Research Article

Azhar Hussain, Muhammad Adeel, Hassen Aydi*, and Dumitru Baleanu

Global optimization and applications to a variational inequality problem

<https://doi.org/10.1515/math-2021-0104>

received November 27, 2020; accepted June 26, 2021

Abstract: In the present paper, we study the existence and convergence of the best proximity point for cyclic Θ -contractions. As consequences, we extract several fixed point results for such cyclic mappings. As an application, we present some solvability theorems in the topic of variational inequalities.

Keywords: cyclic contraction, best proximity point, variational inequality

MSC 2020: 65K10, 47H10

1 Introduction and preliminaries

Let $\mathcal{C}$ and $\mathcal{D}$ be two non-empty closed subsets of a complete metric space $(\mathcal{X}, \varpi)$. A mapping $\Psi : \mathcal{C} \cup \mathcal{D} \rightarrow \mathcal{C} \cup \mathcal{D}$ is called cyclic if $\Psi(\mathcal{C}) \subseteq \mathcal{D}$ and $\Psi(\mathcal{D}) \subseteq \mathcal{C}$. In the case that $\mathcal{C} \cap \mathcal{D} \neq \emptyset$, the mapping Ψ satisfying the contractive condition

$$\varpi(\Psi x, \Psi y) \leq \gamma \varpi(x, y), \quad (1)$$

where $0 < \gamma < 1$, has a fixed point in $\mathcal{C} \cap \mathcal{D}$ (note that Ψ need not to be continuous [1]). On the other hand, an element $x^* \in \mathcal{C} \cup \mathcal{D}$ is a best proximity point of Ψ if $\varpi(x^*, \Psi x^*) = \text{dist}(\mathcal{C}, \mathcal{D}) = \inf\{\varpi(\zeta_1, \zeta_2) : \zeta_1 \in \mathcal{C}, \zeta_2 \in \mathcal{D}\}$.

The existence and approximation of best proximity points is an important theme in optimization theory. Elder and Veeramani [2] extended the result of Kirk et al. [1] and proved the existence and uniqueness of best proximity points in complete metric spaces.

Definition 1.1. [2] Let $\mathcal{C}$ and $\mathcal{D}$ be non-empty subsets of $(\mathcal{X}, \varpi)$. A map $\Psi : \mathcal{C} \cup \mathcal{D} \rightarrow \mathcal{C} \cup \mathcal{D}$ is a cyclic contraction map if $\Psi(\mathcal{C}) \subseteq \mathcal{D}$, $\Psi(\mathcal{D}) \subseteq \mathcal{C}$ and for some $0 < \gamma < 1$,

$$\varpi(\Psi x, \Psi y) \leq \gamma \varpi(x, y) + (1 - \gamma) \text{dist}(\mathcal{C}, \mathcal{D}). \quad (2)$$

* **Corresponding author: Hassen Aydi**, Université de Sousse, Institut Supérieur d'Informatique et des Techniques de Communication, H. Sousse 4000, Tunisia; Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, South Africa; China Medical University Hospital, China Medical University, Taichung, 40402, Taiwan, e-mail: hassen.aydi@isima.rnu.tn

Azhar Hussain: Department of Mathematics, University of Sargodha, Sargodha-40100, Pakistan, e-mail: hafiziqbal30@yahoo.com

Muhammad Adeel: Department of Mathematics, University of Central Punjab, Lahore, Pakistan, e-mail: adeel.uosmaths@gmail.com

Dumitru Baleanu: Department of Mathematics, Cankaya University, Ankara, Turkey; Institute of Space Sciences, Magurele, Bucharest, Romania; Department of Medical Research, China Medical University Hospital, Taiwan, e-mail: dumitru@cankaya.edu.tr

Thagafi and Shahzad [3] provided a constructive answer to the question proposed by Elder and Veeramani [2] on the existence of a best proximity point for a cyclic contraction map on a reflexive Banach space. Abkar and Gabeleh [4] have shown best proximity point theorems for cyclic contractions in ordered metric spaces. For more work in this direction, one can follow [5–11].

On the other hand, in 2014 the contractive condition of Banach was generalized by Jleli and Samet [12] in the following way:

$$\Theta(\varpi(\Psi x, \Psi y)) \leq [\Theta(\varpi(x, y))]^k, \quad (3)$$

for $0 < k < 1$ and $x, y \in \mathfrak{X}$, where $\Theta : [0, \infty) \rightarrow [1, \infty)$ satisfies

(Θ_1) Θ is non-decreasing;

(Θ_2) for the sequence $\{y_n\} \subseteq \mathbb{R}^+$, $\lim_{n \rightarrow \infty} \Theta(y_n) = 1$ if and only if $\lim_{n \rightarrow \infty} (y_n) = 0$;

(Θ_3) there exist $0 < r < 1$ and $l \in (0, \infty)$ such that $\lim_{y \rightarrow 0^+} \frac{\Theta(y) - 1}{y^r} = l$.

They [12] proved a version of Banach contraction principle for such mappings. The condition (Θ_2) can be replaced by the simpler condition:

$$(\Theta_2) : \Theta(y) > 1 \quad \text{for any } y > 0.$$

Ahmad et al. [13] have used the following alternative condition:

(Θ'_3) Θ is continuous on $(0, \infty)$.

The class of functions satisfying (Θ_1), (Θ_2) and (Θ'_3) is denoted by Λ . Many researchers have generalized (3) and have shown fixed point theorems for single and multivalued contractive mappings in many directions (see [13–19]).

In this article, we attempt to generalize the results of [1] and [2] for cyclic Θ -contractions. We also obtain certain fixed point results for such type of contractions. Some few examples are presented to prove the significance of our findings. Moreover, as an application, we prove solvability theorems for variational inequality problems.

2 Best proximity point results

In this section, we present the notion of cyclic Θ -contractions and discuss the existence and convergence of best proximity points for this kind of mappings in complete metric spaces.

Definition 2.1. Let $\mathfrak{C}, \mathfrak{D}$ be two non-empty subsets of $(\mathfrak{X}, \varpi)$. Let $\Theta \in \Lambda$ be a continuous function. A mapping $\Psi : \mathfrak{C} \cup \mathfrak{D} \rightarrow \mathfrak{C} \cup \mathfrak{D}$ is called a cyclic Θ -contraction if $\Psi(\mathfrak{C}) \subseteq \mathfrak{D}$, $\Psi(\mathfrak{D}) \subseteq \mathfrak{C}$ and for all $x \in \mathfrak{C}$ and $y \in \mathfrak{D}$, we have

$$\Theta(\varpi^*(\Psi x, \Psi y)) \leq [\Theta(\varpi^*(x, y))]^k, \quad (4)$$

where $k \in (0, 1)$ and $\varpi^*(x, y) = \varpi(x, y) - \text{dist}(\mathfrak{C}, \mathfrak{D})$.

Remark 2.1. Every cyclic contraction in a classical sense is a cyclic Θ -contraction for $\Theta(t) = e^t$ with $k \in (0, 1)$. Indeed, let Ψ be a cyclic contraction on $\mathfrak{C} \cup \mathfrak{D}$, so for all $x \in \mathfrak{C}$ and $y \in \mathfrak{D}$, we have

$$\varpi(\Psi x, \Psi y) \leq k\varpi(x, y) + (1 - k)\text{dist}(\mathfrak{C}, \mathfrak{D}).$$

This implies that

$$\varpi(\Psi x, \Psi y) - \text{dist}(\mathfrak{C}, \mathfrak{D}) \leq k(\varpi(x, y) - \text{dist}(\mathfrak{C}, \mathfrak{D})), \quad (5)$$

which further yields by taking exponential on both sides

$$e^{\varpi^*(\Psi x, \Psi y)} \leq e^{k\varpi^*(x, y)}.$$

Hence,

$$\Theta(\varpi^*(\Psi x, \Psi y)) \leq [\Theta(\varpi^*(x, y))]^k.$$

Remark 2.2. From (5), one has $\varpi(\Psi x, \Psi y) \leq \varpi(x, y)$ for all $x \in \mathfrak{C}$ and $y \in \mathfrak{D}$.

Example 2.1. Let $\mathfrak{X} = \{1, 2, 5, 6, 9\}$ be endowed with the metric ϖ given as

$$\varpi(x, y) = \begin{cases} 5 & \text{if } x, y \in \{1, 5\}, \\ 6 & \text{if } x, y \in \{2, 6\}, \\ 0 & \text{if } x = y, \\ 1 & \text{otherwise.} \end{cases}$$

Then $(\mathfrak{X}, \varpi)$ is a complete metric space. Let $\mathfrak{C} = \{1, 2\}$ and $\mathfrak{D} = \{5, 6\}$ be subsets of $\mathfrak{X}$. We have $\text{dist}(\mathfrak{C}, \mathfrak{D}) = 1$. Define $\Psi : \mathfrak{C} \cup \mathfrak{D} \rightarrow \mathfrak{C} \cup \mathfrak{D}$ by $\Psi(1) = 5$, $\Psi(2) = 6$, $\Psi(5) = 2$ and $\Psi(6) = 1$. Then $\Psi(\mathfrak{C}) = \mathfrak{D}$, $\Psi(\mathfrak{D}) = \mathfrak{C}$. If we take $\Theta(t) = 5^t$, then clearly, Ψ is a cyclic Θ -contraction for all $k \in (0, 1)$.

First, we give an approximation result.

Lemma 2.1. Let $\mathfrak{C}$ and $\mathfrak{D}$ be non-empty subsets of a metric space $(\mathfrak{X}, \varpi)$ and $\Psi : \mathfrak{C} \cup \mathfrak{D} \rightarrow \mathfrak{C} \cup \mathfrak{D}$ be a cyclic Θ -contraction. For $x_0 \in \mathfrak{C} \cup \mathfrak{D}$, the sequence $\{x_n\}$ defined by $x_{n+1} = \Psi x_n$, $n = 0, 1, 2, \dots$, verifies

$$\lim_{n \rightarrow \infty} \varpi(x_n, x_{n+1}) = \text{dist}(\mathfrak{C}, \mathfrak{D}). \quad (6)$$

Proof. If there is an integer n_0 so that $\varpi(x_{n_0}, x_{n_0+1}) = \text{dist}(\mathfrak{C}, \mathfrak{D})$, the proof is done. Assume that for each n , $\varpi(x_n, x_{n+1}) > \text{dist}(\mathfrak{C}, \mathfrak{D})$, so $\varpi^*(x_n, x_{n+1}) > 0$ for each $n \in \mathbb{N}$. One writes

$$\begin{aligned} 1 &< \Theta(\varpi^*(x_n, x_{n+1})) \leq [(\Theta(\varpi^*(x_{n-1}, x_n)))^k] \\ &\leq [(\Theta(\varpi^*(x_{n-2}, x_{n-1})))^{k^2}] \\ &\vdots \\ &\leq [(\Theta(\varpi^*(x_0, x_1)))^{k^n}]. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$, we have

$$\Theta(\varpi^*(x_n, x_{n+1})) \rightarrow 1.$$

By (Θ_2) , we obtain

$$\lim_{n \rightarrow \infty} \varpi^*(x_n, x_{n+1}) = 0. \quad (7)$$

This gives that

$$\lim_{n \rightarrow \infty} \varpi(x_n, x_{n+1}) = \text{dist}(\mathfrak{C}, \mathfrak{D}). \quad \square$$

Next, the existence of a best proximity point for a cyclic Θ -contraction mapping is as follows.

Lemma 2.2. Let $\mathfrak{C}$ and $\mathfrak{D}$ be non-empty subsets of a metric space $(\mathfrak{X}, \varpi)$ and $\Psi : \mathfrak{C} \cup \mathfrak{D} \rightarrow \mathfrak{C} \cup \mathfrak{D}$ be a cyclic Θ -contraction. If $\{x_{2n}\}$ and $\{x_{2n+1}\}$ have convergent subsequences in $\mathfrak{C}$ and $\mathfrak{D}$, respectively, then there exists $(x, y) \in \mathfrak{C} \times \mathfrak{D}$ such that

$$\varpi(x, \Psi x) = \text{dist}(\mathfrak{C}, \mathfrak{D}) = \varpi(y, \Psi y) = \varpi(x, y).$$

Proof. Since $\Psi(\mathfrak{C}) \subseteq \mathfrak{D}$ and $\Psi(\mathfrak{D}) \subseteq \mathfrak{C}$, likewise in Lemma 2.1, we have a sequence $\{x_n\}$ in $\mathfrak{C} \cup \mathfrak{D}$ such that $\{x_{2n}\}$ is contained in $\mathfrak{C}$ and $\{x_{2n+1}\}$ is contained in $\mathfrak{D}$. Let $\{x_{2n_k}\}$ be a subsequence of $\{x_{2n}\}$ such that

$$\lim_{k \rightarrow \infty} x_{2n_k} = x \quad (8)$$

for some $x \in \mathfrak{C}$. One writes

$$\text{dist}(\mathfrak{C}, \mathfrak{D}) \leq \varpi(x, x_{2n_k-1}) \leq \varpi(x, x_{2n_k}) + \varpi(x_{2n_k}, x_{2n_k-1}). \quad (9)$$

Taking the limit as $k \rightarrow \infty$ in (9) and using Lemma 2.1 with (8), we get

$$\lim_{k \rightarrow \infty} \varpi(x, x_{2n_k-1}) = \text{dist}(\mathfrak{C}, \mathfrak{D}). \quad (10)$$

Since Ψ is a cyclic Θ -contraction, we have

$$\Theta[\varpi^*(\Psi x_{2n_k-1}, \Psi x)] \leq [\Theta(\varpi^*(x_{2n_k-1}, x))]^k < \Theta(\varpi^*(x_{2n_k-1}, x)). \quad (11)$$

That is,

$$\varpi(\Psi x_{2n_k-1}, \Psi x) < \varpi(x_{2n_k-1}, x). \quad (12)$$

Now,

$$\text{dist}(\mathfrak{C}, \mathfrak{D}) \leq \varpi(\Psi x_{2n_k-1}, \Psi x) < \varpi(x_{2n_k-1}, x), \quad (13)$$

so, by letting $k \rightarrow \infty$ in (13) and using (8) and (10), we get $\varpi(x, \Psi x) = \text{dist}(\mathfrak{C}, \mathfrak{D})$. Similarly, if $\{x_{2n_k+1}\}$ is a subsequence of $\{x_{2n+1}\}$ such that $x_{2n_k+1} \rightarrow y \in \mathfrak{D}$ as $k \rightarrow \infty$. It can be proven that $\varpi(y, \Psi y) = \text{dist}(\mathfrak{C}, \mathfrak{D})$. Furthermore,

$$\varpi(x, y) = \lim_{n \rightarrow \infty} \varpi(x_{2n_k}, x_{2n_k+1}) = \text{dist}(\mathfrak{C}, \mathfrak{D}). \quad \square$$

Example 2.2. Let $\mathfrak{X} = \mathbb{R}$ be equipped with the usual metric

$$\varpi(x, y) = |y - x|.$$

Consider $\mathfrak{C} = [0, \frac{3}{10}]$ and $\mathfrak{D} = [\frac{7}{10}, 1]$, then $\text{dist}(\mathfrak{C}, \mathfrak{D}) = \frac{2}{5}$. Define $\Psi : \mathfrak{C} \cup \mathfrak{D} \rightarrow \mathfrak{C} \cup \mathfrak{D}$ by

$$\Psi(x) = \begin{cases} \frac{7}{10} & \text{if } x \in [0, \frac{3}{20}) \subset \mathfrak{C}; \\ 1 - x & \text{if } x \in [\frac{3}{20}, \frac{3}{10}] \subset \mathfrak{C}; \\ \frac{3}{10} & \text{if } x \in [\frac{7}{10}, \frac{41}{50}) \subset \mathfrak{D}; \\ \frac{x}{2} - \frac{1}{5} & \text{if } x \in [\frac{41}{50}, 1] \subset \mathfrak{D}; \end{cases}$$

and $\Theta : (0, \infty) \rightarrow (1, \infty)$ by $\Theta(t) = \exp[t]$. Note that $\Psi(\mathfrak{C}) \subseteq \mathfrak{D}$ and $\Psi(\mathfrak{D}) \subseteq \mathfrak{C}$. Now, for $x = \frac{3}{20} \in \mathfrak{C}$ and $y = \frac{41}{50} \in \mathfrak{D}$, we have

$$\begin{aligned} \Theta[\varpi^*(\Psi x, \Psi y)] &= \Theta\left[\varpi^*\left(1 - x, \frac{y}{2} - \frac{1}{5}\right)\right] \\ &= \Theta\left[\varpi\left(1 - x, \frac{y}{2} - \frac{1}{5}\right) - \varpi(\mathfrak{C}, \mathfrak{D})\right] \\ &= \Theta\left[\left|\left(\frac{y}{2} + x - \frac{6}{5}\right)\right| - \frac{2}{5}\right] \\ &= \exp\left[\frac{6}{25}\right] \end{aligned}$$

and

$$\begin{aligned} [\Theta(\varpi^*(x, y))]^k &= [\Theta(\varpi(x, y) - \varpi(\mathfrak{C}, \mathfrak{D}))]^k \\ &= [\Theta|y - x| - \varpi(\mathfrak{C}, \mathfrak{D})]^k \\ &= \left[\Theta\left(\frac{27}{100}\right) \right]^k < \Theta\left(\frac{27}{100}\right) \\ &= \exp\left[\frac{27}{100}\right]. \end{aligned}$$

This implies that

$$\Theta[\varpi^*(\Psi x, \Psi y)] = \exp\left[\frac{6}{25}\right] < \exp\left[\frac{27}{100}\right] = [\Theta(\varpi^*(x, y))]^k.$$

Hence, Ψ is a cyclic Θ -contraction. Similarly, the inequality (4) holds for the remaining cases. Also, the sequences $\{x_{2n} = \Psi^{2n}(0)\}$ and $\{x_{2n+1} = \Psi^{2n+1}(0)\}$ have convergent subsequences in $\mathfrak{C}$ and $\mathfrak{D}$, respectively. Thus, all the conditions of Lemma 2.2 are satisfied and so there exists $\frac{3}{10} \in \mathfrak{C}$ such that

$$\varpi(x, \Psi x) = \varpi\left(\frac{3}{10}, 1 - \frac{3}{10}\right) = \left|\frac{7}{10} - \frac{3}{10}\right| = \text{dist}(\mathfrak{C}, \mathfrak{D}).$$

Lemma 2.3. Let $\mathfrak{C}$ and $\mathfrak{D}$ be non-empty subsets of a metric space $(\mathfrak{X}, \varpi)$ and $\Psi : \mathfrak{C} \cup \mathfrak{D} \rightarrow \mathfrak{C} \cup \mathfrak{D}$ be a cyclic Θ -contraction. Then for every $x_0 \in \mathfrak{C}$, likewise in Lemma 2.1, there exists a sequence $\{x_n\}$ in $\mathfrak{C} \cup \mathfrak{D}$, which is bounded.

Proof. Suppose that $x_0 \in \mathfrak{C}$, by Lemma 2.1, there exists a sequence $\{x_n\}$ in $\mathfrak{C} \cup \mathfrak{D}$ such that $\{\varpi(x_{2n}, x_{2n+1})\}$ converges to $\text{dist}(\mathfrak{C}, \mathfrak{D})$. We claim that $\{x_{2n}\}$ is bounded. On contrary, assume that $\{x_{2n}\}$ is not bounded, then there exists $q \in \mathbb{N}$ satisfying $M < \varpi^*(x_{2n}, x_{2(n+q)+1})$ and $\varpi^*(x_{2n}, x_{2(n+q)-1}) \leq M$, where $M = 2q \text{dist}(\mathfrak{C}, \mathfrak{D})$. Thus, we have

$$\begin{aligned} M &< \varpi^*(x_{2n}, x_{2(n+q)+1}) \\ &= \varpi(x_{2n}, x_{2(n+q)+1}) - \text{dist}(\mathfrak{C}, \mathfrak{D}) \\ &\leq \varpi(x_{2n}, x_{2n+1}) + \varpi(x_{2n+1}, x_{2n+2}) + \cdots + \varpi(x_{2(n+q)}, x_{2(n+q)+1}) - \text{dist}(\mathfrak{C}, \mathfrak{D}) \\ &= (\varpi(x_{2n}, x_{2n+1}) - \text{dist}(\mathfrak{C}, \mathfrak{D})) + (\varpi(x_{2n+1}, x_{2n+2}) - \text{dist}(\mathfrak{C}, \mathfrak{D})) \\ &\quad + \cdots + (\varpi(x_{2(n+q)}, x_{2(n+q)+1}) - \text{dist}(\mathfrak{C}, \mathfrak{D})) + (2q \text{dist}(\mathfrak{C}, \mathfrak{D}) - \text{dist}(\mathfrak{C}, \mathfrak{D})) \\ &\leq \varpi^*(x_{2n}, x_{2n+1}) + \varpi^*(x_{2n+1}, x_{2n+2}) + \cdots + \varpi^*(x_{2(n+q)}, x_{2(n+q)+1}) + (2q - 1) \text{dist}(\mathfrak{C}, \mathfrak{D}). \end{aligned} \tag{14}$$

Letting $n \rightarrow \infty$ in (14) and using Lemma 2.1, we get

$$M < M - \text{dist}(\mathfrak{C}, \mathfrak{D}),$$

which leads to a contradiction. We can also prove that $\{x_{2n+1}\}$ is bounded and thus $\{x_n\}$ is bounded. $\square$

Theorem 2.1. Let $\mathfrak{C}$ and $\mathfrak{D}$ be non-empty subsets of a metric space $(\mathfrak{X}, \varpi)$ and $\Psi : \mathfrak{C} \cup \mathfrak{D} \rightarrow \mathfrak{C} \cup \mathfrak{D}$ be a cyclic Θ -contraction. If either $\mathfrak{C}$ or $\mathfrak{D}$ is boundedly compact, then there exists $x \in \mathfrak{C} \cup \mathfrak{D}$ such that $\varpi(x, \Psi x) = \text{dist}(\mathfrak{C}, \mathfrak{D})$.

Proof. It follows from Lemmas 2.2 and 2.3. $\square$

3 Particular case: fixed point results

In this section, we deduce some of the fixed point results for cyclic Θ -contractions.

Theorem 3.1. Let $\mathcal{C}$ and $\mathcal{D}$ be non-empty subsets of a metric space $(\mathfrak{X}, \varpi)$ with $\text{dist}(\mathcal{C}, \mathcal{D}) = 0$ and $\Psi : \mathcal{C} \cup \mathcal{D} \rightarrow \mathcal{C} \cup \mathcal{D}$ be a mapping satisfying $\Psi(\mathcal{C}) \subseteq \mathcal{D}$ and $\Psi(\mathcal{D}) \subseteq \mathcal{C}$. Assume that there exists $\Theta \in \Lambda$ such that

$$\Theta(\varpi(\Psi x, \Psi y)) \leq [\Theta(\varpi(x, y))]^k. \quad (15)$$

Then Ψ has a unique fixed point in $\mathcal{C} \cap \mathcal{D}$ provided that $\{x_{2n}\}$ and $\{x_{2n+1}\}$ have convergent subsequences in $\mathcal{C}$ and $\mathcal{D}$, respectively, where $\{x_n\}$ is the sequence of Lemma 2.1, for some $x_0 \in \mathcal{C}$.

Proof. Since $\text{dist}(\mathcal{C}, \mathcal{D}) = 0$, equation (15) implies that Ψ is a cyclic Θ -contraction. Hence, from Lemma 2.2, we have $x \in \mathcal{C}$ and $y \in \mathcal{D}$ with $x = y$ such that $\varpi(x, \Psi x) = 0$. Consequently, $\mathcal{C} \cap \mathcal{D} \neq \emptyset$ and Ψ has a fixed point in $\mathcal{C} \cap \mathcal{D}$. For uniqueness, suppose that $u \in \mathcal{C} \cup \mathcal{D}$ such that $\Psi u = u$ with $x \neq u$, then by (15), we get

$$\Theta(\varpi(x, u)) = \Theta(\varpi(\Psi x, \Psi u)) \leq (\Theta(\varpi(x, u)))^k < \Theta(\varpi(x, u)).$$

It is a contradiction. Thus, $x = u$. $\square$

Theorem 3.2. Let $\mathcal{C}$ and $\mathcal{D}$ be non-empty subsets of a metric space $(\mathfrak{X}, \varpi)$ with $\text{dist}(\mathcal{C}, \mathcal{D}) = 0$ and $\Psi : \mathcal{C} \cup \mathcal{D} \rightarrow \mathcal{C} \cup \mathcal{D}$ be a mapping such that $\Psi(\mathcal{C}) \subseteq \mathcal{D}$ and $\Psi(\mathcal{D}) \subseteq \mathcal{C}$ satisfying (15). If $\mathcal{C}$ and $\mathcal{D}$ are boundedly compact, then Ψ has a unique fixed point in $\mathcal{C} \cap \mathcal{D}$.

Proof. It follows from Lemma 2.3 and Theorem 3.1. $\square$

Remark 3.1. Recently, Radenović [9] showed that fixed point results for mappings satisfying cyclical contractive conditions appearing in [1] are equivalent to fixed point results for classical contractive mappings. It can be noted in the proof of Theorem 2.5 of [9] that to obtain cyclical-type theorems from classical results, completeness of metric space and closedness of subsets $\mathcal{C}_i$ for all i are necessary, otherwise $(\bigcap_{i=1}^m \mathcal{C}_i, \varpi)$ is not a complete metric space. Therefore, Theorem 3.2 cannot be obtained from classical fixed point theorems by using the approach of Radenović [9].

Lemma 3.1. Let $\mathcal{C}$ and $\mathcal{D}$ be non-empty subsets of a metric space $(\mathfrak{X}, \varpi)$ such that $\text{dist}(\mathcal{C}, \mathcal{D}) = 0$. If $\Psi : \mathcal{C} \cup \mathcal{D} \rightarrow \mathcal{C} \cup \mathcal{D}$ is a cyclic Θ -contraction. Then likewise in Lemma 2.1, there exists a Cauchy sequence $\{x_{2n}\}$ in $\mathcal{C}$.

Proof. Let $x_0 \in \mathcal{C}$. Given that $\text{dist}(\mathcal{C}, \mathcal{D}) = 0$, so for all $n \in \mathbb{N}$, from Lemma 2.1 we get

$$\lim_{n \rightarrow \infty} \varpi(x_n, x_{n+1}) = 0. \quad (16)$$

We claim that for $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\varpi(x_{2n}, x_{2m+1}) < \varepsilon, \quad (17)$$

with $n, m \geq N$. If not, then there exist two divergent sequences $\{2n_k\}, \{2m_k + 1\} \subseteq \mathbb{N}$ such that

$$\varpi(x_{2n_k}, x_{2m_k+1}) \geq \varepsilon, \quad (18)$$

for all $k \in \mathbb{N}$. We assume that $2m_k + 1$ is a minimal index for which (18) holds. Then for all $k \in \mathbb{N}$,

$$\varpi(x_{2n_k}, x_{2m_k-1}) < \varepsilon. \quad (19)$$

From (18) and (19), we have

$$\begin{aligned} \varepsilon &\leq \varpi(x_{2n_k}, x_{2m_k+1}) \\ &\leq \varpi(x_{2n_k}, x_{2m_k-1}) + \varpi(x_{2m_k-1}, x_{2m_k}) + \varpi(x_{2m_k}, x_{2m_k+1}) \\ &< \varepsilon + \varpi(x_{2m_k-1}, x_{2m_k}) + \varpi(x_{2m_k}, x_{2m_k+1}). \end{aligned} \quad (20)$$

Letting $k \rightarrow \infty$ in (20) and using (16), we get

$$\lim_{k \rightarrow \infty} \varpi(x_{2n_k}, x_{2m_k+1}) = \varepsilon. \quad (21)$$

Now, by a triangular inequality, we have

$$\varpi(x_{2n_k}, x_{2m_k+1}) \leq \varpi(x_{2n_k}, x_{2n_k-1}) + \varpi(x_{2n_k-1}, x_{2m_k}) + \varpi(x_{2m_k}, x_{2m_k+1}) \quad (22)$$

and

$$\varpi(x_{2n_k-1}, x_{2m_k}) \leq \varpi(x_{2n_k-1}, x_{2n_k}) + \varpi(x_{2n_k}, x_{2m_k+1}) + \varpi(x_{2m_k+1}, x_{2m_k}). \quad (23)$$

Letting $k \rightarrow \infty$ in (22) and (23) and using (16) and (21), we get

$$\lim_{k \rightarrow \infty} \varpi(x_{2n_k-1}, x_{2m_k}) = \varepsilon. \quad (24)$$

In view of (21) and (24), there is an integer k_0 so that $\varpi(x_{2n_k}, x_{2m_k+1}) > 0$ and $\varpi(x_{2n_k-1}, x_{2m_k}) > 0$ for each $k \geq k_0$. Thus, since $\Theta \in \Lambda$, we have

$$\Theta(\varpi(x_{2n_k}, x_{2m_k+1})) \leq (\Theta(\varpi(x_{2n_k-1}, x_{2m_k})))^k.$$

Letting k tend to ∞ and using again (21) and (24) lead to a contradiction. Hence, $\{x_{2n}\}$ is a Cauchy sequence in $\mathfrak{C}$. $\square$

Theorem 3.3. Let $\mathfrak{C}$ and $\mathfrak{D}$ be non-empty closed subsets of a complete metric space $(\mathfrak{X}, \varpi)$ such that $\text{dist}(\mathfrak{C}, \mathfrak{D}) = 0$. If $\Psi : \mathfrak{C} \cup \mathfrak{D} \rightarrow \mathfrak{C} \cup \mathfrak{D}$ is a cyclic Θ -contraction, then Ψ has a unique fixed point in $\mathfrak{C} \cap \mathfrak{D}$.

Proof. Consider $x_0 \in \mathfrak{C}$, then by Lemma 3.1, there exists a Cauchy sequence $\{x_{2n}\}$ in $\mathfrak{C}$. Since $\mathfrak{X}$ is complete and $\mathfrak{C}$ is a closed subset of $\mathfrak{X}$, there exists $x^* \in \mathfrak{C}$ such that $x_{2n} \rightarrow x^*$ as $n \rightarrow \infty$. Now, we have

$$0 \leq \varpi(x_{2n-1}, x^*) \leq \varpi(x_{2n-1}, x_{2n}) + \varpi(x_{2n}, x^*),$$

so taking limit as $n \rightarrow \infty$, we get that

$$\varpi(x_{2n-1}, x^*) \rightarrow 0.$$

Similarly, $\{x_{2n-1}\}$ is a Cauchy sequence in $\mathfrak{D}$, which converges to $x^* \in \mathfrak{D}$ as $\mathfrak{D}$ is closed in the complete metric space $\mathfrak{X}$. Hence, $\{x_{2n}\}$ and $\{x_{2n+1}\}$ have convergent subsequences in $\mathfrak{C}$ and $\mathfrak{D}$, respectively. From Theorem 3.2, Ψ has a unique fixed point in $\mathfrak{C} \cap \mathfrak{D}$. $\square$

If we take $\Theta(t) = e^t$ in Theorem 3.3, we obtain the following main result of [1].

Corollary 3.1. Let $\mathfrak{C}$ and $\mathfrak{D}$ be non-empty closed subsets of a complete metric space $(\mathfrak{X}, \varpi)$ such that $\text{dist}(\mathfrak{C}, \mathfrak{D}) = 0$. If $\Psi : \mathfrak{C} \cup \mathfrak{D} \rightarrow \mathfrak{C} \cup \mathfrak{D}$ is a cyclic contraction, then Ψ has a unique fixed point in $\mathfrak{C} \cap \mathfrak{D}$.

Example 3.1. Let $\mathfrak{X} = \mathbb{R}$ be endowed with the usual metric ϖ . Consider $\mathfrak{C} = [-1, 0]$ and $\mathfrak{D} = [0, 1]$. Define $\Psi : \mathfrak{C} \cup \mathfrak{D} \rightarrow \mathfrak{C} \cup \mathfrak{D}$ such that

$$\Psi(x) = -\frac{x}{3}$$

for all $x \in \mathfrak{C} \cup \mathfrak{D}$ and $\Theta : (0, \infty) \rightarrow (1, \infty)$ by $\Theta(t) = 1 + t$. Note that $\Psi(\mathfrak{C}) \subseteq \mathfrak{D}$ and $\Psi(\mathfrak{D}) \subseteq \mathfrak{C}$. Now, for $x \in \mathfrak{C} \cup \mathfrak{D}$, we have

$$\Theta[\varpi(\Psi x, \Psi y)] = \Theta\left[\varpi\left(-\frac{x}{3}, -\frac{y}{3}\right)\right] = \Theta\left[\frac{|x-y|}{3}\right]$$

and

$$[\Theta(\varpi(x, y))]^k = [\Theta(|x-y|)]^k.$$

This implies

$$\Theta \left[\frac{|x - y|}{3} \right] \leq [\Theta(|x - y|)]^k.$$

Hence, Ψ is a cyclic Θ -contraction for all $k \in (0, 1)$. Thus, all the conditions of Theorem 3.3 are satisfied and so there is a unique $0 \in \mathfrak{C} \cup \mathfrak{D}$ such that $\Psi x = x$.

4 Application to a variational inequality problem

Let $\mathfrak{H}$ be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and an induced norm $\|\cdot\|$. An element $\mu^* \in \Omega$ (where $\emptyset \neq \Omega \subseteq \mathfrak{H}$) is known as a best approximation if $\|\mu - \mu^*\| = \mathfrak{D}(\mu, \Omega)$, where $\mathfrak{D}(\mu, \Omega) = \inf_{v \in \Omega} \|\mu - v\|$ is called the error in approximating μ by Ω . A metric projection map $P_\Omega : \mathfrak{H} \rightarrow \Omega$ such that for all $\mu \in \mathfrak{H}$, $P_\Omega(\mu) = \{v \in \Omega : \|\mu - v\| = \mathfrak{D}(\mu, \Omega)\}$ is the set of all best approximations from μ to Ω . This metric projection plays an important role for solving the variational inequality problem.

For a non-empty, closed and convex subset $\Omega \subseteq \mathfrak{H}$, $f : \Omega \rightarrow \mathfrak{H}$ is a given operator. The Hartman-Stampacchia variational inequality (HSVI) problem is defined as

$$\text{P-I: HSVI}(f, \Omega) := \begin{cases} \text{find } \mu^* \in \Omega \text{ such that} \\ \langle f\mu^*, \mu - \mu^* \rangle \geq 0 \text{ for all } \mu \in \Omega \end{cases}$$

and the Minty variational inequality (MVI) problem is defined by

$$\text{P-II: MVI}(f, \Omega) := \begin{cases} \text{find } \mu^* \in \Omega \text{ such that} \\ \langle f\mu, \mu - \mu^* \rangle \geq 0 \text{ for all } \mu \in \Omega. \end{cases}$$

The HSVI has various applications in the field of engineering, physics and industry, while the MVI facilitates in the study of solvability of HSVI (f, Ω) , see [20].

Definition 4.1. [21] A mapping $f : \Omega \rightarrow \mathfrak{H}$ is called pseudomonotone if for all $\mu, v \in \Omega$, $\langle \mu - v, f(v) \rangle \geq 0$ implies $\langle \mu - v, f(\mu) \rangle \geq 0$.

Definition 4.2. [21] Let $\Omega \subseteq \mathfrak{H}$ be a convex set. A mapping $f : \Omega \rightarrow \mathfrak{H}$ is hemicontinuous from the line segments of Ω to the weak topology of $\mathfrak{H}$.

The following theorem establishes a relation between the problems: P-I and P-II.

Theorem 4.1. [21] Let $\mathfrak{H}(\tau)$ be a locally convex space, $\Omega \subseteq \mathfrak{H}$ be a closed convex set and $f : \Omega \rightarrow \mathfrak{H}$ be a pseudomonotone and hemicontinuous mapping. Then an element $u_0 \in \Omega$ is a solution to the problem HSVI (f, Ω) iff it is a solution to the problem MVI (f, Ω) .

It is known that, for each $u \in \mathfrak{H}$, there exists a unique nearest point $P_\Omega(u) \in \Omega$ such that $\|u - P_\Omega(u)\| \leq \|u - v\|$ for all $v \in \Omega$ [20].

Lemma 4.1. [20] Let Ω be a convex subset of the Hilbert space $\mathfrak{H}$, $z \in \mathfrak{H}$ and $u \in \Omega$. Then $u = P_\Omega(z)$ if and only if $\langle z - u, v - u \rangle \leq 0$ for all $v \in \Omega$.

Lemma 4.2. [20] Let Ω be a convex subset of Hilbert space $\mathfrak{H}$ and $f : \Omega \rightarrow \Omega$. Then $u \in \Omega$ is a solution of $\langle fu, v - u \rangle \geq 0$ for all $v \in \Omega$ if and only if $u = P_\Omega(u - \lambda fu)$, $\lambda > 0$.

Every set is clearly an approximately compact set in terms of itself. So the following findings are proven.

Theorem 4.2. Let Ω be a non-empty, closed and convex subset of a real Hilbert space $\mathfrak{H}$. Assume that for $\mu \in \Omega$, there exists $\Theta \in \Lambda$ such that for $f : \Omega \rightarrow \Omega$, $P_\Omega(I_\Omega - \lambda f) : \Omega \rightarrow \Omega$ satisfies

$$\Theta(\|P_\Omega(I_\Omega - \lambda f)\mu - P_\Omega(I_\Omega - \lambda f)v\|) \leq (\Theta\|\mu - v\|)^k,$$

where I_Ω is the identity operator on Ω . Then for all $v \in \Omega$ there exists a unique element $u \in \Omega$ such that $\langle fu, v - u \rangle \geq 0$. Moreover, for a fixed element $u_0 \in \Omega$, $\lambda > 0$ and $n \in \mathbb{N} \cup \{0\}$, the sequence $\{u_n\}$ with $u_{n+1} = P_\Omega(u_n - \lambda gu_n)$ converges to u .

Proof. Define $g : \Omega \rightarrow \Omega$ by $gx = P_\Omega(x - \lambda fx)$ for all $x \in \Omega$, then g satisfies all the conditions of Theorem 3.3 by setting $\mathfrak{C} = \mathfrak{D} = \Omega$, and so g has a unique fixed point u . Hence, by Lemma 4.2, $\langle fu, v - u \rangle \geq 0$ has a solution $u \in \Omega$ for all $v \in \Omega$ if and only if u is a fixed point of g . This completes the proof. $\square$

Theorem 4.3. Suppose that all the conditions of Theorem 4.2 hold. Then for all $v \in \Omega$, there exists a unique element $u \in \Omega$ such that $\langle fv, v - u \rangle \geq 0$. Moreover, for a fixed element $u_0 \in \Omega$, $\lambda > 0$ and $n \in \mathbb{N} \cup \{0\}$, the sequence $\{u_n\}$ with $u_{n+1} = P_\Omega(u_n - \lambda gu_n)$ converges to u .

Proof. From Theorems 4.1 and 4.2, we have the required results. $\square$

5 Conclusion

In this paper, we presented the existence and convergence of the best proximity point for cyclic Θ -contractions. We derived several fixed point results for such mappings. As an application of the obtained results, we presented some solvability theorems in the topic of variational inequalities.

Funding information: This paper received no external funding.

Author contributions: All authors contributed equally and significantly in writing this article.

Conflict of interest: The authors declare no conflict of interest.

Data availability statement: No data were used to support this study.

References

- [1] W. A. Kirk, P. S. Srinivasan, and P. Veeramani, *Fixed points for mappings satisfying cyclical contractive conditions*, Fixed Point Theory **4** (2003), 79–89.
- [2] A. A. Eldred and P. Veeramani, *Existence and convergence of best proximity points*, J. Math. Anal. Appl. **323** (2006), 1001–1006.
- [3] M. A. Al-Thagafi and N. Shahzad, *Convergence and existence results for best proximity points*, Nonlinear Anal. **70** (2009), 3665–3671.
- [4] A. Abkar and M. Gabeleh, *Best proximity points for cyclic mappings in ordered metric spaces*, J. Optim. Theory Appl. **150** (2011), 188–193.
- [5] N. V. Dung and S. Radenović, *Remarks on theorems for cyclic quasi-contractions in uniformly convex Banach spaces*, Kragujevac J. Math. **40** (2016), no. 2, 272–279.
- [6] V. Parvaneh, M. R. Haddadi, and H. Aydi, *On best proximity point results for some type of mappings*, J. Funct. Spaces **2020** (2020), 6298138.
- [7] H. Aydi, H. Lakzian, Z. D. Mitrović, and S. Radenović, *Best proximity points of MF-cyclic contractions with property UC*, Numer. Funct. Anal. Optim. **41** (2020), no. 7, 871–882.
- [8] Z. Kadelburg and S. Radenović, *A note on Pata-type cyclic contractions*, Sarajevo J. Math. **11** (2015), no. 24, 235–245.

- [9] S. Radenović, *Some remarks on mappings satisfying cyclical contractive conditions*, Afr. Mat. **27** (2016), 291–295.
- [10] S. Radenović, S. Chandok, and W. Shatanawi, *Some cyclic fixed point results for contractive mappings*, Univ. Though Publ. Nature Sci. **6** (2016), no. 2, 38–40.
- [11] H. Aydi, E. Karapinar, and A. F. Roldan Lopez de Hierro, *w-interpolative Ciric-Reich-Rus type contractions*, Mathematics **7** (2019), 57.
- [12] M. Jleli and B. Samet, *A new generalization of the Banach contraction principle*, J. Inequal. Appl. **2014** (2014), 38.
- [13] J. Ahmad, A. E. Al-Mazrooei, Y. J. Cho, and Y. O. Yang, *Fixed point results for generalized Θ -contractions*, J. Nonlinear Sci. Appl. **10** (2017), no. 5, 2350–2358.
- [14] H. A. Hammad, H. Aydi, and N. Mlaiki, *Contributions of the fixed point technique to solve the 2D Volterra integral equations, Riemann? Liouville fractional integrals, and Atangana-Baleanu integral operators*, Adv. Differ. Equ. **2021** (2021), 97.
- [15] H. A. Hammad, H. Aydi, and Y. U. Gaba, *Exciting fixed point results on a novel space with supportive applications*, J. Funct. Spaces **2021** (2021), 6613774.
- [16] X. D. Liu, S. Chang, Y. Xiao, and L. C. Zhao, *Existence of fixed points for Θ -type contraction and Θ -type Suzuki contraction in complete metric spaces*, Fixed Point Theory Appl. **2016** (2016), 8.
- [17] Z. Ma, A. Asif, H. Aydi, S. Ullah Khan, and M. Arshad, *Analysis of F-contractions in function weighted metric spaces with an application*, Open Math. **18** (2020), 582–594.
- [18] A. Ali, H. Isik, H. Aydi, E. Ameer, J. R. Lee, and M. Arshad, *On multivalued Suzuki-type θ -contractions and related applications*, Open Math. **18** (2020), 386–399.
- [19] V. Parvaneh, F. Golkarmanesh, N. Hussain, and P. Salimi, *New fixed point theorems for α -H Θ -contractions in ordered metric spaces*, J. Fixed Point Theory Appl. **18** (2016), 905–925.
- [20] F. Deutsch, *Best Approximation in Inner Product Spaces*, Springer-Verlag, New York, 2001.
- [21] G. Isac, *Leray-Schauder Type Alternatives, Complementarity Problems and Variational Inequalities*, Springer-Verlag, New York, 2006.