2,$$

as $\varepsilon \rightarrow 0$. Similarly, we can show that $\{\Gamma_i\}_{i \in \mathbb{I}}$ is a K -g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$ with frame bounds C_{Γ} and D_{Γ} .

Since $\{\Lambda_i\}_{i \in \sigma_1} \cup \{\Gamma_i\}_{i \in \sigma_2}$ is a K -g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$ with frame bounds C and D for any partition $\{\sigma_1, \sigma_2\}$ of $\mathbb{I}$, it follows that $\{\Lambda_i\}_{i \in \sigma_1} \cup \{\Gamma_i\}_{i \in \sigma_2}$ is a K -g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$ with frame bounds C and D . That is, $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Gamma_i\}_{i \in \mathbb{I}}$ are K -woven in $\mathcal{H}$ with the universal frame bounds C and D . $\square$

We now pay attention to constructing K -woven pair of K -g-frames for subspaces.

Theorem 3.5. Suppose that $T, U \in B(\mathcal{H})$ has closed ranges, that $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Gamma_i\}_{i \in \mathbb{I}}$ are K -g-frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$, that $\text{Ker}(T) = \text{Ker}(U)$ with $KUT^+ = UT^+K$, and that $\text{Range}(K^*) \subset \text{Range}(U)$. If $\{\Lambda_i T^*\}_{i \in \mathbb{I}}$ and $\{\Gamma_i T^*\}_{i \in \mathbb{I}}$ are K -woven in $\text{Range}(T)$, then $\{\Lambda_i U^*\}_{i \in \mathbb{I}}$ and $\{\Gamma_i U^*\}_{i \in \mathbb{I}}$ are K -woven in $\text{Range}(U)$.

Proof. Define the operator $L : \text{Range}(T) \rightarrow \text{Range}(U)$ by $Lf = UT^+f$. Since $\text{Ker } T = \text{Ker } U$, we have $\text{Range}(T^*) = (\text{Ker } T)^\perp = (\text{Ker } U)^\perp = \text{Range}(U^*)$. Thus by Lemma 1.3,

$$\begin{aligned} \text{Ker } UT^+ &= \text{Ker } TT^+ = (\text{Range}(T))^\perp, \\ T^+T &= P_{\text{Range}(T^*)} = P_{\text{Range}(U^*)} = U^+U, \\ LT &= UT^+T = UU^+U = U. \end{aligned} \quad (3.2)$$

From (3.2) we see that

$$\text{Ker } L = \text{Ker } UT^+ \cap \text{Range}(T) = (\text{Range}(T))^\perp \cap \text{Range}(T) = \{0\},$$

implying that L is invertible on $\text{Range}(T)$. Let C' and D' be frame bounds of $\{\Lambda_i T^*\}_{i \in \mathbb{I}}$. Since for each $f \in \text{Range}(U)$, $L^*f \in \text{Range}(T)$, we have

$$\sum_{i \in \mathbb{I}} \|\Lambda_i U^* f\|^2 = \sum_{i \in \mathbb{I}} \|\Lambda_i (LT)^* f\|^2 = \sum_{i \in \mathbb{I}} \|\Lambda_i T^* L^* f\|^2 \geq C' \|K^* L^* f\|^2 = C' \|L^* K^* f\|^2 \geq C' \|L^{-1}\|^{-2} \|K^* f\|^2.$$

It is trivial to show that

$$\sum_{i \in \mathbb{I}} \|\Lambda_i U^* f\|^2 = \sum_{i \in \mathbb{I}} \|\Lambda_i T^* L^* f\|^2 \leq D' \|L^* f\|^2 \leq D' \|L\|^2 \|f\|^2$$

for every $f \in \text{Range}(U)$. Hence, $\{\Lambda_i U^*\}_{i \in \mathbb{I}}$ is a K -g-frame for $\text{Range}(U)$ with respect to $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$. Similarly we can prove that $\{\Gamma_i U^*\}_{i \in \mathbb{I}}$ is a K -g-frame for $\text{Range}(U)$ with respect to $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$ when $\{\Gamma_i T^*\}_{i \in \mathbb{I}}$ is a K -g-frame for $\text{Range}(T)$ with respect to $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$.

Since, for any partition $\{\sigma_1, \sigma_2\}$ of $\mathbb{I}$, $\{\Lambda_i T^*\}_{i \in \sigma_1} \cup \{\Gamma_i T^*\}_{i \in \sigma_2}$ is a K -g-frame for $\text{Range}(T)$ with respect to $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$, it follows that $\{\Lambda_i U^*\}_{i \in \sigma_1} \cup \{\Gamma_i U^*\}_{i \in \sigma_2}$ is a K -g-frame for $\text{Range}(U)$ with respect to $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$. That is, $\{\Lambda_i U^*\}_{i \in \mathbb{I}}$ and $\{\Gamma_i U^*\}_{i \in \mathbb{I}}$ are K -woven in $\text{Range}(U)$. $\square$

The following result tells us that if one K -g-frame is K -woven to two K -g-frames, then it can be K -woven to their sum in $\text{Range}(K)$.

Theorem 3.6. Let $K \in B(\mathcal{H})$ be a positive and closed range operator, and $\{\Lambda_i\}_{i \in \mathbb{I}}$, $\{\Gamma_i\}_{i \in \mathbb{I}}$ and $\{\Theta_i\}_{i \in \mathbb{I}}$ be K -g-frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$. Assume that $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Gamma_i\}_{i \in \mathbb{I}}$, and $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Theta_i\}_{i \in \mathbb{I}}$ are both K -woven in $\mathcal{H}$ with the universal lower frame bounds C' and C'' , respectively. Then $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Gamma_i + \Theta_i\}_{i \in \mathbb{I}}$ are K -woven in $\text{Range}(K)$ with the universal frame bounds $(C' + C'' - (D_\Lambda + 2\sqrt{D_\Gamma D_\Theta})\|K^+\|^2)$ and $D_\Lambda + 2(D_\Gamma + D_\Theta)$, if the following two conditions hold:

- (i) $Kf = \sum_{i \in \mathbb{I}} \Gamma_i^* \Theta_i f$ for each $f \in \mathcal{H}$;
- (ii) $C_\Gamma > D_\Theta \|K^+\|^2$ and $C' + C'' > (D_\Lambda + 2\sqrt{D_\Gamma D_\Theta})\|K^+\|^2$.

Proof. We conclude first that $\{\Gamma_i + \Theta_i\}_{i \in \mathbb{I}}$ is a K -g-frame for $\text{Range}(K)$ with respect to $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$. As a matter of fact, for any $f \in \text{Range}(K)$ we have

$$\begin{aligned}
\left(\sum_{i \in \mathbb{I}} \|(\Gamma_i + \Theta_i)f\|^2 \right)^{1/2} &= \|\{\Gamma_i\}_{i \in \mathbb{I}} + \{\Theta_i\}_{i \in \mathbb{I}}\| \\
&\geq \|\{\Gamma_i\}_{i \in \mathbb{I}}\| - \|\{\Theta_i\}_{i \in \mathbb{I}}\| \\
&\geq \sqrt{C_\Gamma} \|K^*f\| - \sqrt{D_\Theta} \|f\| \\
&= \sqrt{C_\Gamma} \|K^*f\| - \sqrt{D_\Theta} \|KK^*f\| \\
&= \sqrt{C_\Gamma} \|K^*f\| - \sqrt{D_\Theta} \|(K^\dagger)^*K^*f\| \\
&\geq (\sqrt{C_\Gamma} - \sqrt{D_\Theta} \|K^\dagger\|) \|K^*f\|.
\end{aligned}$$

The g -Bessel bound of $\{\Gamma_i + \Theta_i\}_{i \in \mathbb{I}}$ is trivial. Next we show that $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Gamma_i + \Theta_i\}_{i \in \mathbb{I}}$ are K -woven in $\text{Range}(K)$. For any partition $\{\sigma_1, \sigma_2\}$ of $\mathbb{I}$, let

$$T_{\Gamma}^{\sigma_1}(\{g_i\}_{i \in \mathbb{I}}) = \sum_{i \in \sigma_1} \Gamma_i^* g_i \quad \text{and} \quad T_{\Theta}^{\sigma_2}(\{g_i\}_{i \in \mathbb{I}}) = \sum_{i \in \sigma_2} \Theta_i^* g_i \quad (3.3)$$

for any $\{g_i\}_{i \in \mathbb{I}} \in \ell^2(\{\mathcal{H}_i\}_{i \in \mathbb{I}})$. Then it is easy to check that $\|T_{\Gamma}^{\sigma_1}\| \leq \|T_{\Gamma}\| \leq \sqrt{D_\Gamma}$ and $\|T_{\Theta}^{\sigma_2}\| \leq \|T_{\Theta}\| \leq \sqrt{D_\Theta}$. Now for any $f \in \text{Range}(K)$ we compute

$$\begin{aligned}
&\sum_{i \in \sigma_1} \|\Lambda_i f\|^2 + \sum_{i \in \sigma_2} \|(\Gamma_i + \Theta_i)f\|^2 \\
&= \sum_{i \in \sigma_1} \|\Lambda_i f\|^2 + \sum_{i \in \sigma_2} \|\Gamma_i f\|^2 + \sum_{i \in \sigma_2} \|\Theta_i f\|^2 + \sum_{i \in \sigma_2} \langle \Gamma_i^* \Theta_i f, f \rangle + \sum_{i \in \sigma_2} \langle \Theta_i^* \Gamma_i f, f \rangle \\
&= \left(\sum_{i \in \sigma_1} \|\Lambda_i f\|^2 + \sum_{i \in \sigma_2} \|\Gamma_i f\|^2 \right) + \left(\sum_{i \in \sigma_1} \|\Lambda_i f\|^2 + \sum_{i \in \sigma_2} \|\Theta_i f\|^2 \right) \\
&\quad + \langle Kf, f \rangle + \langle K^*f, f \rangle - \left(\sum_{i \in \sigma_1} \langle \Gamma_i^* \Theta_i f, f \rangle + \sum_{i \in \sigma_1} \langle \Theta_i^* \Gamma_i f, f \rangle \right) - \sum_{i \in \sigma_1} \|\Lambda_i f\|^2 \\
&\geq C' \|K^*f\|^2 + C'' \|K^*f\|^2 - D_\Lambda \|f\|^2 - \left| \sum_{i \in \sigma_1} \langle \Gamma_i^* \Theta_i f, f \rangle + \sum_{i \in \sigma_1} \langle \Theta_i^* \Gamma_i f, f \rangle \right| \\
&\geq (C' + C'') \|K^*f\|^2 - D_\Lambda \|f\|^2 - \|f\| \|T_{\Gamma}^{\sigma_1} T_{\Theta}^* f + T_{\Theta}^{\sigma_1} T_{\Gamma}^* f\| \\
&\geq (C' + C'') \|K^*f\|^2 - (D_\Lambda + 2\sqrt{D_\Gamma D_\Theta}) \|f\|^2 \\
&\geq (C' + C'') \|K^*f\|^2 - (D_\Lambda + 2\sqrt{D_\Gamma D_\Theta}) \|K^\dagger\|^2 \|K^*f\|^2 \\
&= (C' + C'' - (D_\Lambda + 2\sqrt{D_\Gamma D_\Theta}) \|K^\dagger\|^2) \|K^*f\|^2.
\end{aligned}$$

The fact that $D_\Lambda + 2(D_\Gamma + D_\Theta)$ is an upper weaving bound is obvious. $\square$

We conclude this section with the result showing that we can use operator K and the corresponding synthesis operators to construct K -woven pair of K -g-frames.

Theorem 3.7. Let $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Gamma_i\}_{i \in \mathbb{I}}$ be two K -g-frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$. For any partition $\{\sigma_1, \sigma_2\}$ of $\mathbb{I}$, let T and S be, respectively, the synthesis operator and g -frame operator of $\{\Lambda_i\}_{i \in \sigma_1} \cup \{\Gamma_i\}_{i \in \sigma_2}$. Then $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Gamma_i\}_{i \in \mathbb{I}}$ are K -woven in $\mathcal{H}$ with the universal frame bounds C and D , if the following two conditions are satisfied:

- (i) $\text{Range}(K) \subset \text{Range}(T)$ and $\text{Range}(S) \subset \text{Range}(T)$;
- (ii) There exist two constants $C, D > 0$ such that

$$C \|K\|^2 \|\{g_i\}_{i \in \mathbb{I}}\|^2 \leq \|T\{g_i\}_{i \in \mathbb{I}}\|^2 \leq D \|\{g_i\}_{i \in \mathbb{I}}\|^2, \quad \forall \{g_i\}_{i \in \mathbb{I}} \in \text{Ker}^\perp(T). \quad (3.4)$$

Proof. We first prove that T has closed range. Suppose that $\{h_n\} \subset \text{Range}(T)$ and $h_n \rightarrow h$ as $n \rightarrow \infty$. Since $\ell^2(\{\mathcal{H}_i\}_{i \in \mathbb{I}}) = \text{Ker}(T) \oplus \text{Ker}^\perp(T)$, we can find $\{x_n\} \subset \text{Ker}^\perp(T)$ such that $h_n = Tx_n$. For any $m, n \in \mathbb{N}$, we see from (3.4) that

$$\|x_n - x_m\|^2 \leq C^{-1} \|K\|^{-2} \|Tx_n - Tx_m\|^2 = C^{-1} \|K\|^{-2} \|h_n - h_m\|^2,$$

it follows that $\{x_n\}$ is a Cauchy sequence. Assume that $x_n \rightarrow x \in \ell^2(\{\mathcal{H}_i\}_{i \in \mathbb{I}})$, then by the continuity of T we have $h_n = Tx_n \rightarrow Tx = h \in \text{Range}(T)$ as $n \rightarrow \infty$. Thus, $\text{Range}(T)$ is closed. For each $f \in \mathcal{H}$, there exist $g \in \text{Range}(T)$ and $h \in (\text{Range}(T))^\perp = \text{Ker}(T^+)$ such that $f = g + h$. Then $T^+f = T^+g \in \text{Range}(T^+) = \text{Ker}^\perp(T)$. It follows, by combining Lemma 1.3 and (3.4), that

$$\begin{aligned}\|T^+f\|^2 &= \|T^+g\|^2 \\ &\leq C^{-1}\|K\|^{-2}\|TT^+g\|^2 \\ &= C^{-1}\|K\|^{-2}\|g\|^2 \\ &\leq C^{-1}\|K\|^{-2}(\|g\|^2 + \|h\|^2) \\ &= C^{-1}\|K\|^{-2}\|f\|^2.\end{aligned}$$

Hence, $\|T^+\|^2 \leq C^{-1}\|K\|^{-2}$. Now for any $g \in \text{Range}(T)$ we have

$$\begin{aligned}\|K^*g\|^2 &\leq \|K\|^2\|g\|^2 \\ &= \|K\|^2\|TT^+g\|^2 \\ &= \|K\|^2\|(T^+)^*T^*g\|^2 \\ &\leq \|K\|^2\|T^+\|^2\|T^*g\|^2 \leq C^{-1}\|T^*g\|^2 \\ &= C^{-1}\left(\sum_{i \in \sigma_1}\|\Lambda_i g\|^2 + \sum_{i \in \sigma_2}\|\Gamma_i g\|^2\right).\end{aligned}$$

Therefore,

$$C\|K^*g\|^2 \leq \sum_{i \in \sigma_1}\|\Lambda_i g\|^2 + \sum_{i \in \sigma_2}\|\Gamma_i g\|^2.$$

For each $f \in \mathcal{H}$ we have $f = g + h$ with $g \in \text{Range}(T)$ and $h \in (\text{Range}(T))^\perp$. Since $\text{Range}(K) \subset \text{Range}(T)$ and $\text{Range}(S) \subset \text{Range}(T)$, it follows that

$$\begin{aligned}\sum_{i \in \sigma_1}\|\Lambda_i f\|^2 + \sum_{i \in \sigma_2}\|\Gamma_i f\|^2 &= \sum_{i \in \sigma_1}\|\Lambda_i(g+h)\|^2 + \sum_{i \in \sigma_2}\|\Gamma_i(g+h)\|^2 \\ &= \sum_{i \in \sigma_1}\|\Lambda_i g\|^2 + \left\langle \sum_{i \in \sigma_1}\Lambda_i^* \Lambda_i g, h \right\rangle + \left\langle h, \sum_{i \in \sigma_1}\Lambda_i^* \Lambda_i g \right\rangle + \left\langle \sum_{i \in \sigma_1}\Lambda_i^* \Lambda_i h, h \right\rangle \\ &\quad + \sum_{i \in \sigma_2}\|\Gamma_i g\|^2 + \left\langle \sum_{i \in \sigma_2}\Gamma_i^* \Gamma_i g, h \right\rangle + \left\langle h, \sum_{i \in \sigma_2}\Gamma_i^* \Gamma_i g \right\rangle + \left\langle \sum_{i \in \sigma_2}\Gamma_i^* \Gamma_i h, h \right\rangle \\ &= \sum_{i \in \sigma_1}\|\Lambda_i g\|^2 + \sum_{i \in \sigma_2}\|\Gamma_i g\|^2 + \langle Sg, h \rangle + \langle h, Sg \rangle + \langle Sh, h \rangle \\ &= \sum_{i \in \sigma_1}\|\Lambda_i g\|^2 + \sum_{i \in \sigma_2}\|\Gamma_i g\|^2 \geq C\|K^*g\|^2 = C\|K^*(g+h)\|^2 = C\|K^*f\|^2.\end{aligned}$$

Next we prove the opposite inequality. For any $g \in \ell^2(\{\mathcal{H}_i\}_{i \in \mathbb{I}})$, there exist $g_1 \in \text{Ker}(T)$ and $g_2 \in \text{Ker}^\perp(T)$ such that $g = g_1 + g_2$. Then the right-hand side of inequality of (3.4) leads to

$$\|Tg\|^2 = \|Tg_2\|^2 \leq D\|g_2\|^2 \leq D(\|g_1\|^2 + \|g_2\|^2) = D\|g\|^2.$$

Thus $\|T\| \leq \sqrt{D}$ and, consequently,

$$\sum_{i \in \sigma_1}\|\Lambda_i f\|^2 + \sum_{i \in \sigma_2}\|\Gamma_i f\|^2 = \|T^+f\|^2 \leq D\|f\|^2, \quad \forall f \in \mathcal{H}.$$

Altogether we obtain

$$C\|K^*f\|^2 \leq \sum_{i \in \sigma_1}\|\Lambda_i f\|^2 + \sum_{i \in \sigma_2}\|\Gamma_i f\|^2 \leq D\|f\|^2, \quad \forall f \in \mathcal{H}.$$

This completes the proof. $\square$

4 A new Paley-Wiener-type perturbation result on the weaving of K -g-frames

The perturbation theory for frames is important in practice and therefore many researchers were interested and some significant results were presented, see, for example, the papers [26,28]. Recently, the authors in [25] proved a Paley-Wiener-type perturbation theorem (Theorem 3) for the weaving of K -g-frames in a subspace of $\mathcal{H}$. We show in the following result, however, that under the perturbation condition involved in that theorem, two K -g-frames can be K -woven in $\mathcal{H}$.

Theorem 4.1. *Let $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Gamma_i\}_{i \in \mathbb{I}}$ be K -g-frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in \mathbb{I}}$. Assume that $\text{Range}(T_\Lambda^{\sigma_1}) \subset \text{Range}(K)$ and $\text{Range}(T_\Gamma^{\sigma_2}) \subset \text{Range}(K)$ for any partition $\{\sigma_1, \sigma_2\}$ of $\mathbb{I}$, where $T_\Lambda^{\sigma_1}$ and $T_\Gamma^{\sigma_2}$ are defined the same way as in (3.3). If there are constants $\alpha, \beta, \gamma \geq 0$ such that $C_\Lambda - (\sqrt{D_\Lambda} + \sqrt{D_\Gamma})(\alpha\sqrt{D_\Lambda} + \beta\sqrt{D_\Gamma} + \gamma)\|\widetilde{K}^{*-1}\|^2 > 0$ and*

$$\left\| \sum_{i \in \mathbb{I}} (\Gamma_i^* - \Lambda_i^*) g_i \right\| \leq \alpha \left\| \sum_{i \in \mathbb{I}} \Lambda_i^* g_i \right\| + \beta \left\| \sum_{i \in \mathbb{I}} \Gamma_i^* g_i \right\| + \gamma \|\{g_i\}_{i \in \mathbb{I}}\| \quad (4.1)$$

for any $\{g_i\}_{i \in \mathbb{I}} \in \ell^2(\{\mathcal{H}_i\}_{i \in \mathbb{I}})$, then $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Gamma_i\}_{i \in \mathbb{I}}$ are K -woven in $\mathcal{H}$ with the universal frame bounds $C_\Lambda - (\sqrt{D_\Lambda} + \sqrt{D_\Gamma})(\alpha\sqrt{D_\Lambda} + \beta\sqrt{D_\Gamma} + \gamma)\|\widetilde{K}^{*-1}\|^2$ and $\sqrt{D_\Lambda} + \sqrt{D_\Gamma}$, where $\widetilde{K}^* : \text{Ker}^\perp(K^*) \rightarrow \text{Range}(K^*)$ is the restriction of K^* on $\text{Ker}^\perp(K^*)$.

Proof. It is easy to check that $\{\Lambda_i\}_{i \in \mathbb{I}}$ and $\{\Gamma_i\}_{i \in \mathbb{I}}$ always admit an universal upper frame bound $\sqrt{D_\Lambda} + \sqrt{D_\Gamma}$. So it only needs to prove the universal lower frame bound condition. Since $\|T_\Lambda\| \leq \sqrt{D_\Lambda}$ and $\|T_\Gamma\| \leq \sqrt{D_\Gamma}$, by using (4.1) we compute that

$$\begin{aligned} \|T_\Gamma\{g_i\}_{i \in \mathbb{I}} - T_\Lambda\{g_i\}_{i \in \mathbb{I}}\| &= \left\| \sum_{i \in \mathbb{I}} (\Gamma_i^* - \Lambda_i^*) g_i \right\| \\ &\leq \alpha \left\| \sum_{i \in \mathbb{I}} \Lambda_i^* g_i \right\| + \beta \left\| \sum_{i \in \mathbb{I}} \Gamma_i^* g_i \right\| + \gamma \|\{g_i\}_{i \in \mathbb{I}}\| \\ &= \alpha \|T_\Lambda\{g_i\}_{i \in \mathbb{I}}\| + \beta \|T_\Gamma\{g_i\}_{i \in \mathbb{I}}\| + \gamma \|\{g_i\}_{i \in \mathbb{I}}\| \\ &\leq \alpha \sqrt{D_\Lambda} \|\{g_i\}_{i \in \mathbb{I}}\| + \beta \sqrt{D_\Gamma} \|\{g_i\}_{i \in \mathbb{I}}\| + \gamma \|\{g_i\}_{i \in \mathbb{I}}\| \\ &= (\alpha \sqrt{D_\Lambda} + \beta \sqrt{D_\Gamma} + \gamma) \|\{g_i\}_{i \in \mathbb{I}}\| \end{aligned}$$

for any $\{g_i\}_{i \in \mathbb{I}} \in \ell^2(\{\mathcal{H}_i\}_{i \in \mathbb{I}})$, which leads to $\|T_\Gamma - T_\Lambda\| \leq \alpha \sqrt{D_\Lambda} + \beta \sqrt{D_\Gamma} + \gamma$. Hence, for any partition $\{\sigma_1, \sigma_2\}$ of $\mathbb{I}$ we get

$$\begin{aligned} \|T_\Lambda^{\sigma_2} T_\Lambda^* - T_\Gamma^{\sigma_2} T_\Gamma^*\| &= \|(T_\Lambda^{\sigma_2} T_\Lambda^* - T_\Lambda^{\sigma_2} T_\Gamma^*) + (T_\Lambda^{\sigma_2} T_\Gamma^* - T_\Gamma^{\sigma_2} T_\Gamma^*)\| \\ &\leq \|T_\Lambda^{\sigma_2} T_\Lambda^* - T_\Lambda^{\sigma_2} T_\Gamma^*\| + \|T_\Lambda^{\sigma_2} T_\Gamma^* - T_\Gamma^{\sigma_2} T_\Gamma^*\| \\ &\leq \|T_\Lambda^{\sigma_2}\| \|T_\Lambda^* - T_\Gamma^*\| + \|T_\Lambda^{\sigma_2} - T_\Gamma^{\sigma_2}\| \|T_\Gamma^*\| \\ &\leq (\|T_\Lambda\| + \|T_\Gamma\|) \|T_\Lambda - T_\Gamma\| \\ &\leq (\sqrt{D_\Lambda} + \sqrt{D_\Gamma})(\alpha \sqrt{D_\Lambda} + \beta \sqrt{D_\Gamma} + \gamma), \end{aligned}$$

where in the third inequality we apply the fact that $\|T_\Lambda^{\sigma_2} - T_\Gamma^{\sigma_2}\| \leq \|T_\Lambda - T_\Gamma\|$. Thus,

$$\begin{aligned} \sum_{i \in \sigma_1} \|\Lambda_i g\|^2 + \sum_{i \in \sigma_2} \|\Gamma_i g\|^2 &= \sum_{i \in \sigma_1} \|\Lambda_i g\|^2 + \sum_{i \in \sigma_2} \|\Lambda_i g\|^2 - \left(\sum_{i \in \sigma_2} \|\Lambda_i g\|^2 - \sum_{i \in \sigma_2} \|\Gamma_i g\|^2 \right) \\ &= \sum_{i \in \mathbb{I}} \|\Lambda_i g\|^2 - \left\langle \sum_{i \in \sigma_2} (\Lambda_i^* \Lambda_i g - \Gamma_i^* \Gamma_i g), g \right\rangle \end{aligned}$$

$$\begin{aligned}
&\geq C_\Lambda \|K^*g\|^2 - \|g\| \left\| \sum_{i \in \sigma_2} \Lambda_i^* \Lambda_i g - \sum_{i \in \sigma_2} \Gamma_i^* \Gamma_i g \right\| \\
&= C_\Lambda \|K^*g\|^2 - \|g\| \|T_\Lambda^{\sigma_2} T_\Lambda^* g - T_\Gamma^{\sigma_2} T_\Gamma^* g\| \\
&\geq C_\Lambda \|K^*g\|^2 - (\sqrt{D_\Lambda} + \sqrt{D_\Gamma})(\alpha\sqrt{D_\Lambda} + \beta\sqrt{D_\Gamma} + \gamma) \|g\|^2 \\
&= C_\Lambda \|K^*g\|^2 - (\sqrt{D_\Lambda} + \sqrt{D_\Gamma})(\alpha\sqrt{D_\Lambda} + \beta\sqrt{D_\Gamma} + \gamma) \|\widetilde{K}^{*-1} \widetilde{K}^* g\|^2 \\
&\geq (C_\Lambda - (\sqrt{D_\Lambda} + \sqrt{D_\Gamma})(\alpha\sqrt{D_\Lambda} + \beta\sqrt{D_\Gamma} + \gamma) \|\widetilde{K}^{*-1}\|^2) \|K^*g\|^2
\end{aligned}$$

for any $g \in \text{Ker}^\perp(K^*)$. Noting that each $f \in \mathcal{H}$ has a decomposition $f = f_1 + f_2$ with $f_1 \in \text{Ker}(K^*)$ and $f_2 \in \text{Ker}^\perp(K^*)$, and that $\text{Range}(T_\Lambda^{\sigma_1}) \subset \text{Range}(K)$ and $\text{Range}(T_\Gamma^{\sigma_2}) \subset \text{Range}(K)$, we obtain

$$\begin{aligned}
\sum_{i \in \sigma_1} \|\Lambda_i f\|^2 + \sum_{i \in \sigma_2} \|\Gamma_i f\|^2 &= \sum_{i \in \sigma_1} \|\Lambda_i(f_1 + f_2)\|^2 + \sum_{i \in \sigma_2} \|\Gamma_i(f_1 + f_2)\|^2 \\
&= \sum_{i \in \sigma_1} \|\Lambda_i f_2\|^2 + \left\langle \sum_{i \in \sigma_1} \Lambda_i^* \Lambda_i f_1, f_1 \right\rangle + 2\text{Re} \left\langle \sum_{i \in \sigma_1} \Lambda_i^* \Lambda_i f_2, f_1 \right\rangle \\
&\quad + \sum_{i \in \sigma_2} \|\Gamma_i f_2\|^2 + \left\langle \sum_{i \in \sigma_2} \Gamma_i^* \Gamma_i f_1, f_1 \right\rangle + 2\text{Re} \left\langle \sum_{i \in \sigma_2} \Gamma_i^* \Gamma_i f_2, f_1 \right\rangle \\
&= \sum_{i \in \sigma_1} \|\Lambda_i f_2\|^2 + \sum_{i \in \sigma_2} \|\Gamma_i f_2\|^2 + \langle T_\Lambda^{\sigma_1} T_\Lambda^* f_1, f_1 \rangle \\
&\quad + 2\text{Re} \langle T_\Lambda^{\sigma_1} T_\Lambda^* f_2, f_1 \rangle + \langle T_\Gamma^{\sigma_2} T_\Gamma^* f_1, f_1 \rangle + 2\text{Re} \langle T_\Gamma^{\sigma_2} T_\Gamma^* f_2, f_1 \rangle \\
&= \sum_{i \in \sigma_1} \|\Lambda_i f_2\|^2 + \sum_{i \in \sigma_2} \|\Gamma_i f_2\|^2 \\
&\geq (C_\Lambda - (\sqrt{D_\Lambda} + \sqrt{D_\Gamma})(\alpha\sqrt{D_\Lambda} + \beta\sqrt{D_\Gamma} + \gamma) \|\widetilde{K}^{*-1}\|^2) \|K^* f_2\|^2 \\
&= (C_\Lambda - (\sqrt{D_\Lambda} + \sqrt{D_\Gamma})(\alpha\sqrt{D_\Lambda} + \beta\sqrt{D_\Gamma} + \gamma) \|\widetilde{K}^{*-1}\|^2) \|K^*(f_1 + f_2)\|^2 \\
&= (C_\Lambda - (\sqrt{D_\Lambda} + \sqrt{D_\Gamma})(\alpha\sqrt{D_\Lambda} + \beta\sqrt{D_\Gamma} + \gamma) \|\widetilde{K}^{*-1}\|^2) \|K^* f\|^2,
\end{aligned}$$

and we have the result. $\square$

Acknowledgments: The author thanks the anonymous reviewers and the editor for their valuable suggestions and comments which have led to a significant improvement of this manuscript.

Funding information: The research was supported by the National Natural Science Foundation of China (Nos 11761057 and 11561057) and the Science Foundation of Jiangxi Education Department (Nos GJJ202302 and GJJ190886).

Author contributions: The author has accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: The author states no conflict of interest.

References

- [1] W. Sun, *G-frames and g-Riesz bases*, J. Math. Anal. Appl. **322** (2006), no. 1, 437–452, DOI: <https://doi.org/10.1016/j.jmaa.2005.09.039>.
- [2] X.-X. Guo, *New characterizations of g-Bessel sequences and g-Riesz bases in Hilbert spaces*, Results Math. **68** (2015), no. 3–4, 361–374, DOI: <https://doi.org/10.1007/s00025-015-0444-4>.
- [3] A. Khosravi and M. Mirzaee Azandaryani, *G-frames and direct sums*, Bull. Malays. Math. Sci. Soc. **36** (2013), no. 2, 313–323.

- [4] A. Najati and A. Rahimi, *Generalized frames in Hilbert spaces*, Bull. Iranian Math. Soc. **35** (2009), no. 1, 97–109.
- [5] M. Rahmani, *Redundancy of g -frames*, Complex Anal. Oper. Theory **14** (2020), no. 1, 23, DOI: <https://doi.org/10.1007/s11785-020-00985-5>.
- [6] L. Găvruta, *Frames for operators*, Appl. Comput. Harmon. Anal. **32** (2012), no. 1, 139–144, DOI: <https://doi.org/10.1016/j.acha.2011.07.006>.
- [7] H. Ellouz, *Some properties of K -frames in quaternionic Hilbert spaces*, Complex Anal. Oper. Theory **14** (2020), no. 1, 8, DOI: <https://doi.org/10.1007/s11785-019-00964-5>.
- [8] X.-X. Guo, *Canonical dual K -Bessel sequences and dual K -Bessel generators for unitary systems of Hilbert spaces*, J. Math. Anal. Appl. **444** (2016), no. 1, 598–609, DOI: <https://doi.org/10.1016/j.jmaa.2016.06.055>.
- [9] M. Jia and Y.-C. Zhu, *Some results about the operator perturbation of a K -frame*, Results Math. **73** (2018), no. 4, 138, DOI: <https://doi.org/10.1007/s00025-018-0902-x>.
- [10] P. S. Johnson and G. Ramu, *Class of bounded operators associated with an atomic system*, Tamkang J. Math. **46** (2015), no. 1, 85–90, DOI: <https://doi.org/10.5556/j.tkjm.46.2015.1601>.
- [11] K. T. Poumai and S. Jahan, *Atomic systems for operators*, Int. J. Wavelets Multiresolut. Inf. Process. **17** (2019), no. 1, 1850066, DOI: <https://doi.org/10.1142/S0219691318500662>.
- [12] X.-C. Xiao, Y.-C. Zhu, Z.-B. Shu, and M.-L. Ding, *G -frames with bounded linear operators*, Rocky Mountain J. Math. **45** (2015), no. 2, 675–693, DOI: <https://doi.org/10.1216/RMJ-2015-45-2-675>.
- [13] M. S. Asgari and H. Rahimi, *Generalized frames for operators in Hilbert spaces*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **17** (2014), no. 2, 1450013, DOI: <https://doi.org/10.1142/S0219025714500131>.
- [14] D.-L. Hua and Y.-D. Huang, *K -g-frames and stability of K -g-frames in Hilbert spaces*, J. Korean Math. Soc. **53** (2016), no. 6, 1331–1345, DOI: <http://dx.doi.org/10.4134/JKMS.j150499>.
- [15] Y.-D. Huang and S.-N. Shi, *New constructions of K -g-frames*, Results Math. **73** (2018), no. 4, 162, DOI: <https://doi.org/10.1007/s00025-018-0924-4>.
- [16] X.-C. Xiao and Y.-C. Zhu, *Exact K -g-frames in Hilbert spaces*, Results Math. **72** (2017), no. 3, 1329–1339, DOI: <https://doi.org/10.1007/s00025-017-0657-9>.
- [17] T. Bemrose, P. G. Casazza, K. Gröchenig, M. C. Lammers, and R. G. Lynch, *Weaving frames*, Oper. Matrices **10** (2016), no. 4, 1093–1116, DOI: <https://doi.org/10.7153/oam-10-61>.
- [18] P. G. Casazza, D. Freeman, and R. G. Lynch, *Weaving Schauder frames*, J. Approx. Theory **211** (2016), 42–60, DOI: <https://doi.org/10.1016/j.jat.2016.07.001>.
- [19] Deepshikha and L. K. Vashisht, *On weaving frames*, Houston J. Math. **44** (2018), no. 3, 887–915.
- [20] A. Rahimi, Z. Samadzadeh, and B. Daraby, *Frame related operators for woven frames*, Int. J. Wavelets Multiresolut. Inf. Process. **17** (2019), no. 3, 1950010, DOI: <https://doi.org/10.1142/S0219691319500103>.
- [21] A. Khosravi and J. S. Banyarani, *Weaving g -frames and weaving fusion frames*, Bull. Malays. Math. Sci. Soc. **42** (2019), no. 6, 3111–3129, DOI: <https://doi.org/10.1007/s40840-018-0647-4>.
- [22] D.-W. Li, J.-S. Leng, T.-Z. Huang, and X.-P. Li, *On weaving g -frames for Hilbert spaces*, Complex Anal. Oper. Theory **14** (2020), no. 2, 33, DOI: <https://doi.org/10.1007/s11785-020-00991-7>.
- [23] L. K. Vashisht, S. Agarwal, and Deepshikha, *On generalized weaving frames in Hilbert spaces*, Rocky Mountain J. Math. **48** (2018), no. 2, 661–685, DOI: <https://doi.org/10.1216/RMJ-2018-48-2-661>.
- [24] Deepshikha and L. K. Vashisht, *Weaving K -frames in Hilbert spaces*, Results Math. **73** (2018), no. 2, 81, DOI: <https://doi.org/10.1007/s00025-018-0843-4>.
- [25] X.-C. Xiao, G.-R. Zhou, and Y.-C. Zhu, *Weaving of K -g-frames in Hilbert spaces*, Sci. Asia **45** (2019), no. 3, 285–291, DOI: <http://dx.doi.org/10.2306/scienceasia1513-1874.2019.45.285>.
- [26] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, Berlin, 2016.
- [27] R. G. Douglas, *On majorization, factorization, and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc. **17** (1966), no. 2, 413–415, DOI: <https://doi.org/10.1090/S0002-9939-1966-0203464-1>.
- [28] P. G. Casazza and G. Kutyniok, *Finite Frames: Theory and Applications*, Birkhäuser, Berlin, 2012.