Research Article

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Primitive and decomposable elements in homology of $\Omega\Sigma\mathbb{C}P^\infty$

<https://doi.org/10.1515/math-2021-0110>

received June 14, 2021; accepted September 27, 2021

Abstract: For each positive integer n , we let $\varphi_n : \Sigma\mathbb{C}P^\infty \rightarrow \Sigma\mathbb{C}P^\infty$ be the self-maps of the suspension of the infinite complex projective space, or the localization of this space at a set of primes which may be an empty set. Furthermore, let $[\varphi_m, \varphi_n] : \Sigma\mathbb{C}P^\infty \rightarrow \Sigma\mathbb{C}P^\infty$ be a commutator of self-maps φ_m and φ_n for any positive integers m and n . In the current study, we show that the image of the homomorphism $[\hat{\varphi}_m, \hat{\varphi}_n]_*$ in homology induced by the adjoint $[\hat{\varphi}_m, \hat{\varphi}_n] : \mathbb{C}P^\infty \rightarrow \Omega\Sigma\mathbb{C}P^\infty$ of the commutator $[\varphi_m, \varphi_n]$ is both primitive and decomposable. As a further support of the above statement, we provide an example.

Keywords: commutator, localization, Leray-Serre spectral sequence

MSC 2020: 55S37, 55P10, 17B01

1 Introduction

In algebraic topology and its applications, the loop functor Ω and suspension functor Σ are examples of adjoint functors in pointed (equivariant) homotopy category. Furthermore, H-spaces [1] and co-H-spaces [2], having dual notions of each other from the Eckmann and Hilton's viewpoints, are the main mathematical objects that are used in research on modern algebraic topology and its applications; see [3–13] for the basic notions of H-spaces, co-H-spaces and their applications to the digital counterparts. An s -cobordism classification of topological 4-manifolds using the group of homotopy self-equivalences was nicely presented in [14]; see also [15] for algebraic loop structures.

In homotopy classification problems, a lot of sets as mathematical objects have the natural group, ring, field, and module structures. For example, the set $[X, \Omega Y]$ of all homotopy classes of base point preserving continuous maps from X to ΩY has a natural group structure originated from the loop structure. Equivalently, the set $[\Sigma X, Y]$ also has a natural group structure induced from suspension structure as an adjointness.

Let $\mathbb{C}P^\infty$ be the infinite complex projective space and let

$$\varphi_n : \Sigma\mathbb{C}P^\infty \rightarrow \Sigma\mathbb{C}P^\infty$$

be the self-maps of $\Sigma\mathbb{C}P^\infty$ for each positive integer n satisfying suitable conditions, and let

$$\hat{\varphi}_n : \mathbb{C}P^\infty \rightarrow \Omega\Sigma\mathbb{C}P^\infty$$

be its adjoint; see Section 2 and [16–20] for more details. We denote

$$[\varphi_m, \varphi_n] : \Sigma\mathbb{C}P^\infty \rightarrow \Sigma\mathbb{C}P^\infty$$

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as a homotopy commutator of self-maps φ_m and φ_n on $\Sigma\mathbb{C}P^\infty$ (or the localization of $\Sigma\mathbb{C}P^\infty_{(P)}$ at a set P of prime numbers, or possibly at the empty set for rationalization [21]); that is,

$$[\varphi_m, \varphi_n] = \varphi_m + \varphi_n - \varphi_m - \varphi_n,$$

where the binary operations “+” and “−” between homotopy classes are originated from the suspension structure. Similarly, we let

$$[\hat{\varphi}_m, \hat{\varphi}_n] = \widehat{[\varphi_m, \varphi_n]} : \mathbb{C}P^\infty \rightarrow \Omega\Sigma\mathbb{C}P^\infty$$

be the adjoint of $[\varphi_m, \varphi_n]$ for $m, n = 1, 2, 3, \dots$, and let

$$\hat{\varphi}_n^{-1} : \mathbb{C}P^\infty \rightarrow \Omega\Sigma\mathbb{C}P^\infty$$

be the loop inverse of $\hat{\varphi}_n$ for each n ; see below. It is well known that $\mathbb{C}P^\infty$ is just the Eilenberg-MacLane space $K(\mathbb{Z}, 2)$ as well as

$$BT^1 = BS^1 = BU(1).$$

This is an infinite dimensional CW-space and its suspension becomes somewhat intractable for us to detect with respect to homotopy in algebraic topology.

In the present study, we develop the commutators of homotopy self-maps on the suspension (and then loop) structures based on the infinite complex projective space, or localization at a set of primes P , or the rationalization of this space. We will show that the image of homomorphisms

$$[\hat{\varphi}_m, \hat{\varphi}_n]_* : H_*(\mathbb{C}P^\infty) \rightarrow H_*(\Omega\Sigma\mathbb{C}P^\infty)$$

on homologies induced by the adjoint of the self-map originated from a commutator

$$[\varphi_m, \varphi_n] : \Sigma\mathbb{C}P^\infty \rightarrow \Sigma\mathbb{C}P^\infty$$

of φ_m and φ_n is both primitive and decomposable for all positive integers m and n . Finally, we provide an example of this result.

2 Some self-maps and description of results

We first consider some self-maps of the suspension of the infinite complex projective space, or localization at a set of primes P , or the rationalization of this space as follows.

Let

$$\hat{\varphi}_1 : \mathbb{C}P^\infty \rightarrow \Omega\Sigma\mathbb{C}P^\infty$$

be the canonical map; that is, the adjoint map of the identity map

$$\varphi_1 : \Sigma\mathbb{C}P^\infty \rightarrow \Sigma\mathbb{C}P^\infty.$$

Following Morisugi [16], we define the maps

$$\hat{\varphi}_{n+1} : \mathbb{C}P^\infty \xrightarrow{\bar{\Delta}} \mathbb{C}P^\infty \wedge \mathbb{C}P^\infty \xrightarrow{\hat{\varphi}_1 \wedge \hat{\varphi}_n} \Omega\Sigma\mathbb{C}P^\infty \wedge \Omega\Sigma\mathbb{C}P^\infty \xrightarrow{\mathcal{H}} \Omega\Sigma\mathbb{C}P^\infty$$

by induction on $n = 1, 2, 3, \dots$, where $\bar{\Delta}$ is the composite of the diagonal map with projection of the cartesian product onto the smash product, and $\mathcal{H}$ is an extension map of adjointness of Hopf construction of $\mathbb{C}P^\infty$.

Note that the map

$$\hat{\varphi}_{n+1} : \mathbb{C}P^\infty \rightarrow \Omega\Sigma\mathbb{C}P^\infty$$

looks like the n th convolution product with a coproduct and a product, which is applicable and deeply studied in a large number of situations in mathematics.

Since $\mathbb{C}P^\infty$ is an H-space, it has the homotopically unique multiplication

$$\mu : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \rightarrow \mathbb{C}P^\infty.$$

We thus have the classical Hopf construction,

$$H(\mu) : \Sigma(\mathbb{C}P^\infty \wedge \mathbb{C}P^\infty) \rightarrow \Sigma\mathbb{C}P^\infty,$$

and the adjoint

$$\widehat{H(\mu)} : \mathbb{C}P^\infty \wedge \mathbb{C}P^\infty \rightarrow \Omega\Sigma\mathbb{C}P^\infty$$

of $H(\mu)$. Using the James model, we see that $\widehat{H(\mu)}$ has the above extension

$$\mathcal{H} : \Omega\Sigma\mathbb{C}P^\infty \wedge \Omega\Sigma\mathbb{C}P^\infty \rightarrow \Omega\Sigma\mathbb{C}P^\infty.$$

We also note that for each positive integer n , the following diagram

$$\begin{array}{ccc} \mathbb{C}P^\infty & \xrightarrow{\quad \hat{\varphi}_n \quad} & \Omega\Sigma\mathbb{C}P^\infty \\ & \searrow p \quad \nearrow f_n & \\ & \mathbb{C}P^\infty / \mathbb{C}P^{n-1} & \end{array}$$

is commutative up to homotopy [16, Theorem 1.4], where

- $p : \mathbb{C}P^\infty \rightarrow \mathbb{C}P^\infty / \mathbb{C}P^{n-1}$ is the projection map, and
- $f_n : \mathbb{C}P^\infty / \mathbb{C}P^{n-1} \rightarrow \Omega\Sigma\mathbb{C}P^\infty$ is the map whose restriction to the $2n$ -cell of the quotient space $\mathbb{C}P^\infty / \mathbb{C}P^{n-1}$ is a homotopy generator in dimension $2n$.

We now consider a self-map

$$\varphi_n : \Sigma\mathbb{C}P^\infty \rightarrow \Sigma\mathbb{C}P^\infty$$

by using the adjointness of

$$\hat{\varphi}_n : \mathbb{C}P^\infty \rightarrow \Omega\Sigma\mathbb{C}P^\infty$$

for each positive integer n . Moreover, we can construct the self-maps

$$\psi^q : \mathbb{C}P^\infty \rightarrow \mathbb{C}P^\infty$$

corresponding to elements q in $\mathbb{Z}$; that is,

$$q \in \mathbb{Z} \cong [\mathbb{C}P^\infty, \mathbb{C}P^\infty] \cong H^2(\mathbb{C}P^\infty; \mathbb{Z}).$$

Taking the suspension of the map ψ^q , we consider the self-maps $\Sigma\psi^q$ of $\Sigma\mathbb{C}P^\infty$ inducing

$$\Sigma\psi^q_* = (q, q^2, q^3, \dots)$$

on homology for any q . Using these maps and the suspension structure, we can also construct self-maps

$$\varphi_n : \Sigma\mathbb{C}P^\infty \rightarrow \Sigma\mathbb{C}P^\infty,$$

which induce

- $\varphi_{1*} = (1, 1, 1, \dots)$;
- $\varphi_{2*} = (0, 2!, \dots)$;
- $\varphi_{3*} = (0, 0, 3!, \dots)$;
- $\vdots$
- $\varphi_{n*} = (0, 0, \dots, 0, n!, \dots)$

for each positive integer n in integral homology groups $H_*(\Sigma\mathbb{C}P^\infty; \mathbb{Z})$. We note that the two constructions of the above self-maps are coincident with each other.

In a practical sense, McGibbon [22] showed that every self-map of $\Sigma\mathbb{C}P^\infty$ is homologous to a linear combination of the maps, $\Sigma\psi^1, \Sigma\psi^2, \dots, \Sigma\psi^n$ up to homology. The self-maps that we consider are as follows:

- $\varphi_{1*} = 1_{H_*(\Sigma\mathbb{C}P^\infty)}$;
- $\varphi_{2*} = \Sigma\psi_*^2 - 2\Sigma\psi_*^1 = (2, 4, 8, \dots) - 2(1, 1, 1, \dots) = (0, 2!, \dots)$;
- $\varphi_{3*} = \Sigma\psi_*^3 - 3\Sigma\psi_*^2 + 3\Sigma\psi_*^1 = (3, 9, 27, \dots) - 3(2, 4, 8, \dots) + 3(1, 1, 1, \dots) = (0, 0, 3!, \dots)$;
- $\varphi_{4*} = \Sigma\psi_*^4 - 4\Sigma\psi_*^3 + 6\Sigma\psi_*^2 - 4\Sigma\psi_*^1 = (4, 16, 64, \dots) - 4(3, 9, 27, \dots) + 6(2, 4, 8, \dots) - 4(1, 1, 1, \dots)$
 $= (0, 0, 0, 4!, \dots)$

and in general,

$$\varphi_{n*} = \sum_{r=0}^{n-1} (-1)^r \binom{n}{r} \Sigma\psi^{n-r}_* = (0, 0, \dots, 0, n!, \dots),$$

where $\binom{n}{r}$ is a binomial coefficient. In the current study, the calculations of the aforementioned self-maps at the homology and homotopy levels will be mostly required.

From the rational homotopy point of view, let $\mathbb{Z}_{(P)}$ be the integers localized at the set of primes P which may be the empty set $\emptyset$. We note that if the set of primes is empty, we speak of *rationalization* instead of 0-localization. We also note that

$$\pi_*(\Sigma\mathbb{C}P^\infty) \otimes \mathbb{Z}_{(\emptyset)} \cong \pi_*(\Sigma\mathbb{C}P^\infty) \otimes \mathbb{Q} \cong \pi_*(\Sigma\mathbb{C}P^\infty_{\mathbb{Q}}) = \pi_*(\Sigma K(\mathbb{Q}, 2)),$$

and that the restriction of

$$f_n : \mathbb{C}P^\infty / \mathbb{C}P^{n-1} \rightarrow \Omega\Sigma\mathbb{C}P^\infty$$

to the bottom cell is a rational homotopy generator of $\pi_*(\Omega\Sigma\mathbb{C}P^\infty) \otimes \mathbb{Q}$ as a rational vector space.

We recall that the reduced homology $\tilde{H}_*(\mathbb{C}P^\infty; \mathbb{Z})$ is isomorphic to the graded $\mathbb{Z}$ -module $\mathbb{Z}\{\beta_1, \beta_2, \dots, \beta_n, \dots\}$ as a graded abelian group, where β_n is a canonical generator of integral homology group $H_{2n}(\mathbb{C}P^\infty; \mathbb{Z})$ for $n = 1, 2, 3, \dots$. From the Bott-Samelson theorem [23], we see that the rational homology of $\Omega\Sigma\mathbb{C}P^\infty$ as the Pontryagin algebra is algebraically isomorphic to the tensor algebra $T[b_1, b_2, \dots, b_n, \dots]$ with basis $\{b_1, b_2, \dots, b_n, \dots\}$, where b_n has the degree $2n$ in homology with diagonal

$$\Delta(b_n) = \sum_{i+j=n} b_i \otimes b_j$$

and $b_n = E_*(\beta_n)$. Here, E is the canonical map

$$E = \hat{\varphi}_1 : \mathbb{C}P^\infty \rightarrow \Omega\Sigma\mathbb{C}P^\infty$$

and b_0 means $1 \in H_0(\Omega\Sigma\mathbb{C}P^\infty)$.

Notation. We will make use of the following notations.

- (1) Let $K = \mathbb{C}P^\infty$, the infinite complex projective space, or the localization $K(\mathbb{Z}_{(P)}, 2)$ at some set of primes P (or at a prime ideal (p) for a prime number p), or the rationalization $K(\mathbb{Q}, 2)$.
- (2) Let $[\varphi_m, \varphi_n] : \Sigma K \rightarrow \Sigma K$ denote a commutator of self-maps φ_m and φ_n on the suspension structure of K , or the localization $\Sigma K(\mathbb{Z}_{(P)}, 2)$ at some set of primes P (or at a prime ideal (p) for a prime number p), or rationalization $\Sigma K(\mathbb{Q}, 2)$ of this space for $m, n = 1, 2, 3, \dots$.
- (3) Let K_i denote the i -skeleton of K .
- (4) We will sometimes write the same notation in homology classes by considering β_n as $b_n = E_*(\beta_n)$ for each n , where E is the canonical map $\hat{\varphi}_1 : K \rightarrow \Omega\Sigma K$.

More generally, it is well known that X is an H-space, e.g., $X = K$ if and only if the canonical map $E : X \rightarrow \Omega\Sigma X$ admits a retraction; see [24, p. 226] and [25].

Let $j_1 : Y \rightarrow Y \times Y$ and $j_2 : Y \rightarrow Y \times Y$ be the first and second inclusions, respectively; that is,

$$j_1(y) = (y, y_0)$$

and

$$j_2(y) = (y_0, y)$$

in the pointed homotopy category, where y_0 is the base point of Y . Recall that an element $x \in H_*(Y)$ is said to be *primitive* if

$$\Delta_*(x) = j_{1*}(x) + j_{2*}(x)$$

in homology, where $\Delta : Y \rightarrow Y \times Y$ is a diagonal map.

An element x of the Pontryagin algebra $H_*(\Omega\Sigma K)$ is said to be *decomposable* if x is the product of two or more homogeneous elements of the positive degree, e.g.,

$$x = \sum_j y_j z_j,$$

where $1 \leq \deg(y_j)$, $\deg(z_j)$, and $\deg(y_j) + \deg(z_j) = \deg(x)$. An element x is said to be *indecomposable* if it is not decomposable; similarly for an element of Lie algebras or Whitehead algebras.

Note that every element of homology of a co-H-space is primitive and that the decomposable elements are important ingredients for a minimal model of a differential graded algebra.

We will now describe the main theorem of the present work.

Theorem 2.1. *Let*

$$[\hat{\varphi}_m, \hat{\varphi}_n] = \widehat{[\varphi_m, \varphi_n]} : K \rightarrow \Omega\Sigma K$$

be the adjoint of $[\varphi_m, \varphi_n]$. Then $[\hat{\varphi}_m, \hat{\varphi}_n]_(\beta_{m+n})$ is both primitive and decomposable in*

$$H_*(\Omega\Sigma K) \cong T[b_1, b_2, \dots, b_{m+n}, \dots]$$

for $m, n = 1, 2, 3, \dots$, where b_{m+n} is a corresponding generator in the Pontryagin algebra over $\mathbb{Z}$, or $\mathbb{Z}_{(p)}$, or $\mathbb{Q}$.

Example 2.2. Let “0” be the additive identity element of the Pontryagin algebra $H_*(\Omega\Sigma K)$. Then we have

- $[\hat{\varphi}_1, \hat{\varphi}_2]_*(\beta_1) = 0$;
- $[\hat{\varphi}_1, \hat{\varphi}_2]_*(\beta_2) = 0$; and
- $[\hat{\varphi}_1, \hat{\varphi}_2]_*(\beta_3) = 2(b_1 b_2 - b_2 b_1)$,

where $b_n = \hat{\varphi}_1_*(\beta_n)$ for $n = 1, 2, 3, \dots$.

Remark 2.3. We will now consider the self-maps of ΣK of the form $1 + [\varphi_m, \varphi_n]$, where “1” is the identity map on ΣK for $m, n = 1, 2, 3, \dots$. These self-maps play a pivotal role and provide a vital clue in that the self-maps $1 + [\varphi_m, \varphi_n]$ of ΣK are actually self-homotopy equivalences; that is, $1 + [\varphi_m, \varphi_n] \in \text{Aut}(\Sigma K)$. We can see that these maps correspond to the homomorphisms

$$\theta : \pi_*(\Sigma K(\mathbb{Q}, 2)) \rightarrow \pi_*(\Sigma K(\mathbb{Q}, 2))$$

of Whitehead algebras such that

$$\theta(x_i) = x_i + \text{the sum of brackets of lower dimensional homotopy classes,}$$

where x_i is the indecomposable generators of $\pi_*(\Sigma K(\mathbb{Q}, 2))$.

3 Proofs

3.1 Proof of Theorem 2.1

Considering the construction of the maps $\hat{\varphi}_n : K \rightarrow \Omega\Sigma K$, it can be seen that the restrictions $\hat{\varphi}_n|_{K_{2n-1}} : K_{2n-1} \rightarrow \Omega\Sigma K$ of the maps $\hat{\varphi}_n : K \rightarrow \Omega\Sigma K$ to the skeletons are inessential for $n = 1, 2, 3, \dots$, as shown in Table 1.

Table 1: On the maps $\hat{\varphi}_n : X \rightarrow \Omega\Sigma X$, $n = 1, 2, 3, \dots$

Maps	$\hat{\varphi}_1$	$\hat{\varphi}_2$	$\hat{\varphi}_3$	$\dots$	$\hat{\varphi}_n$	$\dots$
The $(2n - 1)$ -skeletons	K_1	K_3	K_5	$\dots$	K_{2n-1}	$\dots$
Inessential maps	$\hat{\varphi}_1 _{K_1}$	$\hat{\varphi}_2 _{K_3}$	$\hat{\varphi}_3 _{K_5}$	$\dots$	$\hat{\varphi}_n _{K_{2n-1}}$	$\dots$

Proof. We consider fibration

$$\Omega K \rightarrow \mathcal{P}K \rightarrow K, \quad (1)$$

where $\mathcal{P}K$ is the path space of K . The Leray-Serre spectral sequence on the fibration (1) says that the rational cohomology $H^*(K; \mathbb{Q})$ is isomorphic to the polynomial algebra $\mathbb{Q}[\alpha]$ over $\mathbb{Q}$ generated by α of degree 2, i.e., α is a generator of $H^2(K; \mathbb{Q})$ with

$$\langle \alpha^i, b_j \rangle = \delta_{ij}$$

for $i, j = 1, 2, 3, \dots$. Pictorially, K has a CW-decomposition as follows:

$$K = \underbrace{\underbrace{S^2}_{\mathbb{C}P^1} \cup_{\gamma_1} e^4 \cup_{\gamma_2} e^6 \cup_{\gamma_3} \dots \cup_{\gamma_{n-1}} e^{2n} \cup_{\gamma_n} e^{2(n+1)} \cup_{\gamma_{n+1}} \dots}_{\mathbb{C}P^2} \underbrace{\dots}_{\mathbb{C}P^3} \underbrace{\dots}_{\mathbb{C}P^n} \dots$$

where $\gamma_n : S^{2n+1} \rightarrow \mathbb{C}P^n$ is the attaching maps for $n = 1, 2, 3, \dots$.

To prove Theorem 2.1, we consider the following lemmas as crucial ingredients.

Lemma 3.1. *The restriction of the commutator $[\hat{\varphi}_m, \hat{\varphi}_n] : K \rightarrow \Omega\Sigma K$ to the $(2m + 2n - 1)$ -skeleton of K is inessential for each m and n ; that is,*

$$[\hat{\varphi}_m, \hat{\varphi}_n]|_{K_{2m+2n-1}} \simeq *.$$

Proof. Let

$$\mu : \Omega\Sigma K \times \Omega\Sigma K \rightarrow \Omega\Sigma K$$

be the loop multiplication and let

$$\hat{\varphi}_i^{-1} : K \rightarrow \Omega\Sigma K$$

denote the loop inverse of $\hat{\varphi}_i$. Additionally, let $\Delta : K \rightarrow K \times K$ be the diagonal map. Then we can define a new map

$$C(\hat{\varphi}_m, \hat{\varphi}_n) : K \times K \rightarrow \Omega\Sigma K$$

by the composition of the maps

$$\mu \circ (\mu \times \mu) \circ ((\hat{\varphi}_m \times \hat{\varphi}_n) \times (\hat{\varphi}_m^{-1} \times \hat{\varphi}_n^{-1})) \circ (\Delta \times \Delta) : K \times K \longrightarrow \Omega \Sigma K;$$

that is,

$$C(\hat{\varphi}_m, \hat{\varphi}_n)(k_1, k_2) = \hat{\varphi}_m(k_1) * \hat{\varphi}_n(k_2) * (\hat{\varphi}_m(k_1))^{-1} * (\hat{\varphi}_n(k_2))^{-1},$$

where the binary operation “ $*$ ” is the loop multiplication on the loop structure on $\Omega \Sigma K$ and $k_1, k_2 \in K$. It can be seen that the following diagram

$$\begin{array}{ccc} K & \xrightarrow{[\hat{\varphi}_m, \hat{\varphi}_n]} & \Omega \Sigma K \\ \Delta \downarrow & \simeq & \downarrow = \\ K \times K & \xrightarrow{C(\hat{\varphi}_m, \hat{\varphi}_n)} & \Omega \Sigma K \end{array}$$

commutes up to homotopy. As usual, the CW-structure is put on $K \times K$ as follows:

$$(K \times K)_{2m+2n-1} = \bigcup_{i+j=2m+2n-1} K_i \times K_j.$$

The construction of the maps

$$\hat{\varphi}_m, \hat{\varphi}_n : K \rightarrow \Omega \Sigma K$$

shows that if $i \leq 2m$ or $j \leq 2n$, then all the restrictions $C(\hat{\varphi}_m, \hat{\varphi}_n)|_{K_i \times K_j}$ become inessential; that is, trivial homotopies

$$F_{m,n} : C(\hat{\varphi}_m, \hat{\varphi}_n)|_{K_i \times K_j} \simeq *$$

exist. Using the cellular approximation theorem and the homotopy extension property as dual of the homotopy lifting property, we can extend those trivial homotopies to the $(2m + 2n - 1)$ -skeleton $(K \times K)_{2m+2n-1}$, as required. $\square$

Lemma 3.2. *For each positive integer n , the homomorphic image of*

$$\hat{\varphi}_{n*} : H_{2n}(K) \rightarrow H_{2n}(\Omega \Sigma K)$$

is primitive.

Proof. We note that

$$\hat{\varphi}_n|_{K_{2n-1}} \simeq *,$$

and that the following diagram

$$\begin{array}{ccc} K & \xrightarrow{\hat{\varphi}_n} & \Omega \Sigma K \\ q \downarrow & \simeq & \downarrow = \\ K/K_{2n-1} & \xrightarrow{\Psi_n} & \Omega \Sigma K \end{array}$$

is commutative up to homotopy, where

$$\Psi_n : K/K_{2n-1} \rightarrow \Omega \Sigma K$$

is the induced map from cofibration

$$K_{2n-1} \hookrightarrow K \rightarrow K/K_{2n-1}$$

for each $n = 1, 2, 3, \dots$. Since K/K_{2n-1} is $(2n - 1)$ -connected, every homology class in $H_{2n}(K/K_{2n-1})$ is spherical and thus primitive. Therefore, we can see that the homomorphic images of Ψ_{n*} and $\hat{\varphi}_{n*}$ lie in the set $PH_{2n}(\Omega \Sigma K)$ of primitive homology classes in $H_{2n}(\Omega \Sigma K)$ for each $n = 1, 2, 3, \dots$. $\square$

Lemma 3.3. *The map $[\varphi_m, \varphi_n]|_{(\Sigma K)_{2m+2n}} : (\Sigma K)_{2m+2n} \rightarrow \Sigma K$ is inessential.*

Proof. If we apply the adjointness

$$\widehat{[\varphi_m, \varphi_n]} = [\hat{\varphi}_m, \hat{\varphi}_n] : K \rightarrow \Omega \Sigma K$$

of the commutator $[\varphi_m, \varphi_n]$, then, by Lemma 3.1, the commutator $[\hat{\varphi}_m, \hat{\varphi}_n]$ restricts to the inessential map on the skeleton $K_{2m+2n-1}$. The adjointness and the fact that K has no cells in odd dimensions show the proof. $\square$

Lemma 3.4. *Let $E : K \rightarrow \Omega \Sigma K$ be the canonical map. Then we have*

$$\Omega[\varphi_m, \varphi_n] \circ E = [\hat{\varphi}_m, \hat{\varphi}_n].$$

Proof. As the adjoint of the identity function

$$1 : \Sigma K \rightarrow \Sigma K,$$

the canonical map sends

$$k \mapsto E(k) : I \rightarrow \Sigma K,$$

and

$$E(k)(t) = (k, t) \in \Sigma K.$$

From the map

$$\Omega[\varphi_m, \varphi_n] \circ E : K \rightarrow \Omega \Sigma K,$$

the map

$$(\Omega[\varphi_m, \varphi_n] \circ E)(k) : I \rightarrow \Sigma K$$

sends $t \in I$ to

$$(\Omega[\varphi_m, \varphi_n] \circ E)(k)(t) = [\varphi_m, \varphi_n](E(k)(t)) = [\varphi_m, \varphi_n](k, t).$$

On the other hand,

$$[\hat{\varphi}_m, \hat{\varphi}_n] : K \rightarrow \Omega \Sigma K$$

sends $k \in K$ to

$$[\hat{\varphi}_m, \hat{\varphi}_n](k) : I \rightarrow \Sigma K.$$

Here,

$$([\hat{\varphi}_m, \hat{\varphi}_n](k))(t) = (\widehat{[\varphi_m, \varphi_n]}(k))(t) = [\varphi_m, \varphi_n](k, t),$$

where $(k, t) \in \Sigma K$. Therefore, we have

$$\Omega[\varphi_m, \varphi_n] \circ E = [\hat{\varphi}_m, \hat{\varphi}_n],$$

as required. $\square$

In general, it can be noted that if X and Y are 1-connected CW-spaces, they admit a P -localization [21], and thus, the aforementioned lemmas are still guaranteed for the localized and rationalized counterparts. Indeed, it is well known that a map is an H-map if and only if its localizations are all H-maps.

We now prove the theorem. First, by the lemmas above, we can see that $[\hat{\varphi}_m, \hat{\varphi}_n]_*(b_{m+n})$ is primitive for any positive integers m and n . Second, to show that it is a decomposable element, we will use the homology suspension [26, p. 373]

$$\theta : H_{2m+2n}(\Omega\Sigma K) \longrightarrow H_{2m+2n+1}(\Sigma K).$$

Since the following diagram

$$\begin{array}{ccc} H_{2m+2n}(\Omega S^{2m+2n+1}) & \xrightarrow{\theta'} & H_{2m+2n+1}(S^{2m+2n+1}) \\ \Omega f_* \downarrow & \circlearrowleft & \downarrow f_* \\ H_{2m+2n}(\Omega\Sigma K) & \xrightarrow{\theta} & H_{2m+2n+1}(\Sigma K) \end{array}$$

is strictly commutative for any homotopy class $f : S^{2m+2n+1} \rightarrow \Sigma K$, it can be shown that if $x \in H_{2m+2n+1}(\Sigma K)$ is spherical, then x is an element of the image of the homology suspension $\theta : H_{2m+2n}(\Omega\Sigma K) \rightarrow H_{2m+2n+1}(\Sigma K)$. In particular, an element $s\beta_{m+n}$ of the image of the homology suspension

$$\theta : H_{2m+2n}(\Omega\Sigma K) \rightarrow H_{2m+2n+1}(\Sigma K)$$

is spherical and it sends all the decomposable elements to the zero element. Therefore, since b_{m+n} is an indecomposable homology class in homology dimension $2m + 2n$, we obtain

$$\theta(b_{m+n}) = \lambda s\beta_{m+n}$$

for $m, n = 1, 2, 3, \dots$, where $\lambda \neq 0$. On the other hand, the homomorphism

$$[\varphi_m, \varphi_n]_* : H_{2m+2n+1}(\Sigma K) \rightarrow H_{2m+2n+1}(\Sigma K)$$

induced by the commutator map $[\varphi_m, \varphi_n]$ is trivial except in degree zero. Lemma 3.4 and the following commutative diagram

$$\begin{array}{ccc} H_{2m+2n}(\Omega\Sigma K) & \xrightarrow{\theta} & H_{2m+2n+1}(\Sigma K) \\ \Omega[\varphi_m, \varphi_n]_* \downarrow & \circlearrowleft & \downarrow [\varphi_m, \varphi_n]_* \\ H_{2m+2n}(\Omega\Sigma K) & \xrightarrow{\theta} & H_{2m+2n+1}(\Sigma K) \end{array}$$

indicate that the homology classes $[\hat{\varphi}_m, \hat{\varphi}_n]_*(b_{m+n})$ should be decomposable for $m, n = 1, 2, 3, \dots$, as required. $\square$

3.2 Proof of Example 2.2

We now prove the example in the case of homology class $\beta_1 \in H_2(K)$, where $K = K(\mathbb{Z}, 2), K(\mathbb{Z}_{(p)}, 2)$ or $K(\mathbb{Q}, 2)$.

Proof. We recall that

$$\hat{\varphi}_1(\beta_1) = b_1$$

and that

$$\hat{\varphi}_1 * \hat{\varphi}_1^{-1} = \mu \circ (\hat{\varphi}_1 \times \hat{\varphi}_1^{-1}) \circ \Delta : K \rightarrow \Omega\Sigma K$$

is inessential; that is,

$$\hat{\varphi}_1 * \hat{\varphi}_1^{-1} \simeq c_{k_0},$$

where c_{k_0} is the constant map and μ is the loop multiplication on $\Omega\Sigma K$.

Let “ $\diamond$ ” be the product in the Pontryagin algebra. Then, in the level of homology up to the Pontryagin algebra, we have

$$\begin{aligned}
0 &= c_{k0*}(\beta_1) \\
&= \mu_*(\hat{\varphi}_1 \times \hat{\varphi}_1^{-1})_* \Delta_*(\beta_1) \\
&= \mu_*(\hat{\varphi}_1 \times \hat{\varphi}_1^{-1})_*(\beta_1 \otimes 1 + 1 \otimes \beta_1) \\
&= \mu_*(\hat{\varphi}_{1*}(\beta_1) \otimes 1 + 1 \otimes (\hat{\varphi}_1^{-1})_*(\beta_1)) \\
&= \hat{\varphi}_{1*}(\beta_1) \diamond 1 + 1 \diamond (\hat{\varphi}_1^{-1})_*(\beta_1) \\
&= \hat{\varphi}_{1*}(\beta_1) + (\hat{\varphi}_1^{-1})_*(\beta_1) \\
&= b_1 + (\hat{\varphi}_1^{-1})_*(\beta_1)
\end{aligned}$$

so that

$$(\hat{\varphi}_1^{-1})_*(\beta_1) = -b_1.$$

By using the Künneth theorem in homology, we also obtain

$$\begin{aligned}
[\hat{\varphi}_1, \hat{\varphi}_2]_*(\beta_1) &= (\hat{\varphi}_1 * \hat{\varphi}_2 * \hat{\varphi}_1^{-1} * \hat{\varphi}_2^{-1})_*(\beta_1) \\
&= \mu_*[\mu_*(\hat{\varphi}_1 \times \hat{\varphi}_2)_* \Delta_* \times \mu_*(\hat{\varphi}_1^{-1} \times \hat{\varphi}_2^{-1})_* \Delta_*] \Delta_*(\beta_1) \\
&= \mu_*[\mu_*(\hat{\varphi}_1 \times \hat{\varphi}_2)_* \Delta_* \times \mu_*(\hat{\varphi}_1^{-1} \times \hat{\varphi}_2^{-1})_* \Delta_*](\beta_1 \otimes 1 + 1 \otimes \beta_1) \\
&= \mu_*[\mu_*(\hat{\varphi}_1 \times \hat{\varphi}_2)_* \Delta_*(\beta_1) \otimes 1 + 1 \otimes \mu_*(\hat{\varphi}_1^{-1} \times \hat{\varphi}_2^{-1})_* \Delta_*(\beta_1)] \\
&= \mu_*[\{\mu_*(\hat{\varphi}_1 \times \hat{\varphi}_2)_*(\beta_1 \otimes 1 + 1 \otimes \beta_1)\} \otimes 1 + 1 \otimes \{\mu_*(\hat{\varphi}_1^{-1} \times \hat{\varphi}_2^{-1})_*(\beta_1 \otimes 1 + 1 \otimes \beta_1)\}] \\
&= \mu_*[\{\mu_*(\hat{\varphi}_{1*}(\beta_1) \otimes 1) + \mu_*(1 \otimes \hat{\varphi}_{2*}(\beta_1))\} \otimes 1] + \mu_*[1 \otimes \{\mu_*((\hat{\varphi}_1^{-1})_*(\beta_1) \otimes 1) + \mu_*(1 \otimes (\hat{\varphi}_2^{-1})_*(\beta_1))\}] \\
&= \mu_*[\mu_*(\hat{\varphi}_{1*}(\beta_1) \otimes 1) \otimes 1 + \mu_*(1 \otimes \hat{\varphi}_{2*}(\beta_1)) \otimes 1] + \mu_*[1 \otimes \mu_*((\hat{\varphi}_1^{-1})_*(\beta_1) \otimes 1) \\
&\quad + 1 \otimes \mu_*(1 \otimes (\hat{\varphi}_2^{-1})_*(\beta_1))] \\
&= \mu_*[(\hat{\varphi}_{1*}(\beta_1) \diamond 1) \otimes 1 + (1 \diamond \hat{\varphi}_{2*}(\beta_1)) \otimes 1] + \mu_*[1 \otimes ((\hat{\varphi}_1^{-1})_*(\beta_1) \diamond 1) + 1 \otimes (1 \diamond (\hat{\varphi}_2^{-1})_*(\beta_1))] \\
&= \mu_*[\hat{\varphi}_{1*}(\beta_1) \otimes 1 + \hat{\varphi}_{2*}(\beta_1) \otimes 1] + \mu_*[1 \otimes (\hat{\varphi}_1^{-1})_*(\beta_1) + 1 \otimes (\hat{\varphi}_2^{-1})_*(\beta_1)] \\
&= [\hat{\varphi}_{1*}(\beta_1) \diamond 1 + \hat{\varphi}_{2*}(\beta_1) \diamond 1] + [1 \diamond (\hat{\varphi}_1^{-1})_*(\beta_1) + 1 \diamond (\hat{\varphi}_2^{-1})_*(\beta_1)] \\
&= \hat{\varphi}_{1*}(\beta_1) + \hat{\varphi}_{2*}(\beta_1) + (\hat{\varphi}_1^{-1})_*(\beta_1) + (\hat{\varphi}_2^{-1})_*(\beta_1) \\
&= b_1 + \hat{\varphi}_{2*}(\beta_1) - b_1 + (\hat{\varphi}_2^{-1})_*(\beta_1) \\
&= b_1 + 0 - b_1 + 0 \\
&= 0,
\end{aligned}$$

in the Pontryagin algebra, as required, and similarly for the other cases. $\square$

4 Conclusion

In topology and its applications, the loop functor Ω and the suspension functor Σ are nice examples of adjoint functors in pointed (equivariant) homotopy category. Furthermore, H-spaces and co-H-spaces as dual notions of each other have been the main mathematical objects that are widely used in research on modern topology and its applications.

In this paper, after constructing the self-maps $\hat{\varphi}_n : \Sigma \mathbb{C}P^\infty \rightarrow \Sigma \mathbb{C}P^\infty$ on the suspension of the infinite complex projective space for each positive integer n , we have shown that the image of the homomorphism $[\hat{\varphi}_m, \hat{\varphi}_n]_*$ in homology induced by the adjoint $[\hat{\varphi}_m, \hat{\varphi}_n] : \mathbb{C}P^\infty \rightarrow \Omega \Sigma \mathbb{C}P^\infty$ of the commutator $[\varphi_m, \varphi_n]$ is both primitive and decomposable. To support the main theorem, we have provided an example at the end of this paper.

We hope that our methods will be used to investigate the properties of algebras along with Hopf algebras and to construct a minimal model of a differential graded algebra. We also hope that the results

will be applied to the concepts of various mathematical structures on the (algebraic) objects in many areas of algebra, topology, and differential equations.

Acknowledgements: The author is grateful to the anonymous referees for a careful reading and many helpful suggestions that improved the quality of this paper.

Funding information: This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korean government (MSIT) (No. 2018R1A2B6004407).

Conflict of interest: Author states no conflict of interest.

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