Research Article

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A partial order on transformation semigroups with restricted range that preserve double direction equivalence

<https://doi.org/10.1515/math-2021-0109>

received June 9, 2021; accepted September 20, 2021

Abstract: Let $T(X)$ be the full transformation semigroup on a set X . For an equivalence E on X , let

$$T_E(X) = \{\alpha \in T(X) : \forall x, y \in X, (x, y) \in E \Leftrightarrow (x\alpha, y\alpha) \in E\}.$$

For each nonempty subset Y of X , we denote the restriction of E to Y by E_Y . Let $T_{E^*}(X, Y)$ be the intersection of the semigroup $T_E(X)$ with the semigroup of all transformations with restricted range Y under the condition that $|X/E| = |Y/E_Y|$. Equivalently, $T_{E^*}(X, Y) = \{\alpha \in T_E(X) : X\alpha \subseteq Y\}$, where $|X/E| = |Y/E_Y|$. Then $T_{E^*}(X, Y)$ is a subsemigroup of $T_E(X)$. In this paper, we characterize the natural partial order on $T_{E^*}(X, Y)$. Then we find the elements which are compatible and describe the maximal and minimal elements. We also prove that every element of $T_{E^*}(X, Y)$ lies between maximal and minimal elements. Finally, the existence of an upper cover and a lower cover is investigated.

Keywords: transformation semigroup, equivalence, partial order, compatibility, maximal and minimal

MSC 2020: 20M20

1 Introduction and preliminaries

Let $T(X)$ be the set of all functions from a set X into itself. Then $T(X)$ under the composition of functions is a semigroup which is called the *full transformation semigroup on X* . In 1975, Symons [1] studied the semigroup $T(X, Y)$ defined by

$$T(X, Y) = \{\alpha \in T(X) : X\alpha \subseteq Y\},$$

where Y is a nonempty subset of X . Note that if we let $\alpha \in T(X)$ and $Z \subseteq X$, the notation $Z\alpha$ means $\{z\alpha : z \in Z\}$. Clearly, $X\alpha = \text{ima } \alpha$. In [1], the author studied the automorphism of $T(X, Y)$ and the isomorphism between two semigroups $T(X_1, Y_1)$ and $T(X_2, Y_2)$. In 2008, Sanwong and Sommanee [2] found a necessary and sufficient condition for $T(X, Y)$ to be regular. Moreover, the largest regular subsemigroup was obtained. Later on, Sangkhanan and Sanwong [3] and, independently, Sun and Sun [4] endowed $T(X, Y)$ with $\leq$, the natural partial order, and determined when two elements of $T(X, Y)$ related under this order, then found out elements of $T(X, Y)$ which are compatible with $\leq$ on $T(X, Y)$. They also described its maximal and minimal elements and proved that every element in $T(X, Y)$ must lie between maximal and minimal elements.

Let E be an equivalence on X . In 2005, Pei [5] defined a subsemigroup $T_E(X)$ of $T(X)$ by

$$T_E(X) = \{\alpha \in T(X) : \forall x, y \in X, (x, y) \in E \Rightarrow (x\alpha, y\alpha) \in E\}.$$

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Then $T_E(X)$ is exactly $S(X)$, the semigroup of all continuous self-maps of the topological space X for which all E -classes form a basis. In [5], the author investigated regularity of elements and Green's relations for $T_E(X)$. In 2008, Sun et al. [6] described $T_E(X)$ with the natural partial order and gave a characterization for two elements of $T_E(X)$ related under this order. They found out elements of $T_E(X)$ which are compatible with $\leq$ on $T_E(X)$. In addition, the maximal, minimal and covering elements were described.

Recently, Sangkhanan and Sanwong [7] defined a subsemigroup $T_E(X, Y)$ of $T_E(X)$ where $Y \subseteq X$ by

$$T_E(X, Y) = \{\alpha \in T(X, Y) : \forall (x, y) \in E, (x\alpha, y\alpha) \in E\}.$$

Then $T_E(X, Y)$ is the semigroup of all continuous self-maps of the topological space X for which all E -classes form a basis carrying X into a subspace Y . In [7], they gave a necessary and sufficient condition for $T_E(X, Y)$ to be regular and characterized Green's relations on $T_E(X, Y)$.

In 2010, Deng et al. [8] introduced a subsemigroup $T_{E^*}(X)$ of $T(X)$ by

$$T_{E^*}(X) = \{\alpha \in T(X) : \forall x, y \in X, (x, y) \in E \Leftrightarrow (x\alpha, y\alpha) \in E\}.$$

Then $T_{E^*}(X)$ is a semigroup of continuous self-maps of the topological space X for which all E -classes form a basis. The authors studied regularity of elements and Green's relations for $T_{E^*}(X)$. In 2013, Sun and Sun [9] endowed $T_{E^*}(X)$ with the natural partial order. They determined when two elements are related under this order and found the elements which are compatible. Moreover, the maximal and minimal elements were described. Finally, they studied the existence of the greatest lower bound of two elements.

Now, we define a subset $T_{E^*}(X, Y)$ of $T_{E^*}(X)$ by

$$T_{E^*}(X, Y) = \{\alpha \in T_{E^*}(X) : X\alpha \subseteq Y\} = T_{E^*}(X) \cap T(X, Y)$$

under the condition that $|X/E| = |Y/E_Y|$ where $E_Y = E \cap (Y \times Y)$. In 2019, Chaichompoo and Sangkhanan [10] showed that $T_{E^*}(X, Y)$ is a subsemigroup of $T_{E^*}(X)$ under composition of functions and studied the regularity of $T_{E^*}(X, Y)$. Furthermore, they characterized Green's relations on this semigroup. We see that if $X = Y$, then $T_{E^*}(X, Y) = T_{E^*}(X)$. Hence, $T_{E^*}(X)$ is a special case of $T_{E^*}(X, Y)$. Furthermore, if E is the universal relation, $E = X \times X$, then $T_{E^*}(X, Y)$ becomes $T(X, Y)$. Moreover, it is not difficult to check that $T_{E^*}(X, Y)$ is a semigroup of continuous self-maps of the topological space X for which all E classes form a basis carrying X into a subspace Y , and is referred to as a semigroup of continuous functions (see [11] for details).

In this paper, we aim to generalize the results of [3, 4, 9]. Actually, we study the natural partial order on $T_{E^*}(X, Y)$ which extends the results on $T_{E^*}(X)$ and $T(X, Y)$ and improve some results in these semigroups. For example, in Section 4, we characterize the left and right compatible elements instead of the *strictly* compatibility which was studied in [9]. Moreover, in Section 5, we show that every element in $T_{E^*}(X, Y)$ lies between maximal and minimal elements which have never been studied before in $T_{E^*}(X)$.

We first state some notations and results that will be used later. For each $\alpha \in T_{E^*}(X, Y)$ and $A \subseteq X$, the restriction of α to A is denoted by $\alpha|_A$. We adopt the notation introduced in [12], namely, if $\alpha \in T_{E^*}(X, Y)$, then we write

$$\alpha = \begin{pmatrix} X_i \\ a_i \end{pmatrix}$$

and the subscript i belongs to some unmentioned index set I , the abbreviation $\{a_i\}$ denotes $\{a_i : i \in I\}$ and that $X\alpha = \{a_i\}$ and $a_i\alpha^{-1} = X_i$.

Let X/E be the quotient set, where E is an equivalence on X , the set of all equivalence classes on X . For each $\alpha \in T_{E^*}(X, Y)$, the symbol $\pi(\alpha)$ will denote the decomposition of X induced by the map α , namely,

$$\pi(\alpha) = \{x\alpha^{-1} : x \in X\alpha\}.$$

Then $\pi(\alpha) = X/\ker(\alpha)$ where $\ker(\alpha) = \{(x, y) \in X \times X : x\alpha = y\alpha\}$. For a subset A of X , we put

$$\pi_A(\alpha) = \{M \in \pi(\alpha) : M \cap A \neq \emptyset\}.$$

We also define $\tilde{\pi}_A(\alpha) = \pi_{A \cap Y}(\alpha)$. It is clear that $\tilde{\pi}_A(\alpha)$ is an appropriate extension of $\pi_A(\alpha)$ in the sense that if $Y = X$, then $\tilde{\pi}_A(\alpha) = \pi_A(\alpha)$. Obviously, $\tilde{\pi}_A(\alpha) \subseteq \pi_A(\alpha)$.

We define the restriction of the equivalence E on a subset M of X by

$$E_M = \{(x, y) \in M \times M : (x, y) \in E\} = E \cap (M \times M).$$

It is easy to see that E_M is also an equivalence on M and

$$M/E_M = \{A \cap M : A \in X/E, A \cap M \neq \emptyset\}.$$

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two collections of subsets of X . If for each $A \in \mathfrak{A}$, there exists $B \in \mathfrak{B}$ such that $A \subseteq B$, then $\mathfrak{A}$ is said to *refine* $\mathfrak{B}$.

Note that, by the definition of $T_{E^*}(X, Y)$ studied in [10], we will assume that Y is a subset of X such that $|Y/E_Y| = |X/E|$ in the remaining of this paper.

Now, we deal with the natural partial order or Mitsch's order [13] on any semigroup S defined, for $a, b \in S$, by

$$a \leq b \text{ if and only if } a = xb = by, \quad xa = a \text{ for some } x, y \in S^1.$$

Equivalently,

$$a \leq b \text{ if and only if } a = wb = bz, \quad az = a \text{ for some } w, z \in S^1, \quad (1)$$

where the notation S^1 denotes a monoid obtained from S by adjoining an identity 1 if necessary ($S^1 = S$ for a monoid S). In this paper, we use (1) to define the partial order on the semigroup $T_{E^*}(X, Y)$, that is, for each $\alpha, \beta \in T_{E^*}(X, Y)$,

$$\alpha \leq \beta \text{ if and only if } \alpha = \gamma\beta = \beta\mu, \quad \alpha = \alpha\mu \text{ for some } \gamma, \mu \in T_{E^*}(X, Y)^1.$$

Note that if $Y \subsetneq X$, then $T_{E^*}(X, Y)$ has no identity elements. Hence, in this case, $T_{E^*}(X, Y)^1 \neq T_{E^*}(X, Y)$. It is worth studying the natural partial order on $T_{E^*}(X, Y)$ in its own right since, in general, $\leq$ on $T_{E^*}(X, Y)$ does not coincide with the restriction of $\leq$ on $T_{E^*}(X)$. In other words, for each $\alpha, \beta \in T_{E^*}(X, Y)$, if $\alpha \leq \beta$ on $T_{E^*}(X)$, then it is not necessary that $\alpha \leq \beta$ on $T_{E^*}(X, Y)$. For example, let $X = \{1, 2, 3, 4\}$, $Y = \{1, 2, 3\}$ and $X/E = \{\{1, 2\}, \{3, 4\}\}$. Define

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 2 & 1 \end{pmatrix}.$$

If we let

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 4 & 4 \end{pmatrix} \quad \text{and} \quad \mu = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 3 \end{pmatrix},$$

then $\alpha = \gamma\beta = \beta\mu$, $\alpha = \alpha\mu$, which implies that $\alpha \leq \beta$ in $T_{E^*}(X)$ but we cannot find $\gamma' \in T_{E^*}(X, Y)^1$ such that $\alpha = \gamma'\beta$. Hence, $\alpha \not\leq \beta$ in $T_{E^*}(X, Y)$.

Let us refer to the following corollary which will prove useful.

Corollary 1.1. [9, Corollary 2.2] *Let $\alpha, \beta \in T_{E^*}(X)$ and $\alpha \leq \beta$. Then the following statements hold.*

1. *If $X\alpha = X\beta$, then $\alpha = \beta$.*
2. *For each $P \in \pi(\alpha)$, there exists some $Q \in \pi(\beta)$ such that $Q \subseteq P$ and $P\alpha = Q\beta$.*

2 Characterization

Now, we start this section with the characterization of $\leq$ on $T_{E^*}(X, Y)$, which extends Theorem 2.1 of [9].

Theorem 2.1. *Let $\alpha, \beta \in T_{E^*}(X, Y)$. Then $\alpha \leq \beta$ if and only if either $\alpha = \beta$ or the following statements hold.*

1. *$\pi(\beta)$ refines $\pi(\alpha)$.*
2. *For each $x \in X$, if $x\beta \in X\alpha$, then $x\alpha = x\beta$.*
3. *For each E -class A , $A\alpha \subseteq (A \cap Y)\beta$.*

Proof. ($\Rightarrow$) Suppose that $\alpha \leq \beta$. Then $\alpha = \gamma\beta = \beta\mu$, $\alpha = \alpha\mu$ for some $\gamma, \mu \in T_E^*(X, Y)^1$. If $\gamma = 1$ or $\mu = 1$, then $\alpha = \beta$. Now, we assume that $\gamma, \mu \in T_E^*(X, Y)$. Then (1) and (2) follow from Theorem 2.1 of [9]. To prove (3), let $A \in X/E$. Then $A\alpha = A\gamma\beta \subseteq (B \cap Y)\beta$ for some class B . We claim that $A = B$. Indeed, let $a \in A$. Then $a\alpha = b\beta$ for some $b \in B \cap Y$. Hence, $a\alpha = a\alpha\mu = b\beta\mu = b\alpha$ and so $(a\alpha, b\alpha) \in E$ implies $(a, b) \in E$. Thus, $b \in A \cap B \neq \emptyset$ and so $A = B$.

($\Leftarrow$) The proof is an appropriate modification of the proof of Theorem 2.1 of [9]. In fact, assume that (1), (2) and (3) hold. Note that for each $A \in X/E$, $A\alpha$ is nonempty. Hence, by (3), $(A \cap Y)\beta \neq \emptyset$ and so $A \cap Y$ is nonempty for each $A \in X/E$. Define a function μ on each class A as follows. If $A \cap X\beta = \emptyset$, fix $y_0 \in A \cap Y$ and define $x\mu = y_0$ for each $x \in A$. If $A \cap X\beta \neq \emptyset$, we first define μ on $A \cap X\beta$. For each $z \in A \cap X\beta$, we have $z = x\beta$ for some $x \in X$, so define $z\mu = x\alpha$. To define μ on $A \setminus X\beta$, we fix $z_0 \in A \cap X\beta$ and define $z\mu = z_0\mu$ for each $z \in A \setminus X\beta$. By using the same method as in the proof of Theorem 2.1 in [9], we can see that $\mu \in T_E^*(X, Y)$ is well-defined and $\alpha = \alpha\mu$, $\alpha = \beta\mu$.

Finally, we define a function γ . For each $A \in X/E$, $A\alpha \subseteq (A \cap Y)\beta$. Thus, for each $x \in A$, we choose $y \in A \cap Y$ such that $x\alpha = y\beta$ and define $x\gamma = y$. It is clear that $X\gamma \subseteq Y$. Similarly, we can show that $\gamma \in T_E^*(X, Y)$ and $\alpha = \gamma\beta$. Therefore, $\alpha \leq \beta$. $\square$

We note that if $X = Y$, then $A \cap Y = A$ for each E -class A . We can see that Theorem 2.1 becomes Theorem 2.1 of [9]. Moreover, if $E = X \times X$, then $X/E = \{X\}$. Hence, this theorem also becomes Theorem 2.1 of [4].

Proposition 2.2. Let Y be a subset of X such that $Y \cap A = \emptyset$ for some class $A \in X/E$ and let $\alpha, \beta \in T_E^*(X, Y)$. Then $\alpha \leq \beta$ if and only if $\alpha = \beta$.

Proof. Let $\alpha \leq \beta$ and suppose that $\alpha < \beta$. Then for each $A \in X/E$, $A\alpha \subseteq (A \cap Y)\beta$ by Theorem 2.1 (3). Hence, $(A \cap Y)\beta$ is nonempty and so $A \cap Y$ is also nonempty for all classes A . It leads to a contradiction. Therefore, $\alpha = \beta$. $\square$

By the above proposition, we can see that if $Y \cap A = \emptyset$ for some class $A \in X/E$, then the natural partial order $\leq$ becomes the equality relation. From now on, we assume that $Y \cap A$ is nonempty for all E -classes A , equivalently, $Y/E_Y = \{A \cap Y : A \in X/E\}$.

3 Compatibility

Recall that an element $\gamma \in T_E^*(X, Y)$ is said to be *strictly left compatible* if $\gamma\alpha < \gamma\beta$ whenever $\alpha < \beta$. *Strictly right compatibility* is defined dually. We note that, in Section 2 of [9], the authors studied the strictly compatibility on $T_E^*(X)$ but, in this paper, we remove the term “strictly,” that is, an element $\gamma \in T_E^*(X, Y)$ is said to be *left compatible* if $\gamma\alpha \leq \gamma\beta$ whenever $\alpha \leq \beta$. *Right compatibility* is defined dually.

Theorem 3.1. $\gamma \in T_E^*(X, Y)$ is left compatible if and only if for each $A \in X/E$, $(A \cap Y)\gamma \in Y/E_Y$.

Proof. ($\Rightarrow$) We prove by contrapositive. Assume that there is $A \in X/E$ such that $(A \cap Y)\gamma \not\subseteq B \cap Y \in Y/E_Y$ for some E -class B . Then there exists $y \in (B \cap Y) \setminus (A \cap Y)\gamma$. Since $Y/E_Y = \{A \cap Y : A \in X/E\}$, we can write $(X/E) \setminus \{B\} = \{B_i : i \in I\}$ and $(Y/E_Y) \setminus \{B \cap Y\} = \{B_i \cap Y : i \in I\}$. Now, we choose $z \in (A \cap Y)\gamma$ and define $\alpha, \beta \in T_E^*(X, Y)$ by

$$\alpha = \begin{pmatrix} B & B_i \\ y & b_i \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} B \setminus \{y\} & \{y\} & B_i \\ z & y & b_i \end{pmatrix},$$

where $b_i \in B_i \cap Y$ for all $i \in I$. It is routine to verify that $\alpha < \beta$ by using Theorem 2.1. We see that $z \notin A\gamma\alpha$ but $z \in A\gamma\beta$, which implies that $\gamma\alpha \neq \gamma\beta$. Moreover, we obtain $A\gamma\alpha \subseteq (B \cap Y)\alpha = \{y\} \not\subseteq \{z\} = (A \cap Y)\gamma\beta$. Hence, $\gamma\alpha \not\leq \gamma\beta$. Therefore, γ is not left compatible.

($\Leftarrow$) Suppose that $(A \cap Y)\gamma \in Y/E_Y$ for each $A \in X/E$. Let $\alpha, \beta \in T_{E^*}(X, Y)$ be such that $\alpha \leq \beta$ with $\alpha \neq \beta$. We show that $\gamma\alpha \leq \gamma\beta$ by using Theorem 2.1. Assume that $\gamma\alpha \neq \gamma\beta$.

- (1) Let $U = \gamma(\gamma\beta)^{-1} \in \pi(\gamma\beta)$. Then $U\gamma\beta = \{y\}$ and so $U\gamma \subseteq \gamma\beta^{-1}$. Hence, $\gamma\beta^{-1} \subseteq z\alpha^{-1} \in \pi(\alpha)$ since $\pi(\beta)$ refines $\pi(\alpha)$. We obtain $U\gamma \subseteq z\alpha^{-1}$, which implies that $U\gamma\alpha = \{z\}$. Therefore, $U \subseteq z(\gamma\alpha)^{-1} \in \pi(\gamma\alpha)$ and so $\pi(\gamma\beta)$ refines $\pi(\gamma\alpha)$.
- (2) Let $x\gamma\beta \in X\gamma\alpha$. Then $x\gamma\beta \in X\alpha$, which implies that $x\gamma\beta = x\gamma\alpha$ since $\alpha \leq \beta$.
- (3) Let $A \in X/E$. Then $(A \cap Y)\gamma = B \cap Y$ for some $B \in X/E$ by the assumption and so $A\gamma = B \cap Y$. We see that $B\alpha \subseteq (B \cap Y)\beta$ since $\alpha \leq \beta$ and hence $A\gamma\alpha = (B \cap Y)\alpha \subseteq B\alpha \subseteq (B \cap Y)\beta = (A \cap Y)\gamma\beta$. Therefore, $\gamma\alpha \leq \gamma\beta$. $\square$

Theorem 3.2. Let $\gamma \in T_{E^*}(X, Y)$. Then γ is right compatible if and only if for each $A \in X/E$, $\gamma|_{A \cap Y}$ is injective or $|(A \cap Y)\gamma| = 1$.

Proof. Assume that γ is right compatible and suppose to the contrary that there is $A \in X/E$ such that $\gamma|_{A \cap Y}$ is not injective and $|(A \cap Y)\gamma| > 1$. Then there exist $s, t \in A \cap Y$ with $s \neq t$ and $sy = ty = w$ for some $w \in Y$. Moreover, there is $u \in A \cap Y$ such that $s \neq u \neq t$ and $uy = v$ for some $v \neq w$ in Y . Since $Y/E_Y = \{C \cap Y : C \in X/E\}$, we can write $(X/E) \setminus \{A\} = \{A_i : i \in I\}$ and $(Y/E_Y) \setminus \{A \cap Y\} = \{A_i \cap Y : i \in I\}$. Define $\alpha, \beta \in T_{E^*}(X, Y)$ by

$$\alpha = \begin{pmatrix} \{s\} & A \setminus \{s\} & A_i \\ s & u & a_i \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} \{s\} & \{t\} & A \setminus \{s, t\} & A_i \\ s & t & u & a_i \end{pmatrix},$$

where $a_i \in A_i \cap Y$ for each $i \in I$. It is straightforward to show that $\alpha < \beta$. We can see that $s\alpha y = sy = w = ty = t\beta y$, which implies that $t\beta y \in X\alpha y$ but $t\alpha y = uy = v \neq w = ty = t\beta y$. Thus, $\alpha y \not\leq \beta y$, which contradicts to the right compatibility of γ . Therefore, $\gamma|_{A \cap Y}$ is injective or $|(A \cap Y)\gamma| = 1$.

Conversely, assume that for each $A \in X/E$, $\gamma|_{A \cap Y}$ is injective or $|(A \cap Y)\gamma| = 1$. Let $\alpha, \beta \in T_{E^*}(X, Y)$ be such that $\alpha \leq \beta$ with $\alpha \neq \beta$. We show that $\alpha\gamma \leq \beta\gamma$ by using Theorem 2.1.

(1) Let $U = \gamma(\beta\gamma)^{-1} \in \pi(\beta\gamma)$. We can see that $U\beta \subseteq A \cap Y$ for some class $A \in X/E$. Then $(U\beta)\gamma|_{A \cap Y} = U\beta\gamma = \{y\}$ and so $U\beta \subseteq \gamma(\gamma|_{A \cap Y})^{-1}$. We consider the following two cases.

Case 1: $\gamma|_{A \cap Y}$ is injective. Then $\gamma(\gamma|_{A \cap Y})^{-1} = \{x\}$ for some $x \in A \cap Y$ and hence $U\beta = \{x\}$. Thus, $U \subseteq x\beta^{-1} \in \pi(\beta)$ and then there is $z\alpha^{-1} \in \pi(\alpha)$ such that $U \subseteq x\beta^{-1} \subseteq z\alpha^{-1}$ since $\pi(\beta)$ refines $\pi(\alpha)$. We obtain $U\alpha = \{z\}$, which implies that $U\alpha\gamma = \{z\gamma\}$ and so $U \subseteq (z\gamma)(\alpha\gamma)^{-1} \in \pi(\alpha\gamma)$.

Case 2: $|(A \cap Y)\gamma| = 1$. Then $(A \cap Y)\gamma = \{y\}$. We claim that $U \in X/E$. Clearly, $U \subseteq (A \cap Y)\beta^{-1} = B$ for some class $B \in X/E$ since $\beta \in T_{E^*}(X, Y)$. Let $b \in B$. Then $b\beta \in A \cap Y$ from which it follows that $b\beta\gamma \in (A \cap Y)\gamma = \{y\}$. Thus, $b \in \gamma(\beta\gamma)^{-1} = U$ and so $B \subseteq U$. We conclude that $U = B \in X/E$. By Theorem 2.1 (3), $U\alpha \subseteq (U \cap Y)\beta \subseteq U\beta$, which implies that $U\alpha\gamma \subseteq U\beta\gamma = \{y\}$. Hence, $U \subseteq \gamma(\alpha\gamma)^{-1} \in \pi(\alpha\gamma)$.

Therefore, $\pi(\beta\gamma)$ refines $\pi(\alpha\gamma)$.

(2) Let $x\beta\gamma \in X\alpha\gamma$. Then $x\beta\gamma = y\alpha\gamma$ for some $y \in X$. Hence, $(x\beta\gamma, y\alpha\gamma) \in E$, which implies that $(x\beta, y\alpha) \in E$. We obtain $x\beta, y\alpha \in A \cap Y$ for some class $A \in X/E$. If $\gamma|_{A \cap Y}$ is injective, then $x\beta = y\alpha \in X\alpha$, which implies that $x\beta = x\alpha$. Thus, $x\beta\gamma = x\alpha\gamma$. Now, suppose that $|(A \cap Y)\gamma| = 1$. By Theorem 2.1 (3), we obtain $y\alpha = z\beta$ for some $z \in Y$. Moreover, we have $z\alpha = z\beta$ since $z\beta = y\alpha \in X\alpha$ and $\alpha \leq \beta$. Thus, $y\alpha = z\alpha$, which implies that $(y, z) \in E$. In addition, we obtain $z\beta = y\alpha \in A \cap Y$ from which it follows that $(x\beta, z\beta) \in E$ since $x\beta \in A \cap Y$. Hence, $(x, z) \in E$ and so $(x, y) \in E$. We have $(x\alpha, y\alpha) \in E$ and $x\alpha \in A \cap Y$ since $y\alpha \in A \cap Y$. Therefore, $x\alpha\gamma = y\alpha\gamma$ since $|(A \cap Y)\gamma| = 1$.

(3) Let $A \in X/E$. By Theorem 2.1 (3), $A\alpha \subseteq (A \cap Y)\beta$ and so $A\alpha\gamma \subseteq (A \cap Y)\beta\gamma$. $\square$

By the above two theorems, we obtain the following two corollaries which appear in [3,4,9].

Corollary 3.3. Let $\gamma \in T_{E^*}(X)$. Then the following statements hold.

1. γ is left compatible if and only if for each $A \in X/E$, $A\gamma \in X/E$.
2. γ is right compatible if and only if for each $A \in X/E$, $\gamma|_A$ is injective or $|A\gamma| = 1$.

Corollary 3.4. Let $\gamma \in T(X, Y)$. Then the following statements hold.

1. γ is left compatible if and only if $Y\gamma = Y$.
2. γ is right compatible if and only if $\gamma|_Y$ is injective or $|Y\gamma| = 1$.

4 Maximal and minimal elements

In this section, we characterize maximal and minimal elements and show that every element of $T_{E^*}(X, Y)$ must lie between maximal and minimal elements. Furthermore, we obtain the existence of an upper cover and a lower cover.

Sanwong and Sommanee [2] defined a subset F of $T(X, Y)$ by

$$F = \{\alpha \in T(X, Y) : X\alpha \subseteq Y\alpha\}$$

and proved that F is the largest regular subsemigroup of $T(X, Y)$. Moreover, in [10], the authors defined the set $F_{E^*} = F \cap T_{E^*}(X, Y)$, equivalently,

$$F_{E^*} = \{\alpha \in T_{E^*}(X, Y) : X\alpha \subseteq Y\alpha\} = \{\alpha \in T_{E^*}(X, Y) : X\alpha = Y\alpha\}.$$

It is clear that F_{E^*} is an appropriate extension of F in the sense that if $E = X \times X$, then $F_{E^*} = F$. In addition, if $X = Y$, then $F_{E^*} = T_{E^*}(X)$. It is straightforward to verify that

$$F_{E^*} = \{\alpha \in T_{E^*}(X, Y) : A\alpha = (A \cap Y)\alpha \text{ for all } A \in X/E\}.$$

In [10], they showed that F_{E^*} is a right ideal of $T_{E^*}(X, Y)$ and so it is a subsemigroup of $T_{E^*}(X, Y)$. In this section, F_{E^*} plays an essential role in the characterization of maximal and minimal elements, as follows.

Lemma 4.1. Let $\alpha \in T_{E^*}(X, Y)$. If $\alpha \notin F_{E^*}$, then α is maximal.

Proof. Let $\alpha \notin F_{E^*}$ and $\beta \in T_{E^*}(X, Y)$ be such that $\alpha \leq \beta$. Assume that $\alpha \neq \beta$. Let $x\alpha \in X\alpha$. Then $x \in A$ for some $A \in X/E$ from which it follows that $x\alpha \in A\alpha \subseteq (A \cap Y)\beta$ by Theorem 2.1 (3). Hence, $x\alpha = a\beta$ for some $a \in A \cap Y$ implies $a\beta = x\alpha \in X\alpha$. By Theorem 2.1 (2), $x\alpha = a\beta = aa \in (A \cap Y)\alpha \subseteq Y\alpha$. We obtain $X\alpha \subseteq Y\alpha$, which contradicts to $\alpha \notin F_{E^*}$. Therefore, $\alpha = \beta$. $\square$

Theorem 4.2. Let $\alpha \in T_{E^*}(X, Y)$. Then α is maximal if and only if one of the following statements hold.

1. $\alpha \notin F_{E^*}$.
2. For each E -class A , $A\alpha \in Y/E_Y$ or $\alpha|_A$ is injective.

Proof. ($\Rightarrow$) We prove by contrapositive. Let $\alpha \in F_{E^*}$. Assume that there is a class $A \in X/E$ such that $A\alpha \not\subseteq B$ for some $B \in Y/E_Y$ and $\alpha|_A$ is not injective. We note that $A\alpha = (A \cap Y)\alpha$ since $\alpha \in F_{E^*}$. Then we can find distinct two elements $p \in A$ and $q \in A \cap Y$ such that $p\alpha = q\alpha$. Now, we choose $b \in B \setminus A\alpha$ and define $\beta : X \rightarrow Y$ by

$$x\beta = \begin{cases} b & \text{if } x = p, \\ x\alpha & \text{otherwise.} \end{cases}$$

It is easy to verify that $\beta \in T_{E^*}(X, Y)$ and $\alpha \neq \beta$. Now, we prove that $\alpha \leq \beta$.

- (1) Let $(x\beta)\beta^{-1} \in \pi(\beta)$. We assert that $(x\beta)\beta^{-1} \subseteq (x\alpha)\alpha^{-1} \in \pi(\alpha)$. Indeed, let $a \in (x\beta)\beta^{-1}$. Then $a\beta = x\beta$. If $x = p$, then $a\beta = p\beta = b$, which implies that $a = p$. Hence, $a\alpha = p\alpha = x\alpha$ and so $a \in (x\alpha)\alpha^{-1}$. If $x \neq p$, then $a\beta = x\beta = x\alpha$, which implies that $a \neq p$. Thus, $a\alpha = a\beta = x\alpha$ and so $a \in (x\alpha)\alpha^{-1}$. Therefore, $\pi(\beta)$ refines $\pi(\alpha)$.
- (2) Let $x\beta \in X\alpha$. Obviously, $x \neq p$ and so $x\alpha = x\beta$.

(3) It is clear that $C\alpha = (C \cap Y)\alpha = (C \cap Y)\beta$ if $C \in X/E$ with $C \neq A$. To show that $A\alpha \subseteq (A \cap Y)\beta$, let $a\alpha \in A\alpha$. Since $a\alpha \in A\alpha = (A \cap Y)\alpha$, there is $y \in A \cap Y$ such that $a\alpha = y\alpha$. If $y = p$, then $a\alpha = p\alpha = q\alpha = q\beta \in (A \cap Y)\beta$. If $y \neq p$, then $a\alpha = y\alpha = y\beta \in (A \cap Y)\beta$.

Thus, α is not maximal.

($\Leftarrow$) If $\alpha \notin F_{E^*}$, then α is maximal by Lemma 4.1. Let $\alpha \in F_{E^*}$. Suppose that for each E -class A , $A\alpha \in Y/E_Y$ or $\alpha|_A$ is injective. Let $\beta \in T_{E^*}(X, Y)$ be such that $\alpha \leq \beta$. Assume that $\alpha \neq \beta$. Then there is $x \in X$ such that $x\alpha \neq x\beta$. Let $x \in A$ for some $A \in X/E$. If $A\alpha = B \in Y/E_Y$, then $B = A\alpha \subseteq (A \cap Y)\beta \subseteq A\beta$, which implies that $x\beta \in A\beta = B = A\alpha \subseteq X\alpha$. By Theorem 2.1 (2), we obtain $x\alpha = x\beta$, which is a contradiction. Hence, $\alpha|_A$ is injective. Moreover, since $A\alpha \subseteq (A \cap Y)\beta$, we have $x\alpha = y\beta$ for some $y \in A \cap Y$. Again by Theorem 2.1 (2), we obtain $x\alpha = y\beta = y\alpha$, which implies $x = y$ since $\alpha|_A$ is injective. Thus, $x\alpha = x\beta$, which is also a contradiction. Therefore, $\alpha = \beta$. $\square$

Now, we show that every element in $T_{E^*}(X, Y)$ lies below a maximal element.

Theorem 4.3. *Let $\alpha \in T_{E^*}(X, Y)$. Then there exists a maximal element $\beta \in T_{E^*}(X, Y)$ such that $\alpha \leq \beta$.*

Proof. It is clear that if $\alpha \notin F_{E^*}$, then α is maximal and $\alpha \leq \alpha$. Now, we suppose that $\alpha \in F_{E^*}$ and α is not maximal. Let $\{A_i : i \in I\}$ be the class of all E -classes such that $A_i\alpha \subsetneq B_i$ for some $B_i \in Y/E_Y$ and $\alpha|_{A_i}$ is not injective. We note by Theorem 4.2 that such class is nonempty since α is not maximal. For each $i \in I$, we define

$$C_i(\alpha) = \{x\alpha^{-1} : x \in B_i \text{ and } |x\alpha^{-1}| > 1\}.$$

It is clear that $C_i(\alpha)$ is nonempty since $\alpha|_{A_i}$ is not injective and $C_i(\alpha) \subseteq \pi(\alpha)$ for each $i \in I$. We claim that for each $C \in C_i(\alpha)$, $C \cap Y \neq \emptyset$. Indeed, let $C \in C_i(\alpha)$. Then $C = x\alpha^{-1}$ for some $x \in B_i$. Hence, $x \in X\alpha = Y\alpha$, which implies that $x = y\alpha$ for some $y \in Y$ and so $y \in x\alpha^{-1} \cap Y = C \cap Y \neq \emptyset$.

For each $i \in I$, we define a function $\beta_i : A_i \rightarrow B_i$ as follows. For each $C \in C_i(\alpha)$, choose $d_C \in C \cap Y$. We can see that $C \setminus \{d_C\}$ is nonempty since $|C| > 1$. Consider the following two cases.

Case 1:

$$\left| \bigcup_{C \in C_i(\alpha)} (C \setminus \{d_C\}) \right| \geq |B_i \setminus A_i\alpha|.$$

Then there is an injection

$$\gamma_i : B_i \setminus A_i\alpha \rightarrow \bigcup_{C \in C_i(\alpha)} (C \setminus \{d_C\}).$$

Let $x\gamma_i^{-1} = \{g_x\}$ for each $x \in \text{im}\gamma_i$. Define $\beta_i : A_i \rightarrow B_i$ by

$$x\beta_i = \begin{cases} g_x & \text{if } x \in \text{im}\gamma_i, \\ x\alpha & \text{otherwise.} \end{cases}$$

Clearly, β_i is surjective.

Case 2:

$$\left| \bigcup_{C \in C_i(\alpha)} (C \setminus \{d_C\}) \right| < |B_i \setminus A_i\alpha|.$$

Then there is an injection

$$\mu_i : \bigcup_{C \in C_i(\alpha)} (C \setminus \{d_C\}) \rightarrow B_i \setminus A_i\alpha.$$

Define $\beta_i : A_i \rightarrow B_i$ by

$$x\beta_i = \begin{cases} x\mu_i & \text{if } x \in \text{dom}\mu_i, \\ x\alpha & \text{otherwise.} \end{cases}$$

Clearly, β_i is injective.

Finally, we define a function $\beta : X \rightarrow Y$ by

$$x\beta = \begin{cases} x\beta_i & \text{if } x \in A_i, \\ x\alpha & \text{otherwise.} \end{cases}$$

It is easy to see that $\beta \in T_E(X, Y)$ and $\alpha \neq \beta$. Moreover, β satisfies Theorem 4.2, which implies that β is maximal. Now, we show that $\alpha \leq \beta$ by using Theorem 2.1.

(1) Let $(x\beta)\beta^{-1} \in \pi(\beta)$. If $x \notin A_i$ for all $i \in I$, then $x\beta = x\alpha$. We claim that $(x\beta)\beta^{-1} \subseteq (x\alpha)\alpha^{-1}$. Indeed, let $z \in (x\beta)\beta^{-1}$. Then $z\beta = x\beta$, which implies that $(x\beta, z\beta) \in E$. Thus, $(x, z) \in E$ and so $z \notin A_i$ for all $i \in I$. Hence, $z\beta = z\alpha$ from which it follows that $z\alpha = x\alpha$ and then $z \in (x\alpha)\alpha^{-1}$. On the other hand, if $x \in A_i$ for some $i \in I$, then $x\beta = x\beta_i$. We consider the following two cases.

If A_i satisfies case 1 as above, then we first suppose that $x \in \text{im}\gamma_i$. Then $x\beta = g_x$. We claim that $(x\beta)\beta^{-1} = \{x\}$. Clearly, $x \in (x\beta)\beta^{-1}$. Let $y \in (x\beta)\beta^{-1}$. Then $y\beta = x\beta = g_x$ and $y \in A_i$, which implies that $y \in \text{im}\gamma_i$. Thus,

$$y\gamma_i^{-1} = \{g_y\} = \{y\beta\} = \{x\beta\} = \{g_x\} = x\gamma_i^{-1}$$

and so $x = y$. Hence, $(x\beta)\beta^{-1} = \{x\} \subseteq (x\alpha)\alpha^{-1} \in \pi(\alpha)$. Now, we assume that $x \notin \text{im}\gamma_i$. Let $z \in (x\beta)\beta^{-1}$. Then $z\beta = x\beta = x\alpha$ from which it follows that $z \in A_i$ and $z \notin \text{im}\gamma_i$. Hence, $z\alpha = z\beta = x\alpha$ and so $z \in (x\alpha)\alpha^{-1} \in \pi(\alpha)$. Therefore, $\pi(\beta)$ refines $\pi(\alpha)$.

If A_i satisfies case 2 as above, then we suppose that $x \in \text{dom}\mu_i$. Then $x\beta = x\mu_i$. We claim that $(x\beta)\beta^{-1} = \{x\}$. Clearly, $x \in (x\beta)\beta^{-1}$. Let $y \in (x\beta)\beta^{-1}$. Then $y\beta = x\beta = x\mu_i$ and $y \in A_i$, which implies that $y \in \text{dom}\mu_i$. Thus,

$$y\mu_i = y\beta = x\beta = x\mu_i$$

and so $x = y$ since μ_i is injective. Hence, $(x\beta)\beta^{-1} = \{x\} \subseteq (x\alpha)\alpha^{-1} \in \pi(\alpha)$. Now, we assume that $x \notin \text{dom}\mu_i$. Let $z \in (x\beta)\beta^{-1}$. Then $z\beta = x\beta = x\alpha$ from which it follows that $z \in A_i$ and $z \notin \text{dom}\mu_i$. Hence, $z\beta = z\alpha$ and so $z\alpha = x\alpha$ implies $z \in (x\alpha)\alpha^{-1} \in \pi(\alpha)$. Therefore, $\pi(\beta)$ refines $\pi(\alpha)$.

(2) By the definition of β , it is easy to see that for each $x \in X$, $x\alpha = x\beta$ if $x\beta \in X\alpha$.

(3) Let $A \in X/E$. We assert that $(A \cap Y)\alpha \subseteq (A \cap Y)\beta$. Indeed, let $x\alpha \in (A \cap Y)\alpha$, where $x \in A \cap Y$. If $A \neq A_i$ for all $i \in I$, then $x\alpha = x\beta \in (A \cap Y)\beta$. Assume that $A = A_i$ for some $i \in I$. We consider the following two cases.

For A_i satisfies case 1 as above, it is clear that $x\alpha = x\beta \in (A_i \cap Y)\beta$ if $x \notin \text{im}\gamma_i$. Now, we assume that $x \in \text{im}\gamma_i$. Then $x \in C \setminus \{d_C\}$ for some $C \in C_i(\alpha)$. We can see that $x\alpha = d_C\alpha$ since $x, d_C \in C \in \pi(\alpha)$. Moreover, since $d_C \notin \text{im}\gamma_i$, we obtain $x\alpha = d_C\alpha = d_C\beta \in (C \cap Y)\beta \subseteq (A_i \cap Y)\beta$.

For A_i satisfies case 2 as above, it is clear that $x\alpha = x\beta \in (A_i \cap Y)\beta$ if $x \notin \text{dom}\mu_i$. Now, we assume that $x \in \text{dom}\mu_i$. Then $x \in C \setminus \{d_C\}$ for some $C \in C_i(\alpha)$. We can see that $x\alpha = d_C\alpha$ since $x, d_C \in C \in \pi(\alpha)$. Moreover, since $d_C \notin \text{dom}\mu_i$, we obtain $x\alpha = d_C\alpha = d_C\beta \in (C \cap Y)\beta \subseteq (A_i \cap Y)\beta$.

Therefore, $(A \cap Y)\alpha \subseteq (A \cap Y)\beta$ and so $A\alpha = (A \cap Y)\alpha \subseteq (A \cap Y)\beta$ since $\alpha \in F_E^+$. $\square$

The proof of the above theorem can be illustrated by the following example.

Example 4.4. Let $X = \{1, 2, 3, \dots\}$, $Y = \{2, 4, 6, \dots\}$ and

$$X/E = \{\{1, 2, 3, 4\}, \{5, 6\}, \{7, 8, 9, 10, 11, 12\}, \{13, 14, 15, 16, \dots\}\}.$$

Let $\alpha \in T_E(X, Y)$ be defined by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & \dots \\ 2 & 2 & 2 & 2 & 8 & 8 & 6 & 6 & 6 & 6 & 6 & 6 & 14 & 14 & 16 & 16 & 18 & 18 & 20 & 20 & \dots \end{pmatrix}.$$

According to the proof of Theorem 4.3, let $A_1 = \{1, 2, 3, 4\}$ and $A_2 = \{5, 6\}$. It is easy to see that $A_1\alpha = \{2\} \subsetneq \{2, 4\} = B_1$ and $A_2\alpha = \{8\} \subsetneq \{8, 10, 12\} = B_2$. Moreover, $\alpha|_{A_1}$ and $\alpha|_{A_2}$ are not injective. By the notation defined in the proof of Theorem 4.3, we obtain $C_1(\alpha) = \{\{1, 2, 3, 4\}\}$ and $C_2(\alpha) = \{\{5, 6\}\}$. We choose $d_{C_1} = 4 \in C_1 \cap Y$ and $d_{C_2} = 6 \in C_2 \cap Y$ and then $|\{1, 2, 3, 4\} \setminus \{d_{C_1}\}| = |\{1, 2, 3\}| \geq |\{4\}| = |B_1 \setminus A_1\alpha|$ and $|\{5, 6\} \setminus \{d_{C_2}\}| = |\{5\}| < |\{10, 12\}| = |B_2 \setminus A_2\alpha|$. By Case 1 given in the proof of the above theorem, there is an injection $\gamma_1 : \{4\} \rightarrow \{1, 2, 3\}$, say $\gamma : 4 \mapsto 3$. Define $\beta_1 : A_1 \rightarrow B_1$ by

$$\beta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 2 \end{pmatrix}.$$

Similarly, by using case 2, there is an injection $\mu_2 : \{5\} \rightarrow \{10, 12\}$, say $\gamma : 5 \mapsto 10$. We define $\beta_2 : A_2 \rightarrow B_2$ by

$$\beta_2 = \begin{pmatrix} 5 & 6 \\ 10 & 8 \end{pmatrix}.$$

Finally, we have

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & \dots \\ 2 & 2 & 4 & 2 & 10 & 8 & 6 & 6 & 6 & 6 & 6 & 6 & 14 & 14 & 16 & 16 & 18 & 18 & 20 & 20 & \dots \end{pmatrix}.$$

Then $\alpha \leq \beta$ and β is maximal by Theorem 4.2.

Now, we deal with minimal elements.

Theorem 4.5. Let $\alpha \in T_E^*(X, Y)$. Then α is minimal if and only if $|A\alpha| = 1$ for each $A \in X/E$.

Proof. ($\Rightarrow$) We prove by contrapositive. Assume that $|A\alpha| > 1$ for some $A \in X/E$. We can find $p \in (A \cap Y)\alpha$ since $A \cap Y$ is nonempty. Write $X/E \setminus \{A\} = \{A_i : i \in I\}$ and choose $a_i \in A_i \cap Y$ for all $i \in I$. Define a function $\beta : X \rightarrow Y$ by

$$\beta = \begin{pmatrix} A & A_i \\ p & a_i\alpha \end{pmatrix}.$$

It is easy to verify that $\beta \in T_E^*(X, Y)$ and $\beta \neq \alpha$. We show that $\beta \leq \alpha$.

- (1) Let $(x\alpha)\alpha^{-1} \in \pi(\alpha)$. We claim that $(x\alpha)\alpha^{-1} \subseteq (x\beta)\beta^{-1} \in \pi(\beta)$. Let $a \in (x\alpha)\alpha^{-1}$. Then $a\alpha = x\alpha$. We note that $(a, x) \in E$. If $x \in A$, then $a \in A$ and $a\beta = p = x\beta$, which implies that $a \in (x\beta)\beta^{-1}$. If $x \in A_i$ for some $i \in I$, then $a \in A_i$ and $a\beta = a_i\alpha = x\beta$. Hence, $a \in (x\beta)\beta^{-1}$. We conclude that $\pi(\alpha)$ refines $\pi(\beta)$.
- (2) Let $x\alpha \in X\beta$. Then $x\alpha = y\beta$ for some $y \in X$. If $y \in A$, then $x\alpha = y\beta = p \in (A \cap Y)\alpha$. We obtain $x \in A$ since $\alpha \in T_E^*(X, Y)$. Hence, $x\beta = p = x\alpha$. If $y \in A_i$, then $x\alpha = y\beta = a_i\alpha \in (A_i \cap Y)\alpha$. Similarly, we see that $x \in A_i$ and thus $x\beta = a_i\alpha = x\alpha$.
- (3) Clearly, $A\beta = \{p\} \subseteq (A \cap Y)\alpha$ and $A_i\beta = \{a_i\alpha\} \subseteq (A_i \cap Y)\alpha$ for all $i \in I$.

Thus, α is not minimal.

($\Leftarrow$) Suppose that $|A\alpha| = 1$ for each $A \in X/E$. Let $\beta \leq \alpha$ be such that $\beta \neq \alpha$. There exists $x \in X$ be such that $x\alpha \neq x\beta$. We have $x \in A$ for some $A \in X/E$ and $|A\alpha| = 1$, which implies $A\alpha = \{x\alpha\}$. By Theorem 2.1 (3), $x\beta \in A\beta \subseteq (A \cap Y)\alpha \subseteq A\alpha = \{x\alpha\}$. Hence, $x\alpha = x\beta$, which is a contradiction. Therefore, $\alpha = \beta$. $\square$

Theorem 4.6. Let $\alpha \in T_E^*(X, Y)$. Then there exists a minimal element $\beta \in T_E^*(X, Y)$ such that $\beta \leq \alpha$.

Proof. Assume that α is not minimal. Then $|A\alpha| > 1$ for some $A \in X/E$. Let β be defined as in the proof of Theorem 4.5. Then $\beta < \alpha$. Moreover, it is clear that $|C\beta| = 1$ for each $C \in X/E$. Hence, β is minimal. $\square$

By the above results, we may conclude the following corollaries which extend the results in [3,4,9].

Corollary 4.7. Let $\alpha \in T_{E^*}(X)$. Then the following statements hold.

1. α is maximal if and only if for each E -class A , $A\alpha \in X/E$ or $\alpha|_A$ is injective.
2. α is minimal if and only if $|A\alpha| = 1$ for each $A \in X/E$.
3. There exists a maximal element $\beta \in T_{E^*}(X)$ such that $\alpha \leq \beta$.
4. There exists a minimal element $\beta \in T_{E^*}(X)$ such that $\beta \leq \alpha$.

Corollary 4.8. Let $\alpha \in T(X, Y)$. Then the following statements hold.

1. α is maximal if and only if $\alpha \notin F$ or $X\alpha = Y$ or α is injective.
2. α is minimal if and only if $|X\alpha| = 1$.
3. There exists a maximal element $\beta \in T(X, Y)$ such that $\alpha \leq \beta$.
4. There exists a minimal element $\beta \in T(X, Y)$ such that $\beta \leq \alpha$.

Finally, the following results are concerned with the existence of an upper cover and a lower cover for elements of $T_{E^*}(X, Y)$. Recall that an element b in a semigroup S is called an *upper cover* for $a \in S$ if $a < b$ and there exists no $c \in S$ such that $a < c < b$. A *lower cover* is defined dually.

Theorem 4.9. Let $\alpha \in T_{E^*}(X, Y)$. If α is not maximal, then α has an upper cover.

Proof. Suppose that α is not maximal. Then there is a class $A \in X/E$ such that $A\alpha \subsetneq B$ for some $B \in Y/E_Y$ and $\alpha|_A$ is not injective. Moreover, we have $\alpha \in F_{E^*}$. Let β be defined as in the proof of Theorem 4.2, and we will show that β is an upper cover of α . Suppose that $\alpha < \gamma \leq \beta$ for some $\gamma \in T_{E^*}(X, Y)$. Then by Corollary 1.1, we conclude that $X\alpha \neq X\gamma$. We obtain $X\alpha \subset X\gamma \subseteq X\beta = X\alpha \cup \{b\}$ and thus $X\gamma = X\beta$. It follows from Corollary 1.1 that $\gamma = \beta$. Hence, β is an upper cover of α . $\square$

Theorem 4.10. Let $\alpha \in T_{E^*}(X, Y)$. If α is not minimal, then α has a lower cover.

Proof. Suppose that α is not minimal. We have two cases to consider.

Case 1: $\alpha \in F_{E^*}$. There exists an E -class A such that $|A\alpha| > 1$. We can find two distinct elements $p, q \in A\alpha$. Define $\beta : X \rightarrow Y$ by

$$x\beta = \begin{cases} p & \text{if } x \in q\alpha^{-1}, \\ x\alpha & \text{otherwise.} \end{cases}$$

It is straightforward to show that $\beta \in T_{E^*}(X, Y)$ and $\beta \neq \alpha$. Now, we prove that $\beta \leq \alpha$ by using Theorem 2.1.

- (1) Let $(x\alpha)\alpha^{-1} \in \pi(\alpha)$. To show that $(x\alpha)\alpha^{-1} \subseteq (x\beta)\beta^{-1}$, let $y \in (x\alpha)\alpha^{-1}$. Then $y\alpha = x\alpha$. First, we assume that $x \notin q\alpha^{-1}$. Then $x\alpha = x\beta$. If $y \in q\alpha^{-1}$, then $x\alpha = y\alpha = q$, which contradicts to $x \notin q\alpha^{-1}$. Hence, $y \notin q\alpha^{-1}$ and so $y\beta = y\alpha = x\beta$. We obtain $y \in (x\beta)\beta^{-1} \in \pi(\beta)$. Now, suppose that $x \in q\alpha^{-1}$. Then $x\alpha = q$ and $x\beta = p$. We obtain $y\alpha = x\alpha = q$, which implies that $y \in q\alpha^{-1}$. Thus, $y\beta = p = x\beta$ and so $y \in (x\beta)\beta^{-1} \in \pi(\beta)$. Therefore, $\pi(\alpha)$ refines $\pi(\beta)$.
- (2) Let $x\alpha \in X\beta$. Then $x\alpha = y\beta$ for some $y \in X$. If $y \in q\alpha^{-1}$, then $x\alpha = y\beta = p$ and hence $x \in p\alpha^{-1} \neq q\alpha^{-1}$. We have $x \notin q\alpha^{-1}$ and so $x\alpha = x\beta$. If $y \notin q\alpha^{-1}$, then $x\alpha = y\beta = y\alpha \neq q$, which implies that $x \notin q\alpha^{-1}$. Thus, $x\alpha = x\beta$.
- (3) Let $C \in X/E$. To show that $C\beta \subseteq (C \cap Y)\alpha$, let $c\beta \in C\beta$. If $c \in q\alpha^{-1}$, then $c \in A$, which implies that $C = A$. Hence, $c\beta = p \in A\alpha \subseteq (A \cap Y)\alpha = (C \cap Y)\alpha$ since $\alpha \in F_{E^*}$. On the other hand, if $c \notin q\alpha^{-1}$, then $c\beta = c\alpha \in C\alpha \subseteq (C \cap Y)\alpha$.

To show that β is a lower cover for α , let $\gamma \in T_{E^*}(X, Y)$ be such that $\beta \leq \gamma < \alpha$. Then by Theorem 2.1, $X\alpha \setminus \{q\} = X\beta \subseteq X\gamma \subset X\alpha$, which implies $X\beta = X\gamma$. By Corollary 1.1 (1), we conclude that $\beta = \gamma$. Consequently, β is a lower cover.

Case 2: $\alpha \notin F_{E^*}$. Let $\{A_i : i \in I\}$ be the set of all E -classes such that $|A_i\alpha| > 1$ and $\pi_{A_i}(\alpha) \setminus \tilde{\pi}_{A_i}(\alpha) \neq \emptyset$. We can see that $\{A_i : i \in I\}$ is nonempty since $\alpha \notin F_{E^*}$ and α is not minimal. For each $i \in I$, we choose and fix $a_i \in (A_i \cap Y)\alpha$ and define a function $\beta_i : A_i \rightarrow A_i\alpha$ by

$$x\beta_i = \begin{cases} a_i & \text{if } x \in \bigcup(\pi_{A_i}(\alpha) \setminus \tilde{\pi}_{A_i}(\alpha)), \\ x\alpha & \text{otherwise.} \end{cases}$$

Let $\beta : X \rightarrow Y$ be defined by

$$x\beta = \begin{cases} x\beta_i & \text{if } x \in A_i, \\ x\alpha & \text{otherwise.} \end{cases}$$

It is clear that $\beta \in T_{E^*}(X, Y)$ and $\beta \neq \alpha$. We aim to prove that $\beta \leq \alpha$ by Theorem 2.1.

(1) Let $(x\alpha)\alpha^{-1} \in \pi(\alpha)$. To show that $(x\alpha)\alpha^{-1} \subseteq (x\beta)\beta^{-1}$, let $y \in (x\alpha)\alpha^{-1}$. Then $y\alpha = x\alpha$. We note that $(x, y) \in E$.

First, we assume that $x \notin \bigcup(\pi_{A_i}(\alpha) \setminus \tilde{\pi}_{A_i}(\alpha))$ for all $i \in I$. Then $x\alpha = x\beta$. If $y \in \bigcup(\pi_{A_i}(\alpha) \setminus \tilde{\pi}_{A_i}(\alpha))$ for some $i \in I$, then $y \in A_i$ and $(x\alpha)\alpha^{-1} \cap Y = (y\alpha)\alpha^{-1} \cap Y = \emptyset$. Hence, $x \in A_i$ and $x \in \bigcup(\pi_{A_i}(\alpha) \setminus \tilde{\pi}_{A_i}(\alpha))$, which is a contradiction. Thus, $y \notin \bigcup(\pi_{A_i}(\alpha) \setminus \tilde{\pi}_{A_i}(\alpha))$ for all $i \in I$. Then $y\beta = y\alpha = x\alpha = x\beta$ and hence $y \in (x\beta)\beta^{-1} \in \pi(\beta)$.

Now, suppose that $x \in \bigcup(\pi_{A_i}(\alpha) \setminus \tilde{\pi}_{A_i}(\alpha))$ for some $i \in I$. Then $x \in A_i$ and $x\beta = a_i$. We note that $y \in A_i$ since $(x, y) \in E$. If $y \notin \bigcup(\pi_{A_i}(\alpha) \setminus \tilde{\pi}_{A_i}(\alpha))$, then $(x\alpha)\alpha^{-1} \cap Y = (y\alpha)\alpha^{-1} \cap Y \neq \emptyset$. Hence, $x \in \tilde{\pi}_{A_i}(\alpha)$, which is a contradiction. Hence, $y \in \bigcup(\pi_{A_i}(\alpha) \setminus \tilde{\pi}_{A_i}(\alpha))$. We have $y\beta = a_i = x\beta$ and so $y \in (x\beta)\beta^{-1} \in \pi(\beta)$. We conclude that $\pi(\alpha)$ refines $\pi(\beta)$.

(2) Let $x\alpha \in X\beta$. Then $x\alpha = y\beta$ for some $y \in X$. If $y \in \bigcup(\pi_{A_i}(\alpha) \setminus \tilde{\pi}_{A_i}(\alpha))$ for some $i \in I$, then $x\alpha = y\beta = a_i \in (A_i \cap Y)\alpha$. There is $b \in A_i \cap Y$ such that $x\alpha = b\alpha$ from which it follows that $x \in (b\alpha)\alpha^{-1} \in \tilde{\pi}_{A_i}(\alpha)$. We have $x \notin \bigcup(\pi_{A_i}(\alpha) \setminus \tilde{\pi}_{A_i}(\alpha))$ and so $x\alpha = x\beta$. If $y \notin \bigcup(\pi_{A_i}(\alpha) \setminus \tilde{\pi}_{A_i}(\alpha))$ for all $i \in I$, then $x\alpha = y\beta = y\alpha$ and hence $(x\alpha)\alpha^{-1} \cap Y = (y\alpha)\alpha^{-1} \cap Y \neq \emptyset$. We conclude that $x \notin \bigcup(\pi_{A_i}(\alpha) \setminus \tilde{\pi}_{A_i}(\alpha))$ for all $i \in I$. Consequently, $x\alpha = x\beta$.

(3) Let $C \in X/E$. To show that $C\beta \subseteq (C \cap Y)\alpha$, let $c\beta \in C\beta$. If $c \in \bigcup(\pi_{A_i}(\alpha) \setminus \tilde{\pi}_{A_i}(\alpha))$ for some $i \in I$, then $A_i = C$ and $c\beta = a_i \in (A_i \cap Y)\alpha = (C \cap Y)\alpha$. If $c \notin \bigcup(\pi_{A_i}(\alpha) \setminus \tilde{\pi}_{A_i}(\alpha))$ for all $i \in I$, then $c\beta = c\alpha$. Moreover, we have $c \in \tilde{\pi}_C(\alpha)$, which implies that $c \in (d\alpha)\alpha^{-1}$ for some $(d\alpha)\alpha^{-1} \in \tilde{\pi}_C(\alpha)$. Hence, there is $y \in (d\alpha)\alpha^{-1} \cap Y$ and so $c\alpha = d\alpha = y\alpha$. It is clear that $y \in C$. We obtain $c\beta = c\alpha = y\alpha \in (C \cap Y)\alpha$.

To show that β is a lower cover for α , let $\gamma \in T_{E^*}(X, Y)$ be such that $\beta < \gamma < \alpha$. Then $X\beta \subset X\gamma \subset X\alpha$ and so there is $xy \in X\gamma \setminus X\beta$. Moreover, since $\gamma \neq \alpha$, we obtain $xy \in X\gamma \subseteq Y\alpha$ from which it follows that there exists $y \in Y$ such that $xy = y\alpha$. It is clear that $y \notin \bigcup(\pi_{A_i}(\alpha) \setminus \tilde{\pi}_{A_i}(\alpha))$ for all $i \in I$. Hence, $xy = y\alpha = y\beta \in X\beta$, which is a contradiction. Consequently, β is a lower cover. $\square$

Example 4.11. Let $X = \{1, 2, 3, \dots\}$, $Y = \{2, 4, 6, \dots\}$ and $X/E = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \{9, 10, 11, \dots\}\}$. We define $\alpha \in T_{E^*}(X, Y)$ by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & \dots \\ 2 & 2 & 4 & 2 & 6 & 8 & 8 & 8 & 10 & 10 & 12 & 12 & 14 & 14 & 16 & 16 & 18 & 18 & 20 & 20 & \dots \end{pmatrix}.$$

It is clear that $\alpha \notin F_{E^*}$. Let $A_1 = \{1, 2, 3, 4\}$ and $A_2 = \{5, 6, 7, 8\}$. We can see that $\pi_{A_1}(\alpha) = \{\{1, 2, 4\}, \{3\}\}$, $\tilde{\pi}_{A_1}(\alpha) = \{\{1, 2, 4\}\}$, $\pi_{A_2}(\alpha) = \{\{5\}, \{6, 7, 8\}\}$ and $\tilde{\pi}_{A_2}(\alpha) = \{\{6, 7, 8\}\}$, which imply that $\pi_{A_1}(\alpha) \setminus \tilde{\pi}_{A_1}(\alpha) = \{\{3\}\} \neq \emptyset$ and $\pi_{A_2}(\alpha) \setminus \tilde{\pi}_{A_2}(\alpha) = \{\{5\}\} \neq \emptyset$. Now, we choose $2 = 2\alpha \in (A_1 \cap Y)\alpha$ and $8 = 6\alpha \in (A_2 \cap Y)\alpha$. By Case 2 in the proof of Theorem 4.10, we define

$$\beta_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix} \quad \text{and} \quad \beta_2 = \begin{pmatrix} 5 & 6 & 7 & 8 \\ 8 & 8 & 8 & 8 \end{pmatrix}.$$

Finally, we have

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & \dots \\ 2 & 2 & 2 & 2 & 8 & 8 & 8 & 8 & 10 & 10 & 12 & 12 & 14 & 14 & 16 & 16 & 18 & 18 & 20 & 20 & \dots \end{pmatrix}$$

is a lower cover of α .

We end this section by briefly summarizing our paper and point out future work. In [3,4,9], the natural partial order on $T(X, Y)$ and $T_{E^*}(X)$ was determined. This paper is concerned with the same problems for the semigroup $T_{E^*}(X, Y)$, that is, we study the compatibility, maximality and minimality of its elements in this semigroup under the natural partial order. Moreover, in the last section of our paper, we show that every element of $T_{E^*}(X, Y)$ lies between maximal and minimal elements. These are results that we have not found in the studies of other transformation semigroups. We also prove that for every $\alpha \in T_{E^*}(X, Y)$, if α is not maximal, then it has an upper cover; and if α is not minimal, then it has a lower cover. However, in [4], the authors investigated the greatest lower bound and the least upper bound of two elements of $T_{E^*}(X)$, it is then natural to ask for this property on $T_{E^*}(X, Y)$, which is an open question.

Acknowledgements: The author is grateful to Prof. Jintana Sanwong for her collaboration. The author thanks the referee for his/her careful review and for a number of helpful suggestions.

Part of this work was presented in the Arbeitstagung Allgemeine Algebra conference (AAA99), Siena, Italy, February 21–23, 2020.

Funding information: This research was supported by the Research Fund for DPST Graduate with First Placement, Year 2016, under grant No. 027/2559 and Chiang Mai University.

Conflict of interest: The author states no conflict of interest.

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