Research Article

Wei Sun*, Ru-Long Xie, and Liang-Yu Xu

Oscillatory hyper-Hilbert transform on Wiener amalgam spaces

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Abstract: We study the boundedness of the oscillatory integral

$$T_{\alpha,\beta}f(x,y) = \int_{Q^2} f(x - \gamma_1(t), y - \gamma_2(s)) e^{-2\pi i t^{-\beta_1} s^{-\beta_2} t^{-\alpha_1-1} s^{-\alpha_2-1}} dt ds$$

on Wiener amalgam spaces, where $Q^2 = [0, 1] \times [0, 1]$ is the unit square in two dimensions, $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$, $\gamma_1(t) = (t^{p_1}, t^{p_2}, \dots, t^{p_n})$, $\gamma_2(s) = (s^{q_1}, s^{q_2}, \dots, s^{q_m})$ are homogeneous curves on $\mathbb{R}^n$ and $\mathbb{R}^m$.

Keywords: oscillatory integral, Wiener amalgam spaces, curves, product domain**MSC 2020:** 42B20, 42B25

1 Introduction

Time-frequency analysis is an important branch of modern harmonic analysis, with relevant applications in wireless communication and signal analysis. In order to measure the time and frequency of a function or distribution, Feichtinger has introduced the Wiener amalgam spaces [1,2] (see Section 2 for details), which describes local and global properties of a function or distribution. In the last decade, Wiener amalgam spaces have been widely used in time-frequency and phase-space analyses, and also employed to study boundedness properties of pseudo-differential operators, Fourier integral, Fourier multiplier operators and well-posedness of solutions to partial differential equations. More details may be found in [3–9].

In harmonic analysis, there is a large interest in the study of the properties of singular integral operators. Our starting point is the well-known fact that the Hilbert transform

$$H_\gamma f(x) = \text{p.v.} \int_{-1}^1 f(x - \gamma(t)) \frac{dt}{t},$$

where $\gamma : (-1, 1) \rightarrow \mathbb{R}^n$ is a continuous curve in $\mathbb{R}^n$. The study of these operators was initiated by Fabes and Rivière [10]. Stein and Wainger [11] proved that H_γ is bounded on L^p ($1 < p < \infty$) if γ is well-curved in $\mathbb{R}^n$. Here, we say that γ is well-curved, if γ is smooth with $\gamma(0) = 0$ and a segment of the curve containing the origin lies in a subspace of $\mathbb{R}^n$ spanned by

$$\left. \frac{d^{(k)}\gamma(t)}{dt} \right|_{t=0}, \quad k = 1, 2, \dots$$

* **Corresponding author: Wei Sun**, School of Mathematics and Statistics, Chaohu University, Hefei Anhui, China, e-mail: 058056@chu.edu.cn

Ru-Long Xie: School of Mathematics and Statistics, Chaohu University, Hefei Anhui, China, e-mail: 058028@chu.edu.cn

Liang-Yu Xu: School of Mathematics and Statistics, Chaohu University, Hefei Anhui, China, e-mail: 058059@chu.edu.cn

One may force the singularity of operator H_γ at the origin by considering the operator

$$H_{\gamma,\alpha}f(x,y) = \text{p.v.} \int_{-1}^1 f(x-t, y-\gamma(t)) \frac{dt}{t|t|^\alpha}, \quad \alpha > 0,$$

where $\gamma(t) = |t|^k$ or $\gamma(t) = |t|^k \text{sgn}(t)$ and $k \geq 2$. In this case, it is not difficult to see that the operator is not bounded even on L^2 , since its Fourier multiplier is not uniformly bounded (see, e.g., [12]). In order to balance the singularity of operator $H_{\gamma,\alpha}$ at the origin, we may introduce an additional oscillatory term $e^{-2\pi i|t|^{-\beta}}$ and define the operator

$$H_{\alpha,\beta}f(x,y) = \text{p.v.} \int_{-1}^1 f(x-t, y-\gamma(t)) \frac{e^{-2\pi i|t|^{-\beta}}}{t|t|^\alpha} dt; \quad \alpha, \beta > 0.$$

These operators have been first studied back in 1985 by Zielinski [13], who proved that the operator $H_{\alpha,\beta}$ is bounded on $L^2(\mathbb{R}^2)$ if and only if $\beta \geq 3\alpha$, where $\gamma(t) = t^2$. In 1996, this result was improved by Chandarana [12], who considered the general homogeneous curves $\gamma(t) = |t|^k$ or $\gamma(t) = |t|^k \text{sgn}(t)$ for $k \geq 2$, and proved that if $\beta > 3\alpha > 0$, then the operator $H_{\alpha,\beta}$ is bounded on $L^p(\mathbb{R}^2)$ for

$$1 + \frac{1 + 3\alpha(\beta + 1)}{\beta(\beta + 1) + (\beta - 3\alpha)} < p < \frac{\beta(\beta + 1) + (\beta - 3\alpha)}{1 + 3\alpha(\beta + 1)} + 1.$$

More recently, Cheng et al. have studied the mapping properties of the operator $H_{\alpha,\beta}$ on Wiener amalgam spaces in [14]. They proved that if $\beta \geq 3\alpha$, then the operator $H_{\alpha,\beta}$ is bounded on $W(\mathcal{FL}^p, L^q)(\mathbb{R}^2)$ for $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $\gamma(t) = |t|^k$ or $\gamma(t) = |t|^k \text{sgn}(t)$ for $k \geq 2$.

Let us first consider the following oscillatory integral along curves in the two-dimensional product space $\mathbb{R} \times \mathbb{R}$

$$Tf(x,y) = \int_{Q^2} \frac{f(x-\gamma_1(t), y-\gamma_2(s))}{t^{1+\alpha_1}s^{1+\alpha_2}} e^{-2\pi i t^{-\beta_1} s^{-\beta_2}} dt ds; \quad (x,y) \in \mathbb{R} \times \mathbb{R}, \quad (1.1)$$

where $Q^2 = [0, 1] \times [0, 1]$ is the unit square in two dimensions, and $\gamma_1(t) = t^k$, $\gamma_2(s) = s^j$.

Our first theorem may be stated as follows.

Theorem 1.1. *Let $1 \leq p < \infty$, $1 \leq q \leq \infty$, and operator T is defined by (1.1). Then we have*

- (i) *Suppose $\beta_1 > \alpha_1 \geq 0$, $\beta_2 > \alpha_2 \geq 0$, if $k = 0, j \geq 0$ or $j = 0, k \geq 0$, then the operator T is bounded on $W(\mathcal{FL}^p, L^q)(\mathbb{R} \times \mathbb{R})$;*
- (ii) *Suppose $\beta_1 > 2\alpha_1 \geq 0$, $\beta_2 > 2\alpha_2 \geq 0$, if $k \geq 1, j \geq 0$ or $j \geq 1, k \geq 0$, then the operator T is bounded on $W(\mathcal{FL}^p, L^q)(\mathbb{R} \times \mathbb{R})$.*

In 2008, Chen et al. extended Chandarana's results to n -dimensional space [15]. In particular, they have shown that if $\beta > (n+1)\alpha$ and $\Gamma(t) = (t^{p_1}, t^{p_2}, \dots, t^{p_n})$, then the operator defined by

$$H_{n,\alpha,\beta}f(x) = \text{p.v.} \int_{-1}^1 f(x - \Gamma(t)) e^{-2\pi i|t|^{-\beta}} \frac{dt}{t|t|^\alpha}$$

is bounded on $L^p(\mathbb{R}^n)$, where

$$\frac{2\beta}{2\beta - (n+1)\alpha} < p < \frac{2\beta}{(n+1)\alpha}.$$

Soon after [16], they investigated the mapping properties of the oscillatory integral along certain curves in high-dimensional product domain on Lebesgue space and defined the operator $T_{\alpha,\beta}$ in $\mathbb{R}^n \times \mathbb{R}^m$ by

$$T_{\alpha,\beta}f(x,y) = \int_{Q^2} \frac{f(x-\gamma_1(t), y-\gamma_2(s))}{t^{1+\alpha_1}s^{1+\alpha_2}} e^{-2\pi i t^{-\beta_1} s^{-\beta_2}} dt ds, \quad (x,y) \in \mathbb{R}^n \times \mathbb{R}^m \quad (1.2)$$

where $\gamma_1(t) = (t^{p_1}, t^{p_2}, \dots, t^{p_n})$, $\gamma_2(s) = (s^{q_1}, s^{q_2}, \dots, s^{q_m})$ and $0 < p_1 < p_2 < \dots < p_n$. They proved that if $\beta_1 > (n+1)\alpha_1$, $\beta_2 > (n+1)\alpha_2$ and $\sigma = \min\{\beta_1 - (n+1)\alpha_1, \beta_2 - (n+1)\alpha_2\}$, if $|2^{-1} - p^{-1}| < \sigma$, then the operator $T_{\alpha, \beta} f(x, y)$ is bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$.

For the operator of $T_{\alpha, \beta}$, we prove the following result.

Theorem 1.2. *Let $\beta_1 > (n+1)\alpha_1 \geq 0$ and $\beta_2 > (n+1)\alpha_2 \geq 0$, and assume that $p_1, p_2, \dots, p_n$ and $q_1, q_2, \dots, q_m$ are two groups of distinct positive real numbers. Then the operator $T_{\alpha, \beta}$, which is defined by (1.2), is bounded on $W(\mathcal{FL}^p, L^q)(\mathbb{R}^n \times \mathbb{R}^m)$, where $1 \leq p < \infty$, $1 \leq q \leq \infty$.*

Throughout this paper, we always denote by C any positive constant that may vary at each occurrence, but it is independent on the essential variable.

This paper is organized as follows. In Section 2, we recall the definitions of Wiener amalgam spaces and modulation space. Some basic properties of Wiener amalgam spaces are also reviewed. In Section 3, we prove our main results. Finally, we discuss two extensions in Section 4.

2 Preliminaries and important lemmas

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of all complex-valued, rapidly decreasing, infinitely differentiable functions and $\mathcal{S}'(\mathbb{R}^n)$ be the topological dual of $\mathcal{S}(\mathbb{R}^n)$. For a function $f \in \mathcal{S}(\mathbb{R}^n)$, its Fourier transform is defined by $\hat{f}(\omega) = \int f(t) e^{-2\pi i \omega \cdot t} dt$, and the inverse Fourier transform is $\check{f}(t) = \hat{f}(-t)$. The translation and modulation operators are defined by

$$T_x f(t) = f(t - x), \quad M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t),$$

for every $x, \omega \in \mathbb{R}^n$.

The short-time Fourier transform (STFT) is defined by

$$V_g f(x, \omega) = \langle f, M_\omega T_x g \rangle = \int e^{-2\pi i \omega \cdot y} f(y) \overline{g(y - x)} dy,$$

i.e., the Fourier transform applied to $f \overline{T_x g}$.

Starting from STFT, let g be a nonzero Schwartz function and $1 \leq p, q \leq \infty$, then the modulation space $M^{p, q}(\mathbb{R}^n)$ is defined as the closure of the Schwartz class with respect to the norm

$$\|f\|_{M^{p, q}} = \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |V_g f(x, \omega)|^p dx \right)^{\frac{q}{p}} d\omega \right)^{\frac{1}{q}},$$

with the general modifications for p or $q = \infty$.

For $1 \leq p \leq \infty$, let $\mathcal{FL}^p(\mathbb{R}^n)$ be the space of tempered distributions with Fourier transforms in $L^p(\mathbb{R}^n)$, that is,

$$\mathcal{FL}^p(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) | \hat{f} \in L^p\}$$

with norm $\|f\|_{\mathcal{FL}^p} = \|\hat{f}\|_{L^p}$.

Using $\mathcal{FL}^p(\mathbb{R}^n)$, Feichtinger [1] introduced Wiener amalgam spaces as follows:

Definition 2.1. For $1 \leq p, q \leq \infty$, a tempered distribution f is in the Wiener amalgam spaces $W(\mathcal{FL}^p, L^q)(\mathbb{R}^n)$, if f is locally in $\mathcal{FL}^p(\mathbb{R}^n)$ and globally in $L^q(\mathbb{R}^n)$, that is, for every non-zero $g \in C_0^\infty(\mathbb{R}^n)$, $\mathcal{F}(f T_x g) \in \mathcal{FL}^p(\mathbb{R}^n)$ and

$$\|f\|_{W(\mathcal{FL}^p, L^q)} = \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\mathcal{F}(f T_x g)(y)|^p dy \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}}$$

is finite, with the general modifications for p or $q = \infty$.

To prove our main results, the following lemmas play an important role.

Lemma 2.1. *Let us consider $1 \leq p_1, p_2, q_2, r_1, r_2 \leq \infty$, $1 \leq q_1 < \infty$ which satisfy the conditions $\frac{1}{q_1} = \frac{1}{p_1} + \frac{1}{r_1}$, $1 + \frac{1}{q_2} = \frac{1}{p_2} + \frac{1}{r_2}$, then*

$$W(\mathcal{FL}^{p_1}, L^{p_2}) * W(\mathcal{FL}^{r_1}, L^{r_2}) \hookrightarrow W(\mathcal{FL}^{q_1}, L^{q_2}).$$

Remark. In particular, if $r_1 = \infty$ and $r_2 = 1$, then $p_1 = q_1 \triangleq p$, $p_2 = q_2 \triangleq q$, and the statement of Lemma 2.1 may be written as

$$W(\mathcal{FL}^p, L^q) * W(\mathcal{FL}^\infty, L^1) \hookrightarrow W(\mathcal{FL}^p, L^q).$$

This conclusion provides us with a method to prove that singular convolution operators are bounded on Wiener amalgam spaces. Suppose that the operator T is defined by $Tf(x) = (k * f)(x)$, where the kernel function of operator T is $k(x)$. If we can prove that the kernel $k \in W(\mathcal{FL}^\infty, L^1)$, then the operator is bounded on Wiener amalgam spaces $W(\mathcal{FL}^p, L^q)$.

Lemma 2.2. $f \in W(\mathcal{FL}^\infty, L^1)$ if and only if $\hat{f} \in M^{\infty,1}$.

Lemma 2.3. Let $K = 2\left(\left[\frac{n}{2}\right] + 1\right)$, and G^K defined as follows:

$$G^K = \left\{ f \in C^K(\mathbb{R}^n) \left| \sum_{|l| \leq K} \|\partial^l f\|_\infty \leq C \right. \right\},$$

then $G^K \subset M^{\infty,1}$.

The proofs of Lemmas 2.1–2.3 can be found in [1,2], and the following lemma is the so-called Van der Corput Lemma, which can be found in [17].

Lemma 2.4. If

$$I(a, b) = \int_a^b e^{ih(t)} dt,$$

then we have

(i) If $|h'(t)| \geq \lambda > 0$ and h' is monotonic on (a, b) , then

$$|I(a, b)| \leq C\lambda^{-1};$$

(ii) If $k \geq 2$, $h \in C^k([a, b])$ and $|h^{(k)}(t)| \geq \lambda > 0$, then

$$|I(a, b)| \leq C\lambda^{-1/k},$$

where the constant C is independent of a, b .

Lemma 2.5. Let $p(t)$ and $q(t)$ be real valued smooth functions on the interval (a, b) and $k \in \mathbb{N}$. If $p(t)$ satisfies $|p^{(k)}(t)| \geq \lambda$ for all $t \in (a, b)$ and (1) $k = 1$, $q'(t)$ is monotonic on (a, b) , or (2) $k \geq 2$, then we have

$$\left| \int_a^b e^{ip(t)} q(t) dt \right| \leq C_k \lambda^{-\frac{1}{k}} \left[\|q\|_{L^\infty(a,b)} + \int_a^b |q'(t)| dt \right].$$

It is not difficult to see that Lemma 2.5 is a generalization of Lemma 2.4, which may also be found in [17].

3 Proofs of main results

Before proving the main results, we first estimate two oscillatory integrals.

Let

$$P_s(\xi) = \int_0^1 t^{k l_1 - \alpha_1 - 1} e^{-2\pi i (\xi t^k + t^{-\beta_1} s^{-\beta_2})} dt, \quad s \in (0, 1),$$

where $\xi \in \mathbb{R}$ and $k, l_1, \alpha_1, \beta_1, \beta_2 \geq 0$, then we have the following estimates for $P_s(\xi)$:

Lemma 3.1. *Existence of a positive constant C such that*

(i) *If $k = 0$ and $\beta_1 > \alpha_1$, then $|P_s(\xi)| \leq C s^{\beta_2}$;*

(ii) *If $k \geq 1$ and $\beta_1 > 2\alpha_1$, then $|P_s(\xi)| \leq C s^{\frac{\beta_2}{2}}$,*

where the constant C is independent of ξ .

Proof. Let $\phi_s(u) = \xi u^k + u^{-\beta_1} s^{-\beta_2}$ and $G(t) = \int_0^t e^{-2\pi i \phi_s(u)} du$.

(i) If $k = 0$, differentiating $\phi_s(u)$ by first order, we get

$$\phi'_s(u) = -\beta_1 u^{-\beta_1 - 1} s^{-\beta_2} < 0,$$

which implies that

$$|\phi'_s(u)| \geq C t^{-\beta_1 - 1} s^{-\beta_2}$$

for $u \in (0, t)$. Lemma 2.4 indicates that

$$|G(t)| \leq C t^{\beta_1 + 1} s^{\beta_2}.$$

Integrating by parts for $P_s(\xi)$, we have

$$P_s(\xi) = \int_0^1 t^{-\alpha_1 - 1} G'(t) dt = [t^{-\alpha_1 - 1} G(t)]_0^1 - (-\alpha_1 - 1) \int_0^1 t^{-\alpha_1 - 2} G(t) dt,$$

and

$$|P_s(\xi)| \leq C s^{\beta_2} \left[[t^{\beta_1 - \alpha_1}]_0^1 + |-\alpha_1 - 1| \int_0^1 t^{\beta_1 - \alpha_1 - 1} dt \right] \leq C s^{\beta_2}$$

for $\beta_1 > \alpha_1$.

(ii) If $k \geq 1$, we consider two cases $\xi \leq 0$ and $\xi > 0$. For $\xi \leq 0$, differentiating $\phi_s(u)$ by first order, then

$$\phi'_s(u) = k \xi u^{k-1} - \beta_1 u^{-\beta_1 - 1} s^{-\beta_2} \leq -\beta_1 u^{-\beta_1 - 1} s^{-\beta_2} < 0,$$

which implies that

$$|\phi'_s(u)| \geq C t^{-\beta_1 - 1} s^{-\beta_2}$$

for $u \in [0, t]$. Repeating the proof of (i), we get

$$|P_s(\xi)| \leq C s^{\beta_2} \leq C s^{\frac{\beta_2}{2}}.$$

For $\xi > 0$, differentiating $\phi_s(u)$ by second order, we now have

$$\phi''_s(u) = k(k-1)\xi u^{k-2} + \beta_1(\beta_1+1)u^{-\beta_1-2}s^{-\beta_2} \geq C t^{-\beta_1-2}s^{-\beta_2}$$

for $u \in [0, t]$. Lemma 2.4 gives that

$$|G(t)| \leq C t^{\frac{\beta_1+2}{2}} s^{\frac{\beta_2}{2}}.$$

Integrating by parts for $P_s(\xi)$, we get

$$P_s(\xi) = \int_0^1 t^{kl_1 - \alpha_1 - 1} G'(t) dt = [t^{kl_1 - \alpha_1 - 1} G(t)]_0^1 - (kl_1 - \alpha_1 - 1) \int_0^1 t^{kl_1 - \alpha_1 - 2} G(t) dt,$$

thus

$$|P_s(\xi)| \leq Cs^{\frac{\beta_2}{2}} \left[\left[t^{\frac{\beta_1 - 2\alpha_1}{2} + kl_1} \right]_0^1 + |kl_1 - \alpha_1 - 1| \int_0^1 t^{\frac{\beta_1 - 2\alpha_1}{2} + kl_1 - 1} dt \right] \leq Cs^{\frac{\beta_2}{2}}$$

for $\beta_1 > 2\alpha_1$. $\square$

Let function $\Phi(t) \in C_0^\infty(\mathbb{R})$ satisfying $\text{supp}\Phi \in [\frac{1}{2}, 2]$, $(\xi, \eta) \in (\mathbb{R}^n \times \mathbb{R}^m)$. For $v = 0, 1, 2, \dots$, define the oscillatory integrals as

$$I_{v,s}(\xi, \eta) = \int_{\frac{1}{2}}^2 \Phi(t) t^{p_1 l_1 + p_2 l_2 + \dots + p_n l_n - \alpha_1 - 1} e^{-2\pi i \phi(2^{-v}t, s)} dt, \quad s \in (0, 1),$$

where the $\phi(t, s) = \xi_1 t^{p_1} + \xi_2 t^{p_2} + \dots + \xi_n t^{p_n} + \eta_1 s^{q_1} + \dots + \eta_m s^{q_m} + t^{-\beta_1} s^{-\beta_2}$, the range of index $p_1, p_2, \dots, p_n, q_1, q_2, \dots, q_m$ and $\alpha_1, \beta_1, \beta_2$ are the same as that in Theorem 1.2. Then we have the following estimates for $I_{v,s}(\xi, \eta)$:

Lemma 3.2. *Existence of a positive constant C such that*

$$|I_{v,s}(\xi, \eta)| \leq C 2^{-\frac{v\beta_1}{n+1}} s^{\frac{\beta_2}{n+1}},$$

where the constant C is independent of (ξ, η) .

Proof. Differentiate $\phi(t, s)$ for t by m order, then for $m = 1, 2, 3, \dots, n+1$, we have

$$\frac{\partial^m \phi}{\partial t^m} = -\beta_1(-\beta_1 - 1) \cdots (-\beta_1 - m + 1) t^{-\beta_1 - m} s^{-\beta_2} + \sum_{j=1}^n p_j(p_j - 1) \cdots (p_j - m + 1) t^{p_j - m} \xi_j.$$

Let $\Delta_j^m = \prod_{i=1}^m (p_j - i - 1)$, $p_0 = -\beta_1$, $\xi_0 = s^{-\beta_2}$, it is easy to see that

$$t^{\beta_1 + m} s^{\beta_2} \frac{\partial^m \phi(t, s)}{\partial t^m} = \sum_{j=0}^n \Delta_j^m t^{p_j + \beta_1} s^{\beta_2} \xi_j.$$

Denote vector $\omega = (\omega_1, \omega_2, \dots, \omega_{n+1})$, $\mu = (\mu_0, \mu_1, \dots, \mu_n)$, where $\omega_m = t^{\beta_1 + m} s^{\beta_2} \frac{\partial^m \phi}{\partial t^m}$, $\mu_j = t^{p_j + \beta_1} s^{\beta_2} \xi_j$, then $\omega = D\mu$ and D is a $(n+1)$ -order Vandermonde matrix with

$$D = \begin{pmatrix} \Delta_0^1 & \Delta_1^1 & \cdots & \Delta_n^1 \\ \Delta_0^2 & \Delta_1^2 & \cdots & \Delta_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_0^{n+1} & \Delta_1^{n+1} & \cdots & \Delta_n^{n+1} \end{pmatrix}.$$

It is not difficult to check that $\det D = \prod_{j=0}^n p_j \prod_{0 \leq i < j \leq n} (p_j - p_i)$, for distinct positive real numbers $p_0, p_1, \dots, p_n$, we know $|\det D| \neq 0$. On the other hand, it is easy to see that the norm of u is larger than $\frac{1}{2}$ for every $t \in [\frac{1}{2}, 2]$. Thus, there is a positive C independent of s, t and $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^m$ such that

$$\left| t^{\beta_1 + m} s^{\beta_2} \frac{\partial^m \phi(t, s)}{\partial t^m} \right| \geq C$$

for at least one $m \in \{1, 2, \dots, n+1\}$. Therefore, for any $\frac{1}{2} \leq t \leq 2$, we have

$$\left| \frac{\partial^m \phi(2^{-\nu} t, s)}{\partial t^m} \right| \geq C 2^{\nu \beta_1} s^{-\beta_2}.$$

If the first order of ϕ satisfies the above inequality, we can split the interval $t \in [\frac{1}{2}, 2]$ into finite number of subintervals such that ϕ' is monotonic on these subintervals, and the number of subintervals depends only on n . So by Lemma 2.5 and $s \in [0, 1]$, we get

$$\begin{aligned} |I_{\nu, s}(\xi, \eta)| &\leq C 2^{-\frac{\nu \beta_1}{m} \frac{\beta_2}{m}} s^{\frac{\beta_2}{m}} \left[\|\Phi(t) t^{p_1 l_1 + p_2 l_2 + \dots + p_n l_n - \alpha_1 - 1}\|_{L^\infty(\frac{1}{2}, 2)} + \int_{\frac{1}{2}}^2 |(\Phi(t) t^{p_1 l_1 + p_2 l_2 + \dots + p_n l_n - \alpha_1 - 1})'| dt \right] \\ &\leq C 2^{-\frac{\nu \beta_1}{m} \frac{\beta_2}{m}} s^{\frac{\beta_2}{m}} \\ &\leq C 2^{-\frac{\nu \beta_1}{n+1} \frac{\beta_2}{n+1}} s^{\frac{\beta_2}{n+1}}. \end{aligned}$$

□

Now, we begin with the proof of Theorem 1.1.

Proof. We take the Fourier transform on T and write T as

$$\widehat{Tf}(\xi_1, \xi_2) = m(\xi_1, \xi_2) \hat{f}(\xi_1, \xi_2),$$

where

$$m(\xi_1, \xi_2) = \int_{Q^2} e^{-2\pi i [\xi_1 \cdot t^k + \xi_2 \cdot s^j + t^{-\beta_1} s^{-\beta_2}]} t^{-1-\alpha_1} s^{-1-\alpha_2} dt ds.$$

By Lemmas 2.1–2.3, we only need to prove $\|\partial^l m\|_\infty \leq C$ for every $|l| \leq 4$. Denote that $l = (l_1, l_2) \in \mathbb{N}^2$ and $|l| = l_1 + l_2$, then

$$\begin{aligned} \partial_{\xi_1}^{l_1} \partial_{\xi_2}^{l_2} m(\xi_1, \xi_2) &= \int_{Q^2} (-2\pi i t^k)^{l_1} (-2\pi i s^j)^{l_2} e^{-2\pi i (\xi_1 t^k + \xi_2 s^j + t^{-\beta_1} s^{-\beta_2})} t^{-1-\alpha_1} s^{-1-\alpha_2} dt ds \\ &= (-2\pi i)^{|l|} \int_0^1 \int_0^1 t^{kl_1 - \alpha_1 - 1} e^{-2\pi i (\xi_1 t^k + t^{-\beta_1} s^{-\beta_2})} dt e^{-2\pi i \xi_2 s^j} s^{jl_2 - \alpha_2 - 1} ds \\ &= (-2\pi i)^{|l|} \int_0^1 P_s(\xi_1) e^{-2\pi i \xi_2 s^j} s^{jl_2 - \alpha_2 - 1} ds. \end{aligned}$$

(i) $k = 0, j \geq 0$, by Lemma 3.1(i), we have

$$\|\partial_{\xi_1}^{l_1} \partial_{\xi_2}^{l_2} m\|_\infty \leq C \int_0^1 s^{\beta_2 - \alpha_2 + jl_2 - 1} ds \leq C$$

for $\beta_2 > \alpha_2$.

(ii) $k \geq 1, j \geq 0$, by Lemma 3.1(ii), we get

$$\|\partial_{\xi_1}^{l_1} \partial_{\xi_2}^{l_2} m\|_\infty \leq C \int_0^1 s^{\frac{\beta_2 - 2\alpha_2}{2} + jl_2 - 1} ds \leq C$$

for $\beta_2 > 2\alpha_2$. Combining the proofs of (i) and (ii), we completed the proof of Theorem 1.1. □

Next, we will give the proof of Theorem 1.2.

Proof. Using Fourier transform for operator $T_{\alpha,\beta}$, write $T_{\alpha,\beta}$ as

$$T_{\alpha,\beta} f^\wedge(\xi, \eta) = m(\xi, \eta) \widehat{f}(\xi, \eta),$$

where $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^m$ and

$$m(\xi, \eta) = \int_{Q^2} e^{-2\pi i [\gamma_1(t)\xi + \gamma_2(s)\eta + t^{-\beta_1}s^{-\beta_2}]} t^{-1-\alpha_1}s^{-1-\alpha_2} dt ds.$$

Note that $L = (L_n, L_m)$, and $L_n = (l_1, l_2, \dots, l_n) \in \mathbb{N}^n$, $L_m = (l'_1, l'_2, \dots, l'_m) \in \mathbb{N}^m$. By Lemmas 2.1–2.3, if we can prove $\partial_\xi^{L_n} \partial_\eta^{L_m} m \in L^\infty$ for any $|L| = |L_n| + |L_m| \leq 2\left(\left\lceil \frac{n+m}{2} \right\rceil + 1\right)$, then we can complete the proof of Theorem 1.2.

$$\begin{aligned} \partial_\xi^{L_n} \partial_\eta^{L_m} m(\xi, \eta) &= \partial_{\xi_1}^{l_1} \partial_{\xi_2}^{l_2} \cdots \partial_{\xi_n}^{l_n} \partial_{\eta_1}^{l'_1} \partial_{\eta_2}^{l'_2} \cdots \partial_{\eta_m}^{l'_m} m(\xi_1, \xi_2, \dots, \xi_n, \eta_1, \eta_2, \dots, \eta_m) \\ &= \int_{Q^2} (-2\pi i t^{p_1})^{l_1} (-2\pi i t^{p_2})^{l_2} \cdots (-2\pi i t^{p_n})^{l_n} (-2\pi i s^{q_1})^{l'_1} (-2\pi i s^{q_2})^{l'_2} \cdots \\ &\quad \times (-2\pi i s^{q_m})^{l'_m} e^{-2\pi i \phi(t,s)} t^{-1-\alpha_1} s^{-1-\alpha_2} dt ds \\ &= (-2\pi i)^{|L|} \int_0^1 \int_0^1 t^{p_1 l_1 + p_2 l_2 + \cdots + p_n l_n - \alpha_1 - 1} e^{-2\pi i \phi(t,s)} ds t^{q_1 l'_1 + q_2 l'_2 + \cdots + q_m l'_m - \alpha_2 - 1} ds, \end{aligned}$$

where $\phi(t, s) = \xi_1 t^{p_1} + \xi_2 t^{p_2} + \cdots + \xi_n t^{p_n} + \eta_1 s^{q_1} + \cdots + \eta_m s^{q_m} + t^{-\beta_1} s^{-\beta_2}$.

Let

$$I_s(\xi, \eta) = \int_0^1 t^{p_1 l_1 + p_2 l_2 + \cdots + p_n l_n - \alpha_1 - 1} e^{-2\pi i \phi(t,s)} dt,$$

choose a test function $\Phi(t) \in C_0^\infty(\mathbb{R})$ satisfying $\text{supp } \Phi \in \left[\frac{1}{2}, 2\right]$ and $\sum_{v=0}^\infty \Phi(2^v t) = 1$. Then $I_s(\xi, \eta) = \sum_{v=0}^\infty I_{v,s}(\xi, \eta)$, where

$$\begin{aligned} I_{v,s}(\xi, \eta) &= \int_0^1 \Phi(2^v t) t^{p_1 l_1 + p_2 l_2 + \cdots + p_n l_n - \alpha_1 - 1} e^{-2\pi i \phi(t,s)} dt \\ &= 2^{v\alpha_1} 2^{-v(p_1 l_1 + \cdots + p_n l_n)} \int_{\frac{1}{2}}^2 \Phi(t) t^{p_1 l_1 + p_2 l_2 + \cdots + p_n l_n - \alpha_1 - 1} e^{-2\pi i \phi(2^{-v} t, s)} dt. \end{aligned}$$

By Lemma 3.2, we have

$$|I_s(\xi, \eta)| \leq \sum_{v=0}^\infty |I_{v,s}(\xi, \eta)| \leq C \sum_{v=0}^\infty 2^{-v(\frac{\beta_1}{n+1} - \alpha_1 + p_1 l_1 + \cdots + p_n l_n)} s^{\frac{\beta_2}{n+1}} \leq C s^{\frac{\beta_2}{n+1}}$$

for $\beta_1 > (n+1)\alpha_1$. So we get

$$|\partial_\xi^{L_n} \partial_\eta^{L_m} m(\xi, \eta)| \leq C \int_0^1 s^{\frac{\beta_2}{n+1} - \alpha_2 + q_1 l'_1 + \cdots + q_m l'_m - 1} ds \leq C$$

for $\beta_2 > (n+1)\alpha_2$. To summarize, we completed the proof of Theorem 1.2. $\square$

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