Research Article

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On a fractional Schrödinger-Poisson system with strong singularity

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Abstract: We investigate a fractional Schrödinger-Poisson system with strong singularity as follows:

$$\begin{cases} (-\Delta)^s u + V(x)u + \lambda \phi u = f(x)u^{-\gamma}, & x \in \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & x \in \mathbb{R}^3, \\ u > 0, & x \in \mathbb{R}^3, \end{cases}$$

where $0 < s \leq t < 1$ with $4s + 2t > 3$, $\lambda > 0$ and $\gamma > 1$. When V and f satisfy certain conditions, existence and uniqueness of positive solution u_λ are established via variational method and Nehari method. We also describe the asymptotic behaviour of u_λ as $\lambda \rightarrow 0$.

Keywords: strong singularity, fractional Schrödinger-Poisson system, uniqueness, variational method, asymptotic behaviour

MSC 2020: 35A15, 35B40, 35B09, 35R11

1 Introduction

Due to their important roles in the real world, nonlinear problems involving fractional Laplacian have attracted a lot of attention in recent years. As the infinitesimal generator of Lévy stable diffusion process, the fractional Laplacian arises in plasma physics, finance, flame propagation, geophysical fluid dynamics and population dynamics, see [1] for example. This paper is concerned with existence, uniqueness and asymptotic behaviour of positive solution to the following singular Schrödinger-Poisson system involving the fractional Laplacian

$$\begin{cases} (-\Delta)^s u + V(x)u + \lambda \phi u = f(x)u^{-\gamma}, & x \in \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & x \in \mathbb{R}^3, \\ u > 0, & x \in \mathbb{R}^3, \end{cases} \quad (\text{SP}_\lambda)$$

where $0 < s \leq t < 1$ with $4s + 2t > 3$, $\lambda > 0$ and $\gamma > 1$. $(-\Delta)^\alpha$ is the fractional Laplacian operator defined as

$$(-\Delta)^\alpha u(x) = C(\alpha) \text{P.V.} \int_{\mathbb{R}^3} \frac{u(x) - u(y)}{|x - y|^{3+2\alpha}} dy, \quad u \in \mathcal{S}(\mathbb{R}^3),$$

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where $C(\alpha)$ is a normalized constant, P.V. stands for the Cauchy principal value, $\mathcal{S}(\mathbb{R}^3)$ is the Schwartz space of rapidly decaying function. We assume that V and f satisfy:

- (V₁) $V \in C(\mathbb{R}^3)$ satisfies $\inf_{x \in \mathbb{R}^3} V(x) > V_0 > 0$, where V_0 is a constant;
- (V₂) $\text{meas}\{x \in \mathbb{R}^3 : -\infty < V(x) \leq h\} < +\infty$ for all $h \in \mathbb{R}$;
- (f₁) $f \in L^{\frac{2}{1+\gamma}}(\mathbb{R}^3)$ is a nonnegative function.

Recently, much attention has been paid to the following more generic fractional Schrödinger-Poisson system:

$$\begin{cases} (-\Delta)^s u + V(x)u + \lambda \phi u = h(x, u), & x \in \mathbb{R}^3, \\ (-\Delta)^t \phi = \lambda u^2, & x \in \mathbb{R}^3. \end{cases} \quad (1.1)$$

Zhang et al. [2] paid attention to the existence and asymptotic behaviour of positive solutions for (1.1) with $V = 0$ and h satisfying subcritical or critical growth conditions. When $V = \lambda = 1$, the existence of nontrivial solutions for (1.1) was considered by Li [3]. The existence and multiplicity of nontrivial solutions for (1.1) with superlinear terms was investigated in [4]. Gu et al. [5] showed that (1.1) with $\lambda = 1$ and $h(x, u) = K(x)h(u) + |u|^{2^*_s-2}u$ has a positive solution. Teng [6,7], Yu et al. [8] and Guo [9] considered the existence of solutions for (1.1) with critical nonlinearity term. Xiang and Wang [10] further studied a critical fractional Schrödinger-Poisson-Kirchhoff system. Under certain assumptions, Wang et al. [11] obtained infinitely many solutions to a fractional Schrödinger-Poisson-Kirchhoff system.

The following singular elliptic equation with fractional Laplacian on bounded domains which can be viewed as a special case of system (SP _{λ}) with $\phi = 0$ has been investigated extensively over recent years

$$\begin{cases} (-\Delta)^s u = k(x)u^{-\gamma} + Mu^q, & u > 0, \quad x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \quad (1.2)$$

When $0 < \gamma < 1$ (i.e. weakly singular case), (1.2) with critical or subcritical nonlinearity was studied in [12–14]. Goyal [15] discussed (1.2) with Hardy potential. Fiscella [16,17] obtained two solutions for (1.2) with Kirchhoff prototype and critical exponent. As for the strongly singular case (i.e. $\gamma > 1$), Barrios et al. [18] studied the existence and multiplicity of solutions for (1.2) when $M > 0$. (1.2) with Kirchhoff prototype was discussed in [19,20].

On a bounded domain $\Omega \subset \mathbb{R}^3$, when $s = t = 1$, system (SP _{λ}) reduces to a singular Schrödinger-Poisson system as follows:

$$\begin{cases} -\Delta u + \lambda \phi u = f(x)u^{-\gamma}, & x \in \Omega, \\ -\Delta \phi = u^2, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = \phi = 0, & x \in \partial\Omega. \end{cases} \quad (1.3)$$

Considering the weakly singular case (i.e. $0 < \gamma < 1$), Zhang [21] studied (1.3) with $\lambda = \pm 1$ and f replaced by a positive parameter μ . Some results in Zhang [21] were generalized to critical case by Lei and Liao [22] lately. When $f = \frac{\mu}{|x|^\beta}$, Wang [23] obtained two positive solutions for (1.3) with $\lambda = -1$ and cubic nonlinearity. Zhang [24], Li et al. [25] and Mu and Lu [26] investigated a class of singular Schrödinger-Kirchhoff-Poisson system. Sun and Li [27] discussed a semilinear elliptic equation with singularity on $\mathbb{R}^N$. Lei et al. [28] investigated a critical Schrödinger-Newton system with singularity. When $\gamma > 1$ (i.e. strongly singular case), Sun and Zhang [29] obtained the existence of solutions of (1.3) with $\lambda = 0$ and analysed the reason why 3 is crucial. Li and Gao [30] and Cong and Han [31] generalized the results obtained by Sun and Zhang [29] to the p -Laplace operator. Sun [32] obtained an optimal compatible condition between f and γ for $H_0^1(\Omega)$ solution of (1.3) with sublinear nonlinearities. Li et al. [33] studied (1.3) with Kirchhoff prototype. Sun and Tan [34] further proved the existence of positive solution to problem (1.3) of Kirchhoff type and sublinear nonlinearities. Schrödinger-Poisson system and Kirchhoff-Schrödinger-Poisson system with strong singularity were studied in [35,36] afterwards.

Motivated by the aforementioned results, the main purpose of this paper is to study the existence of positive solutions for Schrödinger-Poisson system involving fractional Laplacian and strong singularity on $\mathbb{R}^N$. Compared with the results in [19, 29–36], our discussion is in three-dimensional whole spaces $\mathbb{R}^3$, which means we need to overcome the lack of compactness. Moreover, system (SP_λ) is a doubly nonlocal elliptic problem which can cause more difficulties than those of works such as [29–36]. On the other hand, the non-differentiability of the energy functional I_λ on E (see Section 2 for I_λ and E) implies that system (SP_λ) could not be discussed by using the critical point theory directly.

According to [11] for instance, for any $u \in H^s(\mathbb{R}^3)$, there is a unique $\phi_u^t \in \mathcal{D}^{t,2}(\mathbb{R}^3)$ such that $(-\Delta)^t \phi_u^t = u^2$, $x \in \mathbb{R}^3$. Substituting ϕ_u^t in system (SP_λ) leads to

$$\begin{cases} (-\Delta)^s u + V(x)u + \lambda \phi_u^t u = f(x)u^{-\gamma}, & x \in \mathbb{R}^3, \\ u > 0, & x \in \mathbb{R}^3. \end{cases} \quad (1.4)$$

We call $(u, \phi_u^t) \in E \times \mathcal{D}^{t,2}(\mathbb{R}^3)$ a solution of system (SP_λ) if u is a solution of (1.4) i.e. $u > 0$ in $\mathbb{R}^3$ and

$$(u, v)_E + \lambda \int_{\mathbb{R}^3} \phi_u^t u v dx - \int_{\mathbb{R}^3} f u^{-\gamma} v dx = 0, \quad \forall v \in E. \quad (1.5)$$

For simplicity, we call u instead of (u, ϕ_u^t) a solution of system (SP_λ) . Applying variational method and Nehari method, necessary and sufficient conditions on the existence and uniqueness of positive solution are obtained. We also get monotonicity properties of solutions with respect to f and the parameter λ and asymptotic behaviour of solutions as $\lambda \rightarrow 0$.

Theorem 1.1. Assume $\lambda > 0$, $\gamma > 1$ and (V_1) , (V_2) , (f_1) hold. Then system (SP_λ) has a unique positive solution $u_\lambda \in E$ if and only if there exists $u_0 \in E$ satisfying

$$\int_{\mathbb{R}^3} f |u_0|^{1-\gamma} dx < +\infty. \quad (1.6)$$

Theorem 1.2. Assume $\lambda > 0$, $\gamma > 1$ and (V_1) , (V_2) hold, further suppose $f_1, f_2 \in L^{\frac{2}{1+\gamma}}(\mathbb{R}^3)$ are two nonnegative functions satisfying $\int_{\mathbb{R}^3} f_i |u_0|^{1-\gamma} dx < +\infty$, $i = 1, 2$, then $f_1 \geq f_2$ implies $u_1 \geq u_2$, where u_1, u_2 are two solutions of system (SP_λ) corresponding to f_1, f_2 and provided by Theorem 1.1, respectively.

Theorem 1.3. Assume $\gamma > 1$ and (V_1) , (V_2) hold, further suppose that f satisfies (f_1) and (1.6). For $0 < \lambda_1 \leq \lambda_2$, we have $u_{\lambda_1} \geq u_{\lambda_2}$, where $u_{\lambda_1}, u_{\lambda_2}$ are two solutions of system (SP_λ) corresponding to λ_1, λ_2 and provided by Theorem 1.1, respectively.

Theorem 1.4. Assume $\gamma > 1$ and (V_1) , (V_2) , (f_1) hold. For any sequence $\{\lambda_n\} \subset (0, 1)$ with $\lambda_n \rightarrow 0$, the sequence $\{u_{\lambda_n}\}$ of positive solutions provided by Theorem 1.1 strongly converges to w_0 in E , where w_0 is the unique positive solution of

$$\begin{cases} (-\Delta)^s u + V(x)u = f(x)u^{-\gamma}, & x \in \mathbb{R}^3, \\ u > 0, & x \in \mathbb{R}^3. \end{cases} \quad (P_0)$$

2 Preliminaries

We utilize the following notations in this article.

- $L^p(\mathbb{R}^3)$ is the usual Lebesgue space with the norm $\|u\|_p = \left(\int_{\mathbb{R}^3} |u|^p dx \right)^{\frac{1}{p}}$.

- For any $\alpha \in (0, 1)$, $2_\alpha^* = \frac{6}{3-2\alpha}$ is the fractional critical exponent in dimension 3.
- $\rightarrow$ denotes the strong convergence and $\rightharpoonup$ denotes the weak convergence.
- $u^\pm = \max\{\pm u, 0\}$ for any function u .
- C and C_i ($i = 1, 2, \dots$) denote positive constants unless specified.

Let

$$E = \{u \in \mathcal{D}^{s,2}(\mathbb{R}^3) : \|u\|_E < +\infty\}$$

be endowed with the inner product and norm

$$(u, v)_E = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{3+2s}} dx dy + \int_{\mathbb{R}^3} Vuv dx, \quad \|u\|_E = (u, u)_E^{1/2}.$$

Here $\mathcal{D}^{s,2}(\mathbb{R}^3)$ is defined as the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} dx dy \right)^{1/2}.$$

Using conditions (V_1) and (V_2) , we have the following compactness result (see [11, Lemma 2.2]).

Lemma 2.1. Assume $0 < s < 1$ and $(V_1), (V_2)$ hold. Then the embedding $E \hookrightarrow L^p(\mathbb{R}^3)$ is continuous for $p \in [2, 2_s^*]$ and is compact for $p \in [2, 2_s^*)$. Hence, there are constants $C_p > 0$ such that $\|u\|_p \leq C_p \|u\|_E$, $\forall u \in E$, $p \in [2, 2_s^*]$.

For $u \in H^s(\mathbb{R}^3)$, as $0 < s \leq t < 1$ and $4s + 2t > 3$, according to [11], there exists a unique $\phi_u^t \in \mathcal{D}^{t,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(\phi_u^t(x) - \phi_u^t(y))(v(x) - v(y))}{|x - y|^{3+2t}} dx dy = \int_{\mathbb{R}^3} u^2 v dx, \quad \forall v \in \mathcal{D}^{t,2}, \quad (2.1)$$

so, ϕ_u^t satisfies the Poisson equation

$$(-\Delta)^t \phi_u^t = u^2, \quad x \in \mathbb{R}^3,$$

and one can get the following t -Riesz potential

$$\phi_u^t = c_t \int_{\mathbb{R}^3} \frac{u^2(y)}{|x - y|^{3-2t}} dy, \quad x \in \mathbb{R}^3,$$

with $c_t = \frac{\Gamma(3-2t)}{\pi^{3/2} 2^{2t} \Gamma(t)}$. Substituting ϕ_u^t into the first equation of system $(SP)_\lambda$ gives equation (1.4). The energy functional corresponding to (1.4) is given by

$$I_\lambda(u) = \frac{1}{2} \|u\|_E^2 + \frac{\lambda}{4} \int_{\mathbb{R}^3} \phi_u^t |u|^2 dx - \frac{1}{1-\gamma} \int_{\mathbb{R}^3} f|u|^{1-\gamma} dx. \quad (2.2)$$

In order to facilitate the following analysis, we put together some properties of ϕ_u^t from [6,11,37], etc.

Lemma 2.2. For $4s + 2t > 3$ and any $u \in H^s(\mathbb{R}^3)$, the following statements hold

- $\phi_u^t : H^s(\mathbb{R}^3) \rightarrow \mathcal{D}^{t,2}(\mathbb{R}^3)$ is continuous and maps bounded sets into bounded sets;
- $\phi_u^t \geq 0$. Moreover, $\phi_u^t > 0$ if $u \neq 0$;
- For $u \in E$, we have $\|\phi_u^t\|_{\mathcal{D}^{t,2}(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} \phi_u^t u^2 dx \leq C \|u\|_E^4$, where C is positive and independent of u ;
- $\phi_{\tau u}^t = \tau^2 \phi_u^t$, $\forall \tau \in \mathbb{R}$;

(v) Suppose $u_n \rightharpoonup u$ in E , then

$$\int_{\mathbb{R}^3} \phi_{u_n}^t u_n v dx \rightarrow \int_{\mathbb{R}^3} \phi_u^t u v dx, \quad \forall v \in E$$

and

$$\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx \rightarrow \int_{\mathbb{R}^3} \phi_u^t u^2 dx;$$

(vi) For $u, v \in H^s(\mathbb{R}^3)$, $\int_{\mathbb{R}^3} (\phi_u^t u - \phi_v^t v)(u - v) dx \geq \frac{1}{2} \|\phi_u^t - \phi_v^t\|_{\mathcal{D}^{t,2}(\mathbb{R}^3)}^2$.

3 Proofs of main results

First, let us define the following two constraints:

$$\mathcal{N}_1^{(\lambda)} = \left\{ u \in E : \|u\|_E^2 + \lambda \int_{\mathbb{R}^3} \phi_u^t |u|^2 dx - \int_{\mathbb{R}^3} f |u|^{1-\gamma} dx \geq 0 \right\}$$

and

$$\mathcal{N}_2^{(\lambda)} = \left\{ u \in E : \|u\|_E^2 + \lambda \int_{\mathbb{R}^3} \phi_u^t |u|^2 dx - \int_{\mathbb{R}^3} f |u|^{1-\gamma} dx = 0 \right\},$$

for any $\lambda > 0$. Now, let us prove main results of this paper.

Proof of Theorem 1.1. (Necessity) Assume u is a solution to (SP_λ) , that is $u > 0$ and fulfils (1.5). Selecting $v = u$ in (1.5), one then has

$$\int_{\mathbb{R}^3} f u^{1-\gamma} dx = \frac{1}{2} \|u\|_E^2 + \frac{\lambda}{4} \int_{\mathbb{R}^3} \phi_u^t |u|^2 dx < +\infty.$$

(Sufficiency) This portion will be divided into six steps.

Step 1. $\mathcal{N}_i^{(\lambda)} \neq \emptyset$, $i = 1, 2$.

Taking $u \in E$ satisfying $\int_{\mathbb{R}^3} f(x) |u|^{1-\gamma} dx < +\infty$, we have

$$I_\lambda(\eta u) = \frac{\eta^2}{2} \|u\|_E^2 + \frac{\lambda \eta^4}{4} \int_{\mathbb{R}^3} \phi_u^t |u|^2 dx - \frac{\eta^{1-\gamma}}{1-\gamma} \int_{\mathbb{R}^3} f |u|^{1-\gamma} dx, \quad \forall \eta > 0.$$

Define $g(\eta) = \eta \frac{dI_\lambda(\eta u)}{d\eta}$, we have

$$g(\eta) = \eta^2 \|u\|_E^2 + \lambda \eta^4 \int_{\mathbb{R}^3} \phi_u^t |u|^2 dx - \eta^{1-\gamma} \int_{\mathbb{R}^3} f |u|^{1-\gamma} dx.$$

Obviously, $g(\eta)$ is increasing on $(0, +\infty)$ together with $\lim_{\eta \rightarrow 0^+} g(\eta) = -\infty$ and $\lim_{\eta \rightarrow +\infty} g(\eta) = +\infty$. Thus, there exists a unique $\eta(u) > 0$ such that $I_\lambda(\eta(u)u) = \min_{\eta > 0} I_\lambda(\eta u)$ and $g(\eta(u)) = 0$, i.e.

$$\eta^2(u) \|u\|_E^2 + \lambda \eta^4(u) \int_{\mathbb{R}^3} \phi_u^t |u|^2 dx - \eta^{1-\gamma}(u) \int_{\mathbb{R}^3} f |u|^{1-\gamma} dx = 0,$$

which means that $\eta(u)u \in \mathcal{N}_2^{(\lambda)}$. Moreover, (1.6) reveals the existence of $\eta(u_0) > 0$ satisfying $\eta(u_0)u_0 \in \mathcal{N}_2^{(\lambda)} \subset \mathcal{N}_1^{(\lambda)}$. Therefore, $\mathcal{N}_i^{(\lambda)} \neq \emptyset$, $i = 1, 2$ for any $\lambda > 0$.

Step 2. $\mathcal{N}_1^{(\lambda)}$ is unbounded and closed in E with $\|u\|_E \geq C_1$, $\forall u \in \mathcal{N}_1^{(\lambda)}$ for some suitable constant $C_1 > 0$.

By Step 1, $\eta u_0 \in \mathcal{N}_1^{(\lambda)}$ for every $\eta \geq \eta(u_0)$, hence $\mathcal{N}_1^{(\lambda)}$ is an unbounded set in E . Fatou's lemma and Lemma 2.2 (v) show that $\mathcal{N}_1^{(\lambda)}$ is closed directly. Next, we want to prove that $\|u\|_E \geq C_1$, $\forall u \in \mathcal{N}_1^{(\lambda)}$ for some suitable constant $C_1 > 0$. If not, there exists a sequence $\{u_n\} \subset \mathcal{N}_1^{(\lambda)}$ with $u_n \rightarrow 0$ in E . Considering $u_n \in \mathcal{N}_1^{(\lambda)}$ and $\gamma > 1$, it follows from Lemma 2.2 (v) and the reverse form of Hölder's inequality that (together with $u_n \neq 0$ since $\gamma > 1$).

$$\left(\int_{\mathbb{R}^3} f^{\frac{2}{1+\gamma}} dx \right)^{\frac{1+\gamma}{2}} \left(\int_{\mathbb{R}^3} |u_n|^2 dx \right)^{\frac{1-\gamma}{2}} \leq \int_{\mathbb{R}^3} f |u_n|^{1-\gamma} dx \leq \|u_n\|_E^2 + \lambda \int_{\mathbb{R}^3} \phi_{u_n}^t |u_n|^2 dx \rightarrow 0.$$

This together with $\left(\int_{\mathbb{R}^3} f^{\frac{2}{1+\gamma}} dx \right)^{\frac{1+\gamma}{2}} > 0$ results in $\|u_n\|_E^2 \rightarrow +\infty$, which is impossible. Hence, $\|u\|_E \geq C_1$, $\forall u \in \mathcal{N}_1^{(\lambda)}$ for some $C_1 > 0$.

Step 3. *Properties of the minimizing sequence $\{u_n\}$.*

By Step 2, For every $u \in \mathcal{N}_1^{(\lambda)}$, we have $\|u\|_E \geq C_1$. Furthermore, it follows from $\gamma > 1$, $\lambda > 0$, Lemma 2.2 (ii) and (2.2) that

$$I_\lambda(u) = \frac{1}{2} \|u\|_E^2 + \frac{\lambda}{4} \int_{\mathbb{R}^3} \phi_u^t |u|^2 dx - \frac{1}{1-\gamma} \int_{\mathbb{R}^3} f |u|^{1-\gamma} dx \geq \frac{1}{2} \|u\|_E^2, \quad (3.1)$$

which implies that $I_\lambda(u)$ is coercive and bounded from below on $\mathcal{N}_1^{(\lambda)}$. Therefore, $\inf_{\mathcal{N}_1^{(\lambda)}} I_\lambda$ exists. Thus, we apply the Ekeland variational principle to get a minimizing sequence $\{u_n\} \subset \mathcal{N}_1^{(\lambda)}$ such that:

- (1) $I_\lambda(u_n) < \inf_{\mathcal{N}_1^{(\lambda)}} I_\lambda + \frac{1}{n}$;
- (2) $I_\lambda(z) \geq I_\lambda(u_n) - \frac{1}{n} \|u_n - z\|_E$, $\forall z \in \mathcal{N}_1^{(\lambda)}$.

The coercivity of I_λ on $\mathcal{N}_1^{(\lambda)}$ implies the existence of a constant $C_2 > 0$ such that $\|u_n\|_E \leq C_2$. To sum up, $C_1 \leq \|u_n\|_E \leq C_2$. So there exists $u_\lambda \in E$ such that, going if necessary to a subsequence, we have

$$\begin{aligned} u_n &\rightharpoonup u_\lambda, \quad \text{in } E, \\ u_n &\rightarrow u_\lambda, \quad \text{in } L^p(\mathbb{R}^3), \quad p \in [2, 2_s^*), \\ u_n &\rightarrow u_\lambda, \quad \text{a.e. in } \mathbb{R}^3. \end{aligned}$$

Since $I_\lambda(|u|) \leq I_\lambda(u)$, one can assume $u_n \geq 0$, then $u_\lambda \geq 0$. Using Lemma 2.2 (iii), $\{u_n\} \subset \mathcal{N}_1^{(\lambda)}$ and Fatou's lemma, we further get $\int_{\mathbb{R}^3} f u_\lambda^{1-\gamma} dx < +\infty$ which implies $u_\lambda > 0$ a.e. in $\mathbb{R}^3$.

Step 4. $u_\lambda \in \mathcal{N}_2^{(\lambda)}$, $\inf_{\mathcal{N}_1^{(\lambda)}} I_\lambda = I_\lambda(u_\lambda)$ and

$$(u_\lambda, \psi)_E + \lambda \int_{\mathbb{R}^3} \phi_{u_\lambda}^t u_\lambda \psi dx - \int_{\mathbb{R}^3} f u_\lambda^{-\gamma} \psi dx \geq 0, \quad \forall 0 \leq \psi \in E.$$

The above result will be proved by considering two situations on whether $\{u_n\} \subset \mathcal{N}_1^{(\lambda)} \setminus \mathcal{N}_2^{(\lambda)}$ or $\{u_n\} \subset \mathcal{N}_2^{(\lambda)}$.

Case 1. $\{u_n\} \subset \mathcal{N}_1^{(\lambda)} \setminus \mathcal{N}_2^{(\lambda)}$ for n large enough.

Fix $0 \leq \psi \in E$, since f is nonnegative, we can derive from $\{u_n\} \subset \mathcal{N}_1^{(\lambda)} \setminus \mathcal{N}_2^{(\lambda)}$ and $\gamma > 1$ that

$$\int_{\mathbb{R}^3} f (u_n + \eta \psi)^{1-\gamma} dx \leq \int_{\mathbb{R}^3} f u_n^{1-\gamma} dx < \|u_n\|_E^2 + \lambda \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx, \quad \forall \eta \geq 0.$$

Thus, there exists $\eta > 0$ sufficiently small satisfying

$$\int_{\mathbb{R}^3} f (u_n + \eta \psi)^{1-\gamma} dx < \|u_n + \eta \psi\|_E^2 + \lambda \int_{\mathbb{R}^3} \phi_{u_n + \eta \psi}^t (u_n + \eta \psi)^2 dx,$$

so $u_n + \eta\psi \in \mathcal{N}_1^{(\lambda)}$. In virtue of condition (2) by choosing $z = u_n + \eta\psi$, one has

$$\begin{aligned} \frac{\|\eta\psi\|_E}{n} &\geq I_\lambda(u_n) - I_\lambda(u_n + \eta\psi) \\ &= \frac{1}{2}(\|u_n\|_E^2 - \|u_n + \eta\psi\|_E^2) + \frac{\lambda}{4} \int_{\mathbb{R}^3} [\phi_{u_n}^t u_n^2 - \phi_{u_n + \eta\psi}^t (u_n + \eta\psi)^2] dx \\ &\quad + \frac{1}{1-\gamma} \int_{\mathbb{R}^3} f[(u_n + \eta\psi)^{1-\gamma} - (u_n)^{1-\gamma}] dx. \end{aligned}$$

Dividing by $\eta > 0$ and letting $\eta \rightarrow 0^+$, then we deduce from Fatou's lemma that

$$\frac{\|\psi\|_E}{n} + (u_n, \psi)_E + \lambda \int_{\mathbb{R}^3} \phi_{u_n}^t u_n \psi dx \geq \int_{\mathbb{R}^3} \liminf_{\eta \rightarrow 0^+} \frac{f}{1-\gamma} \frac{(u_n + \eta\psi)^{1-\gamma} - (u_n)^{1-\gamma}}{\eta} dx = \int_{\mathbb{R}^3} f u_n^{-\gamma} \psi dx.$$

Letting $n \rightarrow \infty$, one can obtain from Fatou's lemma and Lemma 2.2 (v) that

$$(u_\lambda, \psi)_E + \lambda \int_{\mathbb{R}^3} \phi_{u_\lambda}^t u_\lambda \psi dx \geq \int_{\mathbb{R}^3} f u_\lambda^{-\gamma} \psi dx \quad \text{and} \quad \int_{\mathbb{R}^3} f u_\lambda^{-\gamma} \psi dx < +\infty. \quad (3.2)$$

Choosing $\psi = u_\lambda$ in (3.2) leads to $u_\lambda \in \mathcal{N}_1^{(\lambda)}$, $\int_{\mathbb{R}^3} f u_\lambda^{1-\gamma} dx < +\infty$. Then, by Step 1, there exists a unique $\eta(u_\lambda) > 0$ satisfying $\eta(u_\lambda)u_\lambda \in \mathcal{N}_2^{(\lambda)}$ and $I_\lambda(\eta(u_\lambda)u_\lambda) = \min_{\eta>0} I_\lambda(\eta u_\lambda)$. Therefore, by Fatou's lemma and Lemma 2.2 (v) once again, we derive that

$$\begin{aligned} \inf_{\mathcal{N}_1^{(\lambda)}} I_\lambda &= \lim_{n \rightarrow \infty} I_\lambda(u_n) \\ &= \liminf_{n \rightarrow \infty} \left[\frac{1}{2} \|u_n\|_E^2 + \frac{\lambda}{4} \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx - \frac{1}{1-\gamma} \int_{\mathbb{R}^3} f u_n^{1-\gamma} dx \right] \\ &\geq \liminf_{n \rightarrow \infty} \left[\frac{1}{2} \|u_n\|_E^2 \right] + \liminf_{n \rightarrow \infty} \left[\frac{\lambda}{4} \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx \right] + \liminf_{n \rightarrow \infty} \left[\frac{1}{\gamma-1} \int_{\mathbb{R}^3} f u_n^{1-\gamma} dx \right] \\ &\geq \frac{1}{2} \|u_\lambda\|_E^2 + \frac{\lambda}{4} \int_{\mathbb{R}^3} \phi_{u_\lambda}^t u_\lambda^2 dx + \frac{1}{\gamma-1} \int_{\mathbb{R}^3} f u_\lambda^{1-\gamma} dx \\ &= I_\lambda(u_\lambda) \geq I_\lambda(\eta(u_\lambda)u_\lambda) \geq \inf_{\mathcal{N}_2^{(\lambda)}} I_\lambda \geq \inf_{\mathcal{N}_1^{(\lambda)}} I_\lambda. \end{aligned}$$

This together with the uniqueness of $\eta(u_\lambda)$ leads to $\eta(u_\lambda) = 1$. So, we have

$$u_\lambda \in \mathcal{N}_2^{(\lambda)}, \quad \inf_{\mathcal{N}_1^{(\lambda)}} I_\lambda = I_\lambda(u_\lambda). \quad (3.3)$$

Moreover, we can also obtain that $\liminf_{n \rightarrow \infty} \|u_n\|_E^2 = \|u_\lambda\|_E^2$ and a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) fulfils $\lim_{n \rightarrow \infty} \|u_n\|_E^2 = \|u_\lambda\|_E^2$, which means $u_n \rightarrow u_\lambda$ in E .

Case 2. There is a subsequence of $\{u_n\}$ (we still call u_n) belonging to $\mathcal{N}_2^{(\lambda)}$.

Fix $0 \leq \psi \in E$, since $\{u_n\}$ is bounded, one can obtain from $\gamma > 1$ and Lemma 2.2 (iii) that

$$\int_{\mathbb{R}^3} f(u_n + \eta\psi)^{1-\gamma} dx \leq \int_{\mathbb{R}^3} f u_n^{1-\gamma} dx = \|u_n\|_E^2 + \lambda \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx < +\infty, \quad \forall \eta \geq 0.$$

By Step 1, there are some functions $h_{n,\psi}(\eta) : [0, +\infty) \rightarrow (0, +\infty)$ satisfying

$$h_{n,\psi}(0) = 1, \quad h_{n,\psi}(\eta)(u_n + \eta\psi) \in \mathcal{N}_2^{(\lambda)}, \quad \forall \eta \geq 0.$$

By the dominated convergence theorem, Lemma 2.2 (v), $\int_{\mathbb{R}^3} f(x) u_n^{1-\gamma} dx < +\infty$ and $\gamma > 1$, we know that $h_{n,\psi}(\eta)$ is continuous for all $\eta \geq 0$. To facilitate the following proof, we define

$$h'_{n,\psi}(0) = \lim_{\eta \rightarrow 0^+} \frac{h_{n,\psi}(\eta) - 1}{\eta} \in [-\infty, +\infty].$$

If the right term is nonexistent, one can select $\eta_k \rightarrow 0$ (instead of $\eta \rightarrow 0$) with $\eta_k > 0$ and $h'_{n,\psi}(0) = \lim_{k \rightarrow \infty} \frac{h_{n,\psi}(\eta_k) - 1}{\eta_k} \in [-\infty, +\infty]$. Using Lemma 2.2 (iv), $u_n \in \mathcal{N}_2^{(\lambda)}$ and $h_{n,\psi}(\eta)(u_n + \eta\psi) \in \mathcal{N}_2^{(\lambda)}$, one can obtain

$$\begin{aligned} \|u_n\|_E^2 + \lambda \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx &= \int_{\mathbb{R}^3} f u_n^{1-\gamma} dx, \\ h_{n,\psi}^2(\eta) \|u_n + \eta\psi\|_E^2 + \lambda h_{n,\psi}^4(\eta) \int_{\mathbb{R}^3} \phi_{u_n + \eta\psi}^t (u_n + \eta\psi)^2 dx &= h_{n,\psi}^{1-\gamma}(\eta) \int_{\mathbb{R}^3} f (u_n + \eta\psi)^{1-\gamma} dx. \end{aligned}$$

Consequently, we have

$$\begin{aligned} 0 &= [h_{n,\psi}(\eta) - 1] \left\{ [h_{n,\psi}(\eta) + 1] \|u_n + \eta\psi\|_E^2 - \frac{h_{n,\psi}^{1-\gamma}(\eta) - 1}{h_{n,\psi}(\eta) - 1} \int_{\mathbb{R}^3} f (u_n + \eta\psi)^{1-\gamma} dx \right. \\ &\quad \left. + \lambda [h_{n,\psi}^2(\eta) + 1] [h_{n,\psi}(\eta) + 1] \int_{\mathbb{R}^3} \phi_{u_n + \eta\psi}^t (u_n + \eta\psi)^2 dx \right\} + [\|u_n + \eta\psi\|_E^2 - \|u_n\|_E^2] \\ &\quad + \lambda \int_{\mathbb{R}^3} [\phi_{u_n + \eta\psi}^t (u_n + \eta\psi)^2 - \phi_{u_n}^t u_n^2] dx - \int_{\mathbb{R}^3} f [(u_n + \eta\psi)^{1-\gamma} - u_n^{1-\gamma}] dx. \end{aligned}$$

Dividing by $\eta > 0$ and letting $\eta \rightarrow 0^+$, then one can get from $u_n \in \mathcal{N}_2^{(\lambda)}$, the continuity of $h_{n,\psi}(\eta)$, Lemma 2.2 (v) and $\gamma > 1$ that

$$\begin{aligned} 0 &\geq h'_{n,\psi}(0) \left\{ 2\|u_n\|_E^2 + (\gamma - 1) \int_{\mathbb{R}^3} f(x) u_n^{1-\gamma} dx + 4\lambda \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx \right\} + 2(u_n, \psi)_E + 4\lambda \int_{\mathbb{R}^3} \phi_{u_n}^t u_n \psi dx \\ &= h'_{n,\psi}(0) \left\{ (\gamma + 1) \|u_n\|_E^2 + \lambda(\gamma + 3) \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx \right\} + 2(u_n, \psi)_E + 4\lambda \int_{\mathbb{R}^3} \phi_{u_n}^t u_n \psi dx, \end{aligned}$$

which implies that $h'_{n,\psi}(0) \neq +\infty$. Next, we want to show the existence of $C_3 > 0$ satisfying $h'_{n,\psi}(0) \leq C_3$ uniformly in n . Fix n , without loss of generality, suppose $h'_{n,\psi}(0) \geq 0$. By Lemma 2.2 (ii) and the above inequality, we have

$$0 \geq (\gamma + 1) h'_{n,\psi}(0) \|u_n\|_E^2 + 2(u_n, \psi)_E.$$

This together with $C_1 \leq \|u_n\|_E \leq C_2$ leads to the existence of $C_3 > 0$ satisfying

$$h'_{n,\psi}(0) \leq C_3 \quad \text{uniformly in } n. \quad (3.4)$$

We claim that $h'_{n,\psi}(0)$ is bounded from below uniformly for n large enough satisfying

$$\frac{\|u_n\|_E}{n} - \frac{(\gamma + 1)C_1^2}{\gamma - 1} < 0.$$

Suppose $h'_{n,\psi}(0) \leq 0$, otherwise the declaration is obvious. Hence, $h_{n,\psi}(\eta) < 1$ for $\eta > 0$ small. In virtue of condition (2) again, we obtain

$$\frac{1}{n} [1 - h_{n,\psi}(\eta)] \|u_n\|_E + \frac{\eta}{n} h_{n,\psi}(\eta) \|\psi\|_E \geq \frac{1}{n} \|u_n - h_{n,\psi}(\eta)(u_n + \eta\psi)\|_E \geq I_\lambda(u_n) - I_\lambda[h_{n,\psi}(\eta)(u_n + \eta\psi)]. \quad (3.5)$$

By (3.5), Lemma 2.2 (iv) and $u_n \in \mathcal{N}_2^{(\lambda)}$, one has

$$\begin{aligned} \frac{\|\psi\|_E}{n} h_{n,\psi}(\eta) &\geq \frac{h_{n,\psi}(\eta) - 1}{\eta} \left\{ \frac{\|u_n\|_E}{n} - \left(\frac{1}{2} + \frac{1}{\gamma - 1} \right) [h_{n,\psi}(\eta) + 1] \|u_n + \eta\psi\|_E^2 \right. \\ &\quad \left. - \lambda \left(\frac{1}{4} + \frac{1}{\gamma - 1} \right) [h_{n,\psi}^2(\eta) + 1] [h_{n,\psi}(\eta) + 1] \int_{\mathbb{R}^3} \phi_{u_n + \eta\psi}^t (u_n + \eta\psi)^2 dx \right\} \\ &\quad - \left(\frac{1}{2} + \frac{1}{\gamma - 1} \right) \frac{\|u_n + \eta\psi\|_E^2 - \|u_n\|_E^2}{\eta} - \lambda \left(\frac{1}{4} + \frac{1}{\gamma - 1} \right) \int_{\mathbb{R}^3} \frac{\phi_{u_n + \eta\psi}^t (u_n + \eta\psi)^2 - \phi_{u_n}^t u_n^2}{\eta} dx. \end{aligned}$$

Letting $\eta \rightarrow 0^+$ and combining with $C_1 \leq \|u_n\|_E \leq C_2$, the continuity of $h_{n,\psi}(\eta)$, $\gamma > 1$ and Lemma 2.2 (v), one has

$$\begin{aligned} \frac{\|\psi\|_E}{n} &\geq h'_{n,\psi}(0) \left\{ \frac{\|u_n\|_E}{n} - \left(1 + \frac{2}{\gamma - 1} \right) \|u_n\|_E^2 - \lambda \left(1 + \frac{4}{\gamma - 1} \right) \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx \right\} \\ &\quad - \left(1 + \frac{2}{\gamma - 1} \right) (u_n, \psi)_E - \lambda \left(1 + \frac{4}{\gamma - 1} \right) \int_{\mathbb{R}^3} \phi_{u_n}^t u_n \psi dx \\ &= h'_{n,\psi}(0) \left\{ \frac{\|u_n\|_E}{n} - \frac{1}{\gamma - 1} \left[(\gamma + 1) \|u_n\|_E^2 + \lambda (\gamma + 3) \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx \right] \right\} \\ &\quad - \left(1 + \frac{2}{\gamma - 1} \right) (u_n, \psi)_E - \lambda \left(1 + \frac{4}{\gamma - 1} \right) \int_{\mathbb{R}^3} \phi_{u_n}^t u_n \psi dx \\ &\geq h'_{n,\psi}(0) \left[\frac{\|u_n\|_E}{n} - \frac{(\gamma + 1) C_1^2}{\gamma - 1} \right] - \left(1 + \frac{2}{\gamma - 1} \right) (u_n, \psi)_E - \lambda \left(1 + \frac{4}{\gamma - 1} \right) \int_{\mathbb{R}^3} \phi_{u_n}^t u_n \psi dx, \end{aligned}$$

where we used $h'_{n,\psi}(0) \leq 0$. Thus, the sign of the first coefficient reveals that

$$h'_{n,\psi}(0) \neq -\infty, \quad h'_{n,\psi}(0) \geq C_4 \quad \text{uniformly for large } n \quad (3.6)$$

for suitable constant C_4 . This together with (3.4) results in

$$h'_{n,\psi}(0) \in (-\infty, +\infty), \quad |h'_{n,\psi}(0)| \leq C \quad \text{uniformly for large } n$$

with $C = \max\{C_3, |C_4|\}$. Again, by condition (2), we obtain

$$\begin{aligned} &\frac{|1 - h_{n,\psi}(\eta)|}{\eta} \frac{\|u_n\|_E}{n} + \frac{\|\psi\|_E}{n} h_{n,\psi}(\eta) \\ &\geq \frac{1}{n\eta} \|u_n - h_{n,\psi}(\eta)(u_n + \eta\psi)\|_E \geq \frac{1}{\eta} \{I_\lambda(u_n) - I_\lambda[h_{n,\psi}(\eta)(u_n + \eta\psi)]\} \\ &\geq \frac{h_{n,\psi}(\eta) - 1}{\eta} \left\{ -\frac{h_{n,\psi}(\eta) + 1}{2} \|u_n + \eta\psi\|_E^2 + \frac{h_{n,\psi}^{1-\gamma}(\eta) - 1}{(1 - \gamma)[h_{n,\psi}(\eta) - 1]} \int_{\mathbb{R}^3} f(u_n + \eta\psi)^{1-\gamma} dx \right. \\ &\quad \left. - \frac{\lambda}{4} [h_{n,\psi}^2(\eta) + 1] [h_{n,\psi}(\eta) + 1] \int_{\mathbb{R}^3} \phi_{u_n + \eta\psi}^t (u_n + \eta\psi)^2 dx \right\} - \frac{1}{2} \frac{\|u_n + \eta\psi\|_E^2 - \|u_n\|_E^2}{\eta} \\ &\quad - \frac{\lambda}{4} \int_{\mathbb{R}^3} \frac{\phi_{u_n + \eta\psi}^t (u_n + \eta\psi)^2 - \phi_{u_n}^t u_n^2}{\eta} dx + \frac{1}{1 - \gamma} \int_{\mathbb{R}^3} \frac{f(u_n + \eta\psi)^{1-\gamma} - u_n^{1-\gamma}}{\eta} dx. \end{aligned}$$

Letting $\eta \rightarrow 0^+$, similar to the above discussion, one can get

$$\begin{aligned} \frac{|h'_{n,\psi}(0)| \cdot \|u_n\|_E}{n} + \frac{\|\psi\|_E}{n} &\geq h'_{n,\psi}(0) \left\{ -\|u_n\|_E^2 + \int_{\mathbb{R}^3} f u_n^{1-\gamma} dx - \lambda \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx \right\} \\ &\quad - (u_n, \psi)_E - \lambda \int_{\mathbb{R}^3} \phi_{u_n}^t u_n \psi dx + \liminf_{\eta \rightarrow 0^+} \frac{1}{1-\gamma} \int_{\mathbb{R}^3} f \frac{(u_n + \eta \psi)^{1-\gamma} - u_n^{1-\gamma}}{\eta} dx \\ &\geq - (u_n, \psi)_E - \lambda \int_{\mathbb{R}^3} \phi_{u_n}^t u_n \psi dx + \int_{\mathbb{R}^3} \frac{f}{1-\gamma} \liminf_{\eta \rightarrow 0^+} \frac{(u_n + \eta \psi)^{1-\gamma} - u_n^{1-\gamma}}{\eta} dx \\ &= - (u_n, \psi)_E - \lambda \int_{\mathbb{R}^3} \phi_{u_n}^t u_n \psi dx + \int_{\mathbb{R}^3} f u_n^{-\gamma} \psi dx. \end{aligned}$$

We can further get from $|h'_{n,\psi}(0)| \leq C$ for large n and $C_1 \leq \|u_n\|_E \leq C_2$ that

$$\begin{aligned} \int_{\mathbb{R}^3} f u_n^{-\gamma} \psi dx &\leq \frac{|h'_{n,\psi}(0)| \cdot \|u_n\|_E}{n} + \frac{\|\psi\|_E}{n} + (u_n, \psi)_E + \lambda \int_{\mathbb{R}^3} \phi_{u_n}^t u_n \psi dx \\ &\leq \frac{C \cdot C_2 + \|\psi\|_E}{n} + (u_n, \psi)_E + \lambda \int_{\mathbb{R}^3} \phi_{u_n}^t u_n \psi dx, \quad \text{for large } n. \end{aligned}$$

Letting $n \rightarrow \infty$, it follows from Fatou's lemma and Lemma 2.2 (v) that

$$\int_{\mathbb{R}^3} f u_\lambda^{-\gamma} \psi dx \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^3} f u_n^{-\gamma} \psi dx \leq (u_\lambda, \psi)_E + \lambda \int_{\mathbb{R}^3} \phi_{u_\lambda}^t u_\lambda \psi dx < +\infty, \quad (3.7)$$

for every $0 \leq \psi \in E$. Similar to Case 1, we have

$$u_\lambda \in \mathcal{N}_2^{(\lambda)}, \quad \inf_{\mathcal{N}_1^{(\lambda)}} I_\lambda = I_\lambda(u_\lambda), \quad (3.8)$$

in Case 2. Therefore, we can deduce from (3.2), (3.3), (3.7) and (3.8) that in both cases, up to subsequence, $u_n \rightarrow u_\lambda$ in E , $\inf_{\mathcal{N}_1^{(\lambda)}} I_\lambda = I_\lambda(u_\lambda)$, $u_\lambda \in \mathcal{N}_2^{(\lambda)}$ and

$$(u_\lambda, \psi)_E + \lambda \int_{\mathbb{R}^3} \phi_{u_\lambda}^t u_\lambda \psi dx - \int_{\mathbb{R}^3} f u_\lambda^{-\gamma} \psi dx \geq 0, \quad \forall 0 \leq \psi \in E. \quad (3.9)$$

Step 5. $u_\lambda > 0$ is a solution of system (SP_λ) .

Since $u_\lambda \in \mathcal{N}_2^{(\lambda)}$, we have

$$\|u_\lambda\|_E + \lambda \int_{\mathbb{R}^3} \phi_{u_\lambda}^t u_\lambda^2 dx - \int_{\mathbb{R}^3} f u_\lambda^{1-\gamma} dx = 0. \quad (3.10)$$

For every $\varepsilon > 0$ and $v \in E$, set $\psi_\varepsilon = u_\lambda + \varepsilon v$, we have

$$\begin{aligned} &(u_\lambda(x) - u_\lambda(y))(\psi_\varepsilon^+(x) - \psi_\varepsilon^+(y)) \\ &= (u_\lambda(x) - u_\lambda(y))[\psi_\varepsilon^+(x) + \psi_\varepsilon^-(x) - \psi_\varepsilon^-(y) - \psi_\varepsilon^-(y)] \\ &= |u_\lambda(x) - u_\lambda(y)|^2 + \varepsilon(u_\lambda(x) - u_\lambda(y))(v(x) - v(y)) + (u_\lambda(x) - u_\lambda(y))[\psi_\varepsilon^-(x) - \psi_\varepsilon^-(y)]. \end{aligned} \quad (3.11)$$

In virtue of the proof of [16, Theorem 3.2], we have

$$\liminf_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_\lambda(x) - u_\lambda(y))[\psi_\varepsilon^-(x) - \psi_\varepsilon^-(y)]}{|x - y|^{3+2s}} dx dy \leq 0. \quad (3.12)$$

Setting $\Omega_\varepsilon = \{x \in \mathbb{R}^3 : \psi_\varepsilon \leq 0\}$ and inserting $\psi = \psi_\varepsilon^+$ into (3.9) together with (3.10)–(3.11), we obtain

$$\begin{aligned}
 0 &\leq \frac{1}{\varepsilon} \left\{ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_\lambda(x) - u_\lambda(y))(\psi_\varepsilon^+(x) - \psi_\varepsilon^+(y))}{|x - y|^{3+2s}} dx dy + \int_{\mathbb{R}^3} V u_\lambda \psi_\varepsilon^+ dx + \lambda \int_{\mathbb{R}^3} \phi_{u_\lambda}^t u_\lambda \psi_\varepsilon^+ dx - \int_{\mathbb{R}^3} f u_\lambda^{-\gamma} \psi_\varepsilon^+ dx \right\} \\
 &= \frac{1}{\varepsilon} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_\lambda(x) - u_\lambda(y)|^2}{|x - y|^{3+2s}} dx dy + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_\lambda(x) - u_\lambda(y))(v(x) - v(y))}{|x - y|^{3+2s}} dx dy \\
 &\quad + \frac{1}{\varepsilon} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_\lambda(x) - u_\lambda(y))[\psi_\varepsilon^-(x) - \psi_\varepsilon^-(y)]}{|x - y|^{3+2s}} dx dy + \frac{1}{\varepsilon} \int_{\mathbb{R}^3 \setminus \Omega_\varepsilon} \{V u_\lambda(u_\lambda + \varepsilon v) + \lambda \phi_{u_\lambda}^t u_\lambda(u_\lambda + \varepsilon v) \\
 &\quad - f u_\lambda^{-\gamma}(u_\lambda + \varepsilon v)\} dx \\
 &= \frac{1}{\varepsilon} \left\{ \|u_\lambda\|_E^2 + \lambda \int_{\mathbb{R}^3} \phi_{u_\lambda}^t u_\lambda^2 dx - \int_{\mathbb{R}^3} f u_\lambda^{1-\gamma} dx \right\} + \left\{ (u_\lambda, v)_E + \lambda \int_{\mathbb{R}^3} \phi_{u_\lambda}^t u_\lambda v dx - \int_{\mathbb{R}^3} f u_\lambda^{-\gamma} v dx \right\} \\
 &\quad - \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} \{V u_\lambda(u_\lambda + \varepsilon v) + \lambda \phi_{u_\lambda}^t u_\lambda(u_\lambda + \varepsilon v) - f u_\lambda^{-\gamma}(u_\lambda + \varepsilon v)\} dx + \frac{1}{\varepsilon} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_\lambda(x) - u_\lambda(y))[\psi_\varepsilon^-(x) - \psi_\varepsilon^-(y)]}{|x - y|^{3+2s}} dx dy \\
 &\leq \left\{ (u_\lambda, v)_E + \lambda \int_{\mathbb{R}^3} \phi_{u_\lambda}^t u_\lambda v dx - \int_{\mathbb{R}^3} f u_\lambda^{-\gamma} v dx \right\} - \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} [V u_\lambda(u_\lambda + \varepsilon v) + \lambda \phi_{u_\lambda}^t u_\lambda(u_\lambda + \varepsilon v)] dx \\
 &\quad + \frac{1}{\varepsilon} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_\lambda(x) - u_\lambda(y))[\psi_\varepsilon^-(x) - \psi_\varepsilon^-(y)]}{|x - y|^{3+2s}} dx dy \\
 &\leq \left\{ (u_\lambda, v)_E + \lambda \int_{\mathbb{R}^3} \phi_{u_\lambda}^t u_\lambda v dx - \int_{\mathbb{R}^3} f u_\lambda^{-\gamma} v dx \right\} - \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} [V u_\lambda^2 + \lambda \phi_{u_\lambda}^t u_\lambda^2] dx \\
 &\quad - \int_{\Omega_\varepsilon} [V u_\lambda v + \lambda \phi_{u_\lambda}^t u_\lambda v] dx + \frac{1}{\varepsilon} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_\lambda(x) - u_\lambda(y))[\psi_\varepsilon^-(x) - \psi_\varepsilon^-(y)]}{|x - y|^{3+2s}} dx dy \\
 &\leq \left\{ (u_\lambda, v)_E + \lambda \int_{\mathbb{R}^3} \phi_{u_\lambda}^t u_\lambda v dx - \int_{\mathbb{R}^3} f u_\lambda^{-\gamma} v dx \right\} - \int_{\Omega_\varepsilon} [V u_\lambda v + \lambda \phi_{u_\lambda}^t u_\lambda v] dx \\
 &\quad + \frac{1}{\varepsilon} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_\lambda(x) - u_\lambda(y))[\psi_\varepsilon^-(x) - \psi_\varepsilon^-(y)]}{|x - y|^{3+2s}} dx dy.
 \end{aligned}$$

Letting $\varepsilon \rightarrow 0^+$, thanks to (3.12) and the fact that $|\Omega_\varepsilon| \rightarrow 0$ as $\varepsilon \rightarrow 0^+$, one has

$$(u_\lambda, v)_E + \lambda \int_{\mathbb{R}^3} \phi_{u_\lambda}^t u_\lambda v dx - \int_{\mathbb{R}^3} f u_\lambda^{-\gamma} v dx \geq 0, \quad \forall v \in E.$$

Substituting v for $-v$, one can get the reserved inequality. So we have

$$(u_\lambda, v)_E + \lambda \int_{\mathbb{R}^3} \phi_{u_\lambda}^t u_\lambda v dx - \int_{\mathbb{R}^3} f u_\lambda^{-\gamma} v dx = 0, \quad \forall v \in E. \quad (3.13)$$

Similar to [12, Theorem 6.3], we get $u_\lambda \in C_{loc}^\alpha(\mathbb{R}^3)$ for some $\alpha \in (0, s)$. Since $u_\lambda \geq 0$ and $u_\lambda \not\equiv 0$, by the strong maximum principle in [38, Theorem 2.6], we can obtain that $u_\lambda > 0$ in $\mathbb{R}^3$ and $u_\lambda \in E$ is a solution to $(SP)_\lambda$.

Step 6. The solution u_λ is unique.

Assume $u_* \in E$ also solves $(SP)_\lambda$, then one has

$$(u_*, v)_E + \lambda \int_{\mathbb{R}^3} \phi_{u_*}^t u_* v dx - \int_{\mathbb{R}^3} f u_*^{-\gamma} v dx = 0, \quad \forall v \in E. \quad (3.14)$$

Taking $v = u_\lambda - u_*$ in (3.13)–(3.14), then subtracting (3.13) from (3.14) and using Lemma 2.2 (vi), we get

$$\begin{aligned} 0 &\geq \int_{\mathbb{R}^3} f(u_\lambda^{-\gamma} - u_*^{-\gamma})(u_\lambda - u_*) dx \\ &= \|u_\lambda - u_*\|_E^2 + \lambda \int_{\mathbb{R}^3} (\phi_{u_\lambda}^t u_\lambda - \phi_{u_*}^t u_*)(u_\lambda - u_*) dx \\ &\geq \|u_\lambda - u_*\|_E^2 + \frac{1}{2} \|\phi_{u_\lambda}^t - \phi_{u_*}^t\|_{\mathcal{D}^{t,2}(\mathbb{R}^3)}^2 \geq \|u_\lambda - u_*\|_E^2 \geq 0. \end{aligned}$$

Therefore, $\|u_\lambda - u_*\|_E^2 = 0$, which implies $u_\lambda = u_*$ and the solution u_λ is unique. $\square$

Proof of Theorem 1.2. By direct calculation, for every $\psi \in E$ and a.e. $x, y \in \mathbb{R}^3$, one can obtain

$$\begin{aligned} (\psi(x) - \psi(y))(\psi^+(x) - \psi^+(y)) &= [\psi^+(x) - \psi^-(x) - \psi^+(y) + \psi^-(y)](\psi^+(x) - \psi^+(y)) \\ &= |\psi^+(x) - \psi^+(y)|^2 + \psi^-(x)\psi^+(y) + \psi^-(y)\psi^+(x) \\ &\geq |\psi^+(x) - \psi^+(y)|^2. \end{aligned} \quad (3.15)$$

For every $v \in E$, as u_1, u_2 are two solutions corresponding to f_1, f_2 , we get

$$\begin{aligned} (u_1, v)_E + \lambda \int_{\mathbb{R}^3} \phi_{u_1}^t u_1 v dx &= \int_{\mathbb{R}^3} f_1 u_1^{-\gamma} v dx, \\ (u_2, v)_E + \lambda \int_{\mathbb{R}^3} \phi_{u_2}^t u_2 v dx &= \int_{\mathbb{R}^3} f_2 u_2^{-\gamma} v dx. \end{aligned}$$

Taking $v = (u_2 - u_1)^+$ in the above two equations and subtracting term by term, we obtain from applying (3.15) with $\psi = u_2 - u_1$ that

$$\begin{aligned} 0 &\geq \int_{\mathbb{R}^3} (f_2 u_2^{-\gamma} - f_1 u_1^{-\gamma})(u_2 - u_1)^+ dx \\ &= (u_2 - u_1, (u_2 - u_1)^+)_E + \lambda \int_{\mathbb{R}^3} (\phi_{u_2}^t u_2 - \phi_{u_1}^t u_1)(u_2 - u_1)^+ dx \\ &\geq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{[(u_2 - u_1)^+(x) - (u_2 - u_1)^+(y)]^2}{|x - y|^{3+2s}} dx dy + \int_{\mathbb{R}^3} V[(u_2 - u_1)^+]^2 dx + \lambda \int_{[u_2 - u_1 \geq 0]} (\phi_{u_2}^t u_2 - \phi_{u_1}^t u_1)(u_2 - u_1) dx \\ &= \|(u_2 - u_1)^+\|_E^2 + \lambda \int_{[u_2 - u_1 \geq 0]} (\phi_{u_2}^t u_2 - \phi_{u_1}^t u_1)(u_2 - u_1) dx \\ &\geq \|(u_2 - u_1)^+\|_E^2 \geq 0, \end{aligned}$$

where we used $\gamma > 1$, Lemma 2.2 (vi) and $f_1 \geq f_2$. So $(u_2 - u_1)^+ \equiv 0$ and hence $u_1 \geq u_2$. $\square$

Proof of Theorem 1.3. For every $v \in E$, as $u_{\lambda_1}, u_{\lambda_2}$ are two solutions corresponding to λ_1, λ_2 , we deduce that

$$\begin{aligned} (u_{\lambda_1}, v)_E + \lambda_1 \int_{\mathbb{R}^3} \phi_{u_{\lambda_1}}^t u_{\lambda_1} v dx &= \int_{\mathbb{R}^3} f u_{\lambda_1}^{-\gamma} v dx, \\ (u_{\lambda_2}, v)_E + \lambda_2 \int_{\mathbb{R}^3} \phi_{u_{\lambda_2}}^t u_{\lambda_2} v dx &= \int_{\mathbb{R}^3} f u_{\lambda_2}^{-\gamma} v dx. \end{aligned}$$

Taking $v = (u_{\lambda_2} - u_{\lambda_1})^+$ in the above two equations and subtracting, we get from applying (3.15) with $\psi = u_{\lambda_2} - u_{\lambda_1}$ that

$$\begin{aligned}
0 &\geq \int_{\mathbb{R}^3} f(u_{\lambda_2}^{-\gamma} - u_{\lambda_1}^{-\gamma})(u_{\lambda_2} - u_{\lambda_1})^+ dx \\
&= (u_{\lambda_2} - u_{\lambda_1}, (u_{\lambda_2} - u_{\lambda_1})^+)_E + \int_{\mathbb{R}^3} (\lambda_2 \phi_{u_{\lambda_2}}^t u_{\lambda_2} - \lambda_1 \phi_{u_{\lambda_1}}^t u_{\lambda_1})(u_{\lambda_2} - u_{\lambda_1})^+ dx \\
&\geq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{[(u_{\lambda_2} - u_{\lambda_1})^+(x) - (u_{\lambda_2} - u_{\lambda_1})^+(y)]^2}{|x - y|^{3+2s}} dx dy + \int_{\mathbb{R}^3} V[(u_{\lambda_2} - u_{\lambda_1})^+]^2 dx \\
&\quad + \lambda_1 \int_{[u_{\lambda_2} - u_{\lambda_1} \geq 0]} (\phi_{u_{\lambda_2}}^t u_{\lambda_2} - \phi_{u_{\lambda_1}}^t u_{\lambda_1})(u_{\lambda_2} - u_{\lambda_1}) dx \\
&= \|(u_{\lambda_2} - u_{\lambda_1})^+\|_E^2 + \lambda_1 \int_{[u_{\lambda_2} - u_{\lambda_1} \geq 0]} (\phi_{u_{\lambda_2}}^t u_{\lambda_2} - \phi_{u_{\lambda_1}}^t u_{\lambda_1})(u_{\lambda_2} - u_{\lambda_1}) dx \\
&\geq \|(u_{\lambda_2} - u_{\lambda_1})^+\|_E^2 \geq 0,
\end{aligned}$$

where we used $0 < \lambda_1 \leq \lambda_2$, Lemma 2.2 (vi) and $\gamma > 1$. So $(u_{\lambda_2} - u_{\lambda_1})^+ \equiv 0$ and then $u_{\lambda_1} \geq u_{\lambda_2}$. $\square$

Proof of Theorem 1.4. Since the proof of Theorem 1.1 is also valid for $\lambda = 0$ and $\lambda = 1$, then under the assumptions of Theorem 1.4, we have positive solutions $w_0 \in E$ and $w_1 \in E$ corresponding to (P_0) and (SP_1) , respectively. Hence, for every $v \in E$, we have

$$(w_0, v)_E = \int_{\mathbb{R}^3} f w_0^{-\gamma} v dx \quad \text{and} \quad w_0 \in \mathcal{N}_2^{(0)}, \quad \inf_{\mathcal{N}_1^{(0)}} I_0 = I_0(w_0), \quad (3.16)$$

and

$$(w_1, v)_E + \int_{\mathbb{R}^3} \phi_{w_1}^t w_1 v dx - \int_{\mathbb{R}^3} f w_1^{-\gamma} v dx = 0 \quad \text{and} \quad w_1 \in \mathcal{N}_2^{(1)}, \quad \inf_{\mathcal{N}_1^{(1)}} I_1 = I_1(w_1). \quad (3.17)$$

Step 1. $c_\lambda \leq c_1$ for $\lambda \in [0, 1]$, where $c_\lambda = \inf_{\mathcal{N}_1^{(\lambda)}} I_\lambda$.

According to the proof for the necessity of Theorem 1.1, one has $\int_{\mathbb{R}^3} f w_0^{1-\gamma} dx < +\infty$ and $\int_{\mathbb{R}^3} f w_1^{1-\gamma} dx < +\infty$. Moreover, (3.17) and Step 1 in the proof for Theorem 1.1 reveal the existence of a unique $\eta(w_1) > 0$ such that $\eta(w_1)w_1 \in \mathcal{N}_2^{(\lambda)}$, $I_\lambda(\eta(w_1)w_1) = \min_{\eta>0} I_\lambda(\eta w_1)$ and $c_1 = I_1(w_1) = \min_{\eta>0} I_1(\eta w_1)$. By $\mathcal{N}_2^{(\lambda)} \subset \mathcal{N}_1^{(\lambda)}$ and $\lambda \in [0, 1]$, we obtain

$$c_\lambda = \inf_{\mathcal{N}_1^{(\lambda)}} I_\lambda \leq \inf_{\mathcal{N}_2^{(\lambda)}} I_\lambda \leq I_\lambda(\eta(w_1)w_1) = \min_{\eta>0} I_\lambda(\eta w_1) \leq \min_{\eta>0} I_1(\eta w_1) = c_1.$$

For any sequence $\{\lambda_n\} \subset (0, 1)$ satisfying $\lambda_n \rightarrow 0$, as $\{u_{\lambda_n}\}$ is a sequence of positive solutions to system (SP_{λ_n}) , we have $c_{\lambda_n} \leq c_1$ and for any $v \in E$,

$$(u_{\lambda_n}, v)_E + \lambda_n \int_{\mathbb{R}^3} \phi_{u_{\lambda_n}}^t u_{\lambda_n} v dx = \int_{\mathbb{R}^3} f u_{\lambda_n}^{-\gamma} v dx. \quad (3.18)$$

Using (3.1), we can further get

$$\frac{1}{2} \|u_{\lambda_n}\|_E^2 \leq I_{\lambda_n}(u_{\lambda_n}) = c_{\lambda_n} \leq c_1,$$

so $\{u_{\lambda_n}\}$ is bounded in E . Thus, there exists a nonnegative function $u_0 \in E$ such that, going if necessary to a subsequence, one has

$$\begin{aligned}
u_{\lambda_n} &\rightharpoonup u_0, & \text{in } E, \\
u_{\lambda_n} &\rightarrow u_0, & \text{in } L^p(\mathbb{R}^3), \quad p \in [2, 2_s^*), \\
u_{\lambda_n} &\rightarrow u_0, & \text{a.e. in } \mathbb{R}^3.
\end{aligned} \quad (3.19)$$

Step 2. $u_0 \in \mathcal{N}_2^{(0)}$, $\inf_{\mathcal{N}_1^{(0)}} I_0 = I_0(u_0)$, $u_{\lambda_n} \rightarrow u_0$ in E and

$$(u_0, v)_E \geq \int_{\mathbb{R}^3} f u_0^{-\gamma} v dx, \quad \forall 0 \leq v \in E.$$

Letting $n \rightarrow \infty$ in (3.18), then we can deduce from Fatou's lemma, Lemma 2.2 (v) and (3.19) that

$$(u_0, v)_E - \int_{\mathbb{R}^3} f u_0^{-\gamma} v dx \geq 0 \quad (3.20)$$

for any $0 \leq v \in E$. Similar to Step 4 in the proof for Theorem 1.1, one has $u_0 > 0$ in $\mathbb{R}^3$. Selecting $v = u_0$ in (3.20), then we get $\|u_0\|_E^2 - \int_{\mathbb{R}^3} f u_0^{1-\gamma} dx \geq 0$, i.e. $u_0 \in \mathcal{N}_1^{(0)}$, so $I_0(u_0) \geq c_0$. Similar to Step 1 in the proof for Theorem 1.1, for every $n \in \mathbb{N}$, there exists a unique $\eta_n(w_0) > 0$ satisfying $\eta_n(w_0)w_0 \in \mathcal{N}_2^{(\lambda_n)}$, $I_{\lambda_n}(\eta_n(w_0)w_0) = \min_{\eta > 0} I_{\lambda_n}(\eta w_0)$. Thus,

$$\begin{aligned} c_0 &= I_0(w_0) \\ &= I_{\lambda_n}(w_0) - \frac{\lambda_n}{4} \int_{\mathbb{R}^3} \phi_{w_0}^t w_0^2 dx \\ &\geq I_{\lambda_n}(\eta_n(w_0)w_0) - \frac{\lambda_n}{4} \int_{\mathbb{R}^3} \phi_{w_0}^t w_0^2 dx \\ &\geq c_{\lambda_n} - \frac{\lambda_n}{4} \int_{\mathbb{R}^3} \phi_{w_0}^t w_0^2 dx, \end{aligned}$$

which yields

$$\limsup_{n \rightarrow +\infty} c_{\lambda_n} \leq c_0. \quad (3.21)$$

On the other hand, we can obtain from (2.2) and Lemma 2.2 (ii) that

$$c_{\lambda_n} = I_{\lambda_n}(u_{\lambda_n}) \geq \frac{1}{2} \|u_{\lambda_n}\|_E^2 - \frac{1}{1-\gamma} \int_{\mathbb{R}^3} f u_{\lambda_n}^{1-\gamma} dx.$$

According to (3.19), $I_0(u_0) \geq c_0$ and Fatou's lemma, one has

$$\begin{aligned} \liminf_{n \rightarrow \infty} c_{\lambda_n} &\geq \liminf_{n \rightarrow \infty} \left[\frac{1}{2} \|u_{\lambda_n}\|_E^2 \right] + \liminf_{n \rightarrow \infty} \left[\frac{1}{\gamma-1} \int_{\mathbb{R}^3} f u_{\lambda_n}^{1-\gamma} dx \right] \\ &\geq \frac{1}{2} \|u_0\|_E^2 + \frac{1}{\gamma-1} \int_{\mathbb{R}^3} f u_0^{1-\gamma} dx = I_0(u_0) \geq c_0. \end{aligned} \quad (3.22)$$

This combined with (3.21) leads to $\lim_{n \rightarrow +\infty} c_{\lambda_n} = c_0$. So, $u_{\lambda_n} \rightarrow u_0$ in E , $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} f u_{\lambda_n}^{1-\gamma} dx = \int_{\mathbb{R}^3} f u_0^{1-\gamma} dx$ and $I_0(u_0) = c_0 = \inf_{\mathcal{N}_1^{(0)}} I_0$. Choosing $v = u_{\lambda_n}$ in (3.18) and letting $n \rightarrow +\infty$, we have

$$\|u_0\|_E^2 = \int_{\mathbb{R}^3} f u_0^{1-\gamma} dx, \quad \text{i.e. } u_0 \in \mathcal{N}_2^{(0)}.$$

Step 3. $u_0 = w_0$ and then $u_{\lambda_n} \rightarrow w_0$ in E .

By (3.20) and $u_0 \in \mathcal{N}_2^{(0)}$, similar to Step 5 in the proof for Theorem 1.1, one could further obtain that $0 < u_0 \in E$ is a solution to (P_0) . Then, the uniqueness of solution to (P_0) implies $u_0 = w_0$. Hence $u_{\lambda_n} \rightarrow w_0$ in E and the solution w_0 is unique to problem (P_0) . $\square$

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