Research Article

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A new characterization of the automorphism groups of Mathieu groups

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Abstract: Let $\text{cd}(G)$ be the set of irreducible complex character degrees of a finite group G . $\rho(G)$ denotes the set of primes dividing degrees in $\text{cd}(G)$. For any prime p , let $p^{e_p(G)} = \max\{\chi(1)_p \mid \chi \in \text{Irr}(G)\}$ and $V(G) = \{p^{e_p(G)} \mid p \in \rho(G)\}$. The degree prime-power graph $\Gamma(G)$ of G is a graph whose vertices set is $V(G)$, and two vertices $x, y \in V(G)$ are joined by an edge if and only if there exists $m \in \text{cd}(G)$ such that $xy \mid m$. It is an interesting and difficult problem to determine the structure of a finite group by using its degree prime-power graphs. Qin proved that all Mathieu groups can be uniquely determined by their orders and degree prime-power graphs. In this article, we continue this topic and successfully characterize all the automorphism groups of Mathieu groups by using their orders and degree prime-power graphs.

Keywords: finite group, character degree, degree prime-power graph, automorphism group

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1 Introduction

The groups involved in this paper are all finite groups, and all characters are complex characters.

We use $\text{Irr}(G)$ to denote the set of all complex irreducible characters of a group G , and $\text{cd}(G) = \{\chi(1) \mid \chi \in \text{Irr}(G)\}$ to denote the set of all irreducible character degrees of G forgetting multiplicities. In particular, $\text{cd}^*(G)$ denotes a multiple set whose elements can be repeated and $|\text{cd}^*(G)| = |\text{Irr}(G)|$. Denote by $\rho(G)$ the set of primes dividing degrees in $\text{cd}(G)$. We use $\text{Out}(G)$ to denote the outer automorphism group of a group G . We use $\pi(n)$ to denote the set of all prime divisors of n and n_p the maximum power of p such that $n_p \mid n$, where n is a positive integer and p is a prime. Let $\mathfrak{P}(n) = \{n_p \mid p \in \pi(n)\}$. For convenience, we write $\mathfrak{P}(G) = \mathfrak{P}(|G|)$ and $\pi(G) = \pi(|G|)$. All other symbols and terms are standard (see [1, 2]).

In 2000, Huppert proposed the following conjecture:

Huppert's conjecture: Let M be a non-abelian simple group such that $\text{cd}(G) = \text{cd}(M)$, then $G \cong M \times A$, where A is an abelian group.

Huppert conjectured that all finite non-abelian simple groups can be uniquely determined by their orders and the sets of irreducible character degrees. Huppert checked the conjecture case-by-case for many non-abelian simple groups such as $\text{Sz}(q)$, the alternating groups A_n with $5 \leq n \leq 11$, and most of the sporadic simple groups and some other simple groups of Lie type (see [3–5]). Tong-Viet and Wakefield

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proved that if M is one of $G_2(q)$ ($q \geq 7$), $PSL_3(q)$, $PSU_3(q^2)$, ${}^2G^2(q^2)$, and $PSp_4(q)$ ($q > 7$), then the conjecture holds (see [6–9]). Nguyen continued Huppert's work and proved the conjecture for the alternating groups A_n with $n = 12, 13$ (see [10]). Bessenrodt et al. solved the remaining alternating groups and showed that Huppert's conjecture follows for all the simple alternating groups A_n ($n \geq 14$) (see [11]). In addition, Daneshkhah proved that all the sporadic simple groups Co_1 , Co_2 , Co_3 , and Fi_{23} can be uniquely characterized by the set of their irreducible character degrees (see [12,13]). However, up to now, Huppert's conjecture is still open. So, a natural problem is what the influence on the structure of a finite group is by weakening the condition of Huppert's conjecture. In particular, an interesting question is that if G and M have the same order, when we just consider some subset of $cd(M)$, whether we can determine the structure of such finite groups? Some people found that many non-abelian simple groups can be characterized by their orders and some largest irreducible character degrees. For example, Xu proved that simple K_3 -groups and Mathieu groups can be uniquely determined by their orders and one or two irreducible character degrees (see [14,15]). In addition, Heydari characterized simple K_4 -groups according to their orders and at most three distinct irreducible character degrees (see [16]).

The character degree graph of G , which is denoted by $\Delta(G)$, is a graph with the vertex set $\rho(G)$ and two vertices x and y are adjacent in $\Delta(G)$, if there is some $f \in cd(G)$ such that $xy|f$ (see [17]). Many researchers began to investigate the relationship between $\Delta(G)$ and structure of finite group, trying to know about the properties of $\Delta(G)$. Khosravi et al. proved that A_5 , A_6 , A_7 , A_8 , $L_3(3)$, $L_3(4)$, $L_2(64)$, $L_2(q)$ (where q is an odd prime or a square of an odd prime, and $q \geq 5$), and $L_2(2^\alpha)$ (where α is a positive integer such that $2^\alpha - 1$ or $2^\alpha + 1$ is a prime) can be determined by their orders and character degree graphs (see [18,19]). Furthermore, the authors also proved that some simple groups of orders less than 6,000 are uniquely determined by their orders and character degree graphs (see [20]).

But an interesting fact is that not all non-abelian simple groups can be uniquely determined by their orders and character degree graphs. We knew that the Mathieu M_{11} , M_{22} , and M_{23} can be uniquely determined by their orders and character degree graphs, while M_{12} cannot be determined by the order of M_{12} and the character degree graph $\Delta(M_{12})$ (see [21]). In fact, M_{12} and $A_4 \times M_{11}$ have the same orders and the same character degree graphs. So it is a difficult problem whether there exists some graph such that any Mathieu group can be uniquely determined by the graph. Based on this fact, in 2018, Qin et al. for the first time put forward the degree prime-power graph via the set of irreducible character degrees. Also, the authors successfully characterized all the Mathieu groups and sporadic simple groups just by using their orders and character degree prime-power graphs (see [22,23]). In this article, we continue this topic and prove that all the automorphism groups of Mathieu groups can also be uniquely determined by their orders and character degree prime-power graphs.

We first give the following definition:

Definition 1.1. Let G be a group. For every $p \in \rho(G)$, let $p^{e_p(G)} = \max\{\chi(1)_p \mid \chi \in \text{Irr}(G)\}$ and $V(G) = \{p^{e_p(G)} \mid p \in \rho(G)\}$. Define the degree prime-power graph $\Gamma(G)$ as follows: $V(G)$ is the vertex set, and there is an edge between distinct numbers $x, y \in V(G)$ if xy divides some integer in $cd(G)$. Denote the edge between distinct numbers $x, y \in V(G)$ by $x \sim y$, and the set of all edges of $\Gamma(G)$ by $E(G)$.

Definition 1.2. Let G be a group, for any $p \in \rho(G)$. We define $v_p(G)$ the p -exponent variation of G as follows: $v_p(G) = \min\left\{\log_p\left(\frac{|G|}{\chi(1)}\right) \mid \chi(1) \in \text{Irr}(G)\right\}$. Obviously, $v_p(G) = \log_p(|G|_p) - e_p(G)$.

Definition 1.3. Let G be a group and $p \in \pi(G)$. An irreducible character χ has p -defect zero if $\chi(1)_p = |G|_p$.

In this article, we successfully characterize all the automorphism groups of Mathieu groups by using their orders and degree prime-power graphs. Our main result is:

Theorem 1.4. Let G be a group. Suppose that M is one of the automorphism groups of a Mathieu group. Then $G \cong M$ if and only if $|G| = |M|$ and $\Gamma(G) = \Gamma(M)$.

2 Preliminary results

In what follows, we need to make some preparations for the proof of Theorem 1.4 and we begin with some important lemmas which will be used in what follows.

Lemma 2.1. [22] Suppose there is an irreducible character of G with p -defect zero for some $p \in \pi(G)$. If $N \trianglelefteq G$ and $p \in \pi(N)$, then there is an irreducible character of N which has p -defect zero, and $O_p(N) = 1$.

Lemma 2.2. [22] Let G be a finite solvable group of order $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, where $p_1, p_2, \dots, p_n$ are distinct primes. If $kp_n + 1 \nmid p_i^{\alpha_i}$ for each $1 \leq i \leq n-1$ and $k \in \mathbb{Z}^*$, then the Sylow p_n -subgroup is normal in G .

Lemma 2.3. [22] Let G be a non-solvable group. If T/S is a non-abelian chief factor of G , then there is a normal series $1 \leq H < K \leq G$ such that $K/H \cong T/S$ and $|G/K| \mid |\text{Out}(T/S)|$.

Lemma 2.4. [23] Let G be a group. For any $p \in \pi(G) \setminus (2)$, we have $v_p(G) = 0$, but $v_2(G) = 2$. If N is an arbitrary non-unit and solvable subnormal subgroup of group G , then N satisfies one of the following conditions:

- (1) $|N| = 2$ or 4 ;
- (2) $|N| = 2^3$, and $\text{cd}^*(N) = \{1, 1, 1, 1, 2\}$;
- (3) $|N| = 2^2 \cdot 3$, and $\text{cd}^*(N) = \{1, 1, 1, 3\}$;
- (4) $|N| = 2^3 \cdot 3$, and $\text{cd}^*(N) = \{1, 1, 1, 2, 2, 2, 3\}$ or $\text{cd}^*(N) = \{1, 1, 2, 3, 3\}$;
- (5) $|N| = 2^4 \cdot 3$, and $\text{cd}^*(N) = \{1, 1, 2, 2, 2, 2, 3, 3, 4\}$.

Lemma 2.5. [22] Let G be a group of order $672 = 2^5 \cdot 3 \cdot 7$. If $3, 7 \in V(G)$, then one of the following holds:

- (1) The number of irreducible characters with same degree is less than 11;
- (2) G has a normal abelian Sylow 2-subgroup.

Corollary 2.6. [22] Let G be a non-solvable group. Then there is a subnormal series

$$G = G_0 \geq G_1 > G_2 \geq \cdots \geq G_{2k-1} > G_{2k} \geq 1 \quad (k \geq 1)$$

such that G_{2k} is solvable, G_{2i-1}/G_{2i} is a non-abelian chief factor of G_{2i-2} and $|G_{2i-2}/G_{2i-1}| \mid |\text{Out}(G_{2i-1}/G_{2i})|$ for each $1 \leq i \leq k$.

3 Proof of Theorem 1.4

Remark. If G is a simple group, then $G \cong \text{Inn}(G)$. By [1], it is easy to check that $|\text{Out}(M_{11})| = 1$, $|\text{Out}(M_{23})| = 1$ and $|\text{Out}(M_{24})| = 1$. Hence, in what follows we just need to discuss the automorphism groups of the Mathieu groups M_{12} and M_{22} .

Proof of Theorem 1.4. The necessity of Theorem 1.4 is obvious and so it is enough to prove the sufficiency. In what follows, we write up the proof of what M is case-by-case.

Case 1. $M = \text{Aut}(M_{12})$.

From [1], we have $|G| = 2^7 \cdot 3^3 \cdot 5 \cdot 11$, $V(G) = \{2^5, 3^3, 5, 11\}$, and $E(G) = \{5 \sim 11\}$. By Lemma 2.1, $O_3(G) = O_5(G) = O_{11}(G) = 1$. We claim that G is non-solvable. Otherwise, if the group G is solvable, a Sylow 11-subgroup G_{11} of G is normal in G by Lemma 2.2. This is in contradiction with $O_{11}(G) = 1$.

By Lemma 2.3, there is a normal series $1 \leq H < K \leq G$ such that K/H is a non-abelian chief factor of G and $|G/K| \mid |\text{Out}(K/H)|$. By comparing the order of G and the orders of the simple groups in [1], every non-abelian chief factor of G is isomorphic to one of the following groups: A_5 , A_6 , $L_2(11)$, M_{11} , and M_{12} . And the non-abelian chief factors of G are pairwise non-isomorphic.

If $K/H \cong A_5$, by $|A_5| = 2^2 \cdot 3 \cdot 5$, and $|\text{Out}(A_5)| = 2$, then $|G/K| \geq 2$, which implies that $|H| = 2^\alpha \cdot 3^2 \cdot 11$, where $\alpha = 4$ or 5 . Since the order of the group H cannot be divisible by the order of any non-abelian simple group, H is solvable. By Lemma 2.2 a Sylow 11-subgroup of H is normal in H , and this is in contradiction with $O_{11}(G) = 1$.

If $K/H \cong A_6$, by $|A_6| = 2^3 \cdot 3^2 \cdot 5$, and $|\text{Out}(A_6)| = 4$, we have $|G/K| \geq 4$, and so $|H| = 2^\alpha \cdot 3 \cdot 11$, where $\alpha = 2$, 3 , or 4 . For the same reasons as above, we also get a contradiction.

If $K/H \cong L_2(11)$, since $|L_2(11)| = 2^2 \cdot 3 \cdot 5 \cdot 11$ and $|\text{Out}(L_2(11))| = 2$, then $|G/K| \geq 2$. One has that $|H| = 2^\alpha \cdot 3^2$, where $\alpha = 4$ or 5 . Then H is solvable. By Lemma 2.4, it is easy to deduce that there exists no such group $|H|$ such that H satisfies the above condition, and hence H is unsolvable, which leads to a contradiction.

Assume that $K/H \cong M_{11}$, by $|M_{11}| = 2^4 \cdot 3^2 \cdot 5 \cdot 11$, and $|\text{Out}(M_{11})| = 1$, $|G/K| \geq 1$, we have $|K| = |G|$, $G/H \cong M_{11}$, and $|H| = 2^3 \cdot 3$. By $3^3 \in V(G)$ and Lemma 2.1, there is a character $\theta \in \text{Irr}(H)$ such that $3|\theta(1)|$, then $\theta(1) = 3$. According to $v_3(H) = v_3(G) = 0$, then θ is G -invariant. Since the Schur multiplier $H(M_{11}, \mathbb{C}^\times)$ of M_{11} is 1, θ is extendible to G by [2, Theorem 11.7]. Note that $3^2 \cdot 5 \in \text{cd}(M_{11})$. Then, we have $3^3 \cdot 5 \in \text{cd}(G)$ by [2, Corollary 6.17]. This is in contradiction with $3^3 \sim 5 \notin E(G)$.

Now, assume that $K/H \cong M_{12}$, by $|M_{12}| = 2^6 \cdot 3^3 \cdot 5 \cdot 11$, and $|\text{Out}(M_{12})| = 2$, $|G/K| \geq 2$. This means that $|H| = 2^\alpha$, where $\alpha = 0$ or 1 . By N/C theorem, we have $G/H \leq \text{Aut}(K/H)$. So we get $K/H \leq G/H \leq \text{Aut}(K/H)$, that is, $M_{12} \leq G/H \leq \text{Aut}(M_{12})$.

If $G/H \cong \text{Aut}(M_{12})$, and since $|G| = |\text{Aut}(M_{12})|$, we deduce $H = 1$ and $G \cong \text{Aut}(M_{12})$.

If $G/H \cong M_{12}$, then $|H| = 2$. So $H \leq \mathbf{Z}(G)$. Therefore, G is a central extension of $\mathbf{Z}_2$ by M_{12} and G is isomorphic to one of the following groups:

$$\begin{aligned} &2 \cdot M_{12} \text{ (a non-split extension of } \mathbf{Z}_2 \text{ by } M_{12}), \\ &\mathbf{Z}_2 \times M_{12} \text{ (a split extension of } \mathbf{Z}_2 \text{ by } M_{12}). \end{aligned}$$

If $G \cong 2 \cdot M_{12}$, by [1], we have $\chi \in \text{Irr}(2 \cdot M_{12})$ such that $\chi(1)_2 = 2^6$. This is in contradiction with $V(G)$.

If $G \cong \mathbf{Z}_2 \times M_{12}$, by [1] and [2, Theorem 4.21], we have $\chi \in \text{Irr}(\mathbf{Z}_2 \times M_{12})$ such that $\chi(1)_2 = 2^4$. This is in contradiction with $V(G)$.

Case 2. $M = \text{Aut}(M_{22})$.

From [1], we have $|G| = 2^8 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$, $V(G) = \{2^4, 3^2, 5, 7, 11\}$, and $E(G) = \{2^4 \sim 5, 2^4 \sim 7, 3^2 \sim 5, 3^2 \sim 11, 5 \sim 7, 5 \sim 11, 7 \sim 11\}$. By Lemma 2.1, $O_3(G) = O_5(G) = O_7(G) = O_{11}(G) = 1$. We claim that G is non-solvable. Otherwise, if the group G is solvable, a Sylow 11-subgroup G_{11} of G is normal in G by Lemma 2.2. This is in contradiction with $O_{11}(G) = 1$.

By Lemma 2.3, there is a normal series $1 \leq H < K \leq G$ such that K/H is a non-abelian chief factor of G and $|G/K| \mid |\text{Out}(K/H)|$. Since $|G| = 2^8 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$, the non-abelian chief factors of G are pairwise non-isomorphic, and by [1], we see that K/H is isomorphic to one of the following groups: A_5 , $L_3(2)$, A_6 , $L_2(8)$, $L_2(11)$, A_7 , M_{11} , A_8 , $L_3(4)$, and M_{22} .

The orders of the outer automorphisms of these simple groups are not divisible by 11. We claim that there is a non-abelian simple chief factor of G whose order is divisible by 11. Otherwise, there is a solvable subnormal subgroup N of G such that $11 \nmid |N|$ by Corollary 2.6. Then N has a normal Sylow 11-subgroup by Lemma 2.2. This contradicts $O_{11}(N) = 1$ by Lemma 2.1. Hence, G has a chief factor isomorphic to $L_2(11)$, M_{11} , or M_{22} .

Assume that G has a chief factor isomorphic to $L_2(11)$. By Lemma 2.3, there is a chief factor K/H of G such that $K/H \cong L_2(11)$. According to $|L_2(11)| = 2^2 \cdot 3 \cdot 5 \cdot 11$, and $|\text{Out}(L_2(11))| = 2$, $|G/K| \geq 2$, this shows that $|H| = 2^\alpha \cdot 3 \cdot 7$, where $\alpha = 5$ or 6 . Since the edge $5 \sim 11 \in E(G)$, we have that $5 \sim 11 \in E(K)$ by [2, Corollary 11.29]. Hence, there exists a character $\chi \in K$ such that $55 \mid \chi(1)$. Let $\xi \in \text{Irr}(H)$ be an irreducible constituent of χ_H . By [2, Theorem 6.2], we see that $\chi_H = e \sum_{i=1}^s \xi_i$ where $\xi = \xi_1, \xi_2, \dots, \xi_s$ are the distinct conjugates of ξ in K and $e = [\chi_H, \xi]$. If $s = |K : I_K(\xi)| \neq 1$, then there is a maximal subgroup T/H of K/H such that $I_K(\xi)/H \leq T/H$. Since the index of the maximal subgroup of $L_2(11)$ is 11, 12, or 55 (see [1]), s is divisible by 11, 12, or 55. Since $3^2, 7 \in V(G)$, we know $3, 7 \in V(H)$.

If $\alpha = 5$, $|H| = 2^5 \cdot 3 \cdot 7$. By Lemma 2.5, we have $s < 11$, then $s = 1$, $\chi_H = e\xi$. Since $\xi(1) \nmid |H|$, $5 \cdot 11 \nmid \xi(1)$, then $55|e$ and $55^2 \leq [\chi_H, \chi_H] \leq |K : H|[\chi, \chi] = 2^2 \cdot 3 \cdot 5 \cdot 11$, where χ_H is the restriction of χ to H , a contradiction. Therefore, H has a normal abelian Sylow 2-subgroup H_2 . By [2, Theorem 6.15], we have $2^4 ||G : H_2|$, i.e., $2^4 | 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$, a contradiction too.

If $\alpha = 6$, then $|H| = 2^6 \cdot 3 \cdot 7$. Noting that $|G| = |K|$ and $s = |G : I_G(\xi)|$, we have $\chi(1) = es\xi(1)$. Since $\xi(1) \nmid |H|$, it follows that $5 \cdot 11 \nmid \xi(1)$, and hence $5 \cdot 11 | es$. If H is solvable, there exists $N \trianglelefteq H$ such that the normal subgroup of H/N is elementary abelian, so that $|N|_2 \leq 2^6$ and $|(H/N)|_2 \leq 2^6$. Since $|\text{GL}(6, 2)|$ is indivisible by 11, it implies that $L_2(11)$ induces the identity on N and H/N , and so $L_2(11)$ acts trivially on H . Consequently,

$$G = (H \circ \text{SL}_2(11)).O \text{ or } (H \times L_2(11)).O \text{ with } O \leq 2.$$

It follows that $H \leq I_G(\xi)$, and hence χ can be viewed as an irreducible character of G/H , which is impossible because 55 divides $\chi(1)$.

Thus, H is non-solvable. By Lemma 2.3 and [1], there is a normal series $1 \leq S < T \leq H$ such that T/S is a non-abelian chief factor of H , and $T/S \cong L_2(7)$, we get $|S| = 2^t$, where $t = 3$ or 2 . If $t = 3$, then $H/S \cong L_2(7)$, and hence $H \cong \text{AGL}_3(2)$ or $S \times L_2(7)$. According to $\mathbf{C}_G(H) \leq I_G(\xi)$, then $s ||G : \mathbf{C}_G(H)| = |G : I_G(\xi)| \cdot |I_G(\xi) : \mathbf{C}_G(H)|$. By N/C theorem, we have $G/\mathbf{C}_G(H) \leq \text{Aut}(H)$, that is, $s ||\text{Aut}(H)|$. In either case, $55 \nmid |\text{Aut}(H)|$, then $55|e$, again a contradiction. In a similar fashion, we rule out the possibility where $t = 2$.

Assume that G has a chief factor isomorphic to M_{11} . By $|M_{11}| = 2^4 \cdot 3^2 \cdot 5 \cdot 11$, $|\text{Out}(M_{11})| = 1$, we have $|G/K| \nmid 1$, $|K| = |G|$, then $|H| = 2^4 \cdot 7$. Since $v_7(H) = v_7(G) = 0$, H has a G -invariant irreducible character η of degree 7. Since the Schur multiplier $H(M_{11}, \mathbb{C}^\times)$ of M_{11} is 1, η is extendible to G . Since $3^2 \in V(M_{11})$, we see that $7 \sim 3^2 \in E(G)$ by [2, Theorem 6.17]. This is in contradiction with $7 \sim 3^2 \notin E(G)$.

Now, assume that $K/H \cong M_{22}$, by $|M_{22}| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$, $|\text{Out}(M_{22})| = 2$, we know $|G/K| \nmid 2$, then $|H| = 2^\alpha$, where $\alpha = 0$ or 1 . By N/C theorem, we have $G/H \leq \text{Aut}(K/H)$. So we get $K/H \leq G/H \leq \text{Aut}(K/H)$, that is $M_{22} \leq G/H \leq \text{Aut}(M_{22})$.

If $G/H \cong \text{Aut}(M_{22})$, and since $|G| = |\text{Aut}(M_{22})|$, we deduce $H = 1$ and $G \cong \text{Aut}(M_{22})$.

If $G/H \cong M_{22}$, then $|H| = 2$. So $H \leq \mathbf{Z}(G)$. Therefore, G is a central extension of $\mathbf{Z}_2$ by M_{22} and G is isomorphic to one of the following groups:

$$\begin{aligned} & 2 \cdot M_{22} \text{ (a non-split extension of } \mathbf{Z}_2 \text{ by } M_{22}), \\ & \mathbf{Z}_2 \times M_{22} \text{ (a split extension of } \mathbf{Z}_2 \text{ by } M_{22}). \end{aligned}$$

If $G \cong 2 \cdot M_{22}$, by [1], we have $\chi \in \text{Irr}(2 \cdot M_{22})$ such that $3^2 \cdot 7 | \chi(1)$, then $3^2 \sim 7 \in E(2 \cdot M_{22})$. This is in contradiction with $E(G)$.

If $G \cong \mathbf{Z}_2 \times M_{22}$, by [1] and [2, Theorem 4.21], we have $\chi \in \text{Irr}(\mathbf{Z}_2 \times M_{22})$ such that $\chi(1)_2 = 2^3$, a contradiction to $V(G)$, which completes the proof of Theorem 1.4. $\square$

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