Degenerate binomial and Poisson random variables associated with degenerate Lah-Bell polynomials

Abstract: The aim of this paper is to study the Poisson random variables in relation to the Lah-Bell polynomials and the degenerate binomial and degenerate Poisson random variables in connection with the degenerate Lah-Bell polynomials. Among other things, we show that the rising factorial moments of the degenerate Poisson random variable with parameter $\alpha$ are given by the degenerate Lah-Bell polynomials evaluated at $\alpha$. We also show that the probability-generating function of the degenerate Poisson random variable is equal to the generating function of the degenerate Lah-Bell polynomials. Also, we show similar results for the Poisson random variables. Here the $n$th Lah-Bell number counts the number of ways a set of $n$ elements can be partitioned into non-empty linearly ordered subsets, the Lah-Bell polynomials are natural extensions of the Lah-Bell numbers and the degenerate Lah-Bell polynomials are degenerate versions of the Lah-Bell polynomials.

Keywords: degenerate binomial random variable, degenerate Poisson random variable, degenerate Lah-Bell polynomial, probability

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1 Introduction

The aim of this paper is to study the Poisson random variables in relation to the Lah-Bell polynomials and the degenerate binomial and degenerate Poisson random variables in connection with the degenerate Lah-Bell polynomials. Here Lah-Bell polynomials $B_n^L(x)$ are natural extension of the Lah-Bell numbers $B_n^L$, which are defined as the number of ways a set of $n$ elements can be partitioned into non-empty linearly ordered subsets (see [1]). Thus, we have $B_n^L = \sum_{k=0}^{n} L(n,k)$, where $L(n,k)$ counts the number of ways a set of $n$ elements can be partitioned into $k$ non-empty linearly ordered subsets, called the unsigned Lah numbers (see [2]). The motivation for our introduction of the degenerate binomial and degenerate Poisson random variables is as follows. Let us assume that the probability of success in a trial is $p$. Then we might wonder if the probability of success in the ninth trial is still $p$ after failing eight times in the trial experiment. Because there is a psychological burden for one to be successful, it seems plausible that the probability is less than $p$. This speculation motivated our study of the degenerate binomial and degenerate Poisson random variables.
The outline of our main results is as follows. We derive the expectation and variance of the degenerate binomial and degenerate Poisson random variables. Then we introduce the degenerate Lah-Bell polynomials which are degenerate versions of the Lah-Bell polynomials. Then, among other things, we show that the rising factorial moments of the degenerate Poisson random variable with parameter $\alpha$ are given by the degenerate Lah-Bell polynomials evaluated at $\alpha$. We also show that the probability-generating function of the degenerate Poisson random variable is equal to the generating function of the degenerate Lah-Bell polynomials. In addition, we show that the rising factorial moments of the Poisson random variable with parameter $\alpha$ are given by the Lah-Bell polynomials evaluated at $\alpha$. Furthermore, we show that the probability-generating function of the Poisson random variable is equal to the generating function of the Lah-Bell polynomials.

The novelty of this paper is that it reveals the connection between the rising factorial moments of the Poisson random variable and the Lah-Bell polynomials and that between the rising factorial moments of the degenerate Poisson random variable and the degenerate Lah-Bell polynomials. For the rest of this section, we recall the necessary facts that will be needed throughout this paper.

For any $0 \neq \lambda \in \mathbb{R}$, the degenerate exponential functions are defined by (see [3])

$$e^{\xi}(t) = \sum_{k=0}^{\infty} (x)_{k,\lambda} t^k,$$

where $(x)_{0,1} = 1$, $(x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda), \quad (n \geq 1)$. Note that

$$\lim_{k \to 0} e^{\xi}(t) = e^{\xi t}, \quad e^{\xi}(t) = e^{\xi t}(t).$$

For $n, k \geq 0$, the unsigned Lah numbers are given by (see [1,2,4,5])

$$L(n, k) = \binom{n-1}{k-1} \frac{n!}{k!}.$$  

In [3], the Lah-Bell polynomials are defined by

$$e^{\lambda(t-1)} = \sum_{n=0}^{\infty} B^L_n(x) \frac{t^n}{n!}.$$  

For $x = 1$, $B^L_n = B^L_n(1) \quad (n \geq 0)$, are called the Lah-Bell numbers. Here we recall from [1] that $B^L_n$ counts the number of ways a set of $n$ elements can be partitioned into non-empty linearly ordered subsets. From (3), we note that (see [1])

$$B^L_n(x) = \sum_{k=0}^{n} x^k L(n, k), \quad (n \geq 0).$$

A sample space is the set of all possible outcomes of an experiment and an event is any subset of the sample space. A random variable $X$ is a real valued function on a sample space. If $X$ takes any values in a countable set, then $X$ is called a discrete random variable. If $X$ takes any values in an interval on the real line, then $X$ is called a continuous random variable.

For a discrete random variable $X$, the probability mass function $p(x)$ of $X$ is defined as (see [6–9])

$$p(x) = P(X = a).$$

Suppose that $n$ independent trials, each of which results in a “success” with probability $p$ and in a “failure” with probability $1 - p$, are to be performed. If $X$ denotes the number of successes that occur in $n$ trials, then $X$ is called the binomial random variable with parameter $n, p$, which is denoted by $X \sim B(n, p)$. Let $X \sim B(n, p)$. Then the probability mass function of $X$ is given by

$$p(i) = \binom{n}{i} p^i (1-p)^{n-i}, \quad i = 0, 1, 2, \ldots, n.$$  

A Poisson random variable indicates how many events occurred within a given period of time. A random variable $X$, taking on one of the values $0, 1, 2, \ldots$, is said to be the Poisson random variable with parameter $\alpha(>0)$, if the probability mass function of $X$ is given by (see [8])

$$p(i) = \frac{\alpha^i}{i!} e^{-\alpha}, \quad i = 0, 1, 2, \ldots.$$
\[ p(i) = e^{-\alpha} \frac{\alpha^i}{i!}. \] (7)

Let \( f(x) \) be a real valued function and let \( X \) be a random variable. Then we define (see [8])

\[ E[f(X)] = \sum_{i=0}^{\infty} f(i)p(i), \] (8)

where \( p(x) \) is the probability mass function of \( X \).

It is well known that the Bell polynomials are defined by (see [3,10,11])

\[ e^{te^{t-1}} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \] (9)

Let us take \( f(x) = x^n, \ (n \geq 0). \) Then we have the moments of the Poisson random variable \( X \) with parameter \( \alpha > 0 \) as follows (see [11]):

\[ E[X^n] = B_n(\alpha), \quad (n \geq 0). \] (10)

### 2 Poisson random variables

The falling factorial sequence is given by

\[ (x)_0 = 1, \quad (x)_n = x(x-1)\cdots(x-n+1), \quad (n \geq 1), \]

while the rising factorial sequence is given by (see [1–4,6–8,10–12])

\[ (x)^0 = 1, \quad (x)^n = x(x+1)\cdots(x+n-1), \quad (n \geq 1). \]

Replacing \( t \) by \( \log(1 + t) \) in (9), we get

\[ e^{tx} = \sum_{k=0}^{\infty} B_k(x) \frac{(\log(1 + t))^k}{k!} = \sum_{n=0}^{\infty} B_n(x) \sum_{n-k}^{\infty} S_1(n, k) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} B_k(x)S_1(n, k) \right) \frac{t^n}{n!}, \] (11)

where \( S_1(n, k) \) are the Stirling numbers of the first kind defined by

\[ (x)_n = \sum_{k=0}^{n} S_1(n, k)x^k, \quad (n \geq 0). \] (12)

Therefore, by (11), we obtain the following lemma.

**Lemma 1.** For \( n \geq 0 \), we have

\[ x^n = \sum_{k=0}^{n} S_1(n, k)B_k(x) \]

and

\[ B_n(x) = \sum_{k=0}^{n} S_2(n, k)x^k. \]

Let \( X \) be the Poisson random variable with parameter \( \alpha > 0 \). Then we have

\[ E[(X)_n] = \sum_{k=0}^{n} S_1(n, k)E[X^k] = \sum_{k=0}^{n} S_2(n, k)B_k(\alpha). \] (13)

From Lemma 1 and (13), we note the well-known fact about the falling factorial moments of the random variable \( X \), namely, the expectation of the falling factorial of the random variable \( X \):
\[ E[X_n] = \sum_{k=0}^{n} S_j(n, k) B_k(\alpha) = a^n, \quad (n \geq 0). \tag{14} \]

On the other hand, the rising factorial moment of \( X \), namely the expectation of the rising factorial of \( X \), is given by
\[
E\langle X \rangle_n = \sum_{k=0}^{\infty} \langle k \rangle_n p(k) = e^{-a} \sum_{k=0}^{\infty} \frac{\langle k \rangle_n}{k!} a^k. \tag{15} \]

From (3), we can derive the following equation:
\[
\sum_{n=0}^{\infty} B_n^L(a) \frac{t^n}{n!} = e^{-a} e^{a(\frac{1}{1-t})} \]
\[= e^{-a} \sum_{k=0}^{\infty} a^k \frac{1}{k!} \left( \frac{1}{1-t} \right)^k \]
\[= e^{-a} \sum_{k=0}^{\infty} \frac{a^k}{k!} \sum_{n=0}^{\infty} \frac{\langle k \rangle_n}{n!} t^n \]
\[= \sum_{n=0}^{\infty} \left( e^{-a} \sum_{k=0}^{\infty} \frac{\langle k \rangle_n a^k}{k!} \right) \frac{t^n}{n!}. \tag{16} \]

Comparing the coefficients on both sides of (16), we have the following identity:
\[ B_n^L(a) = e^{-a} \sum_{k=0}^{\infty} \frac{\langle k \rangle_n a^k}{k!}, \tag{17} \]
where \( n \) is a non-negative integer.

Therefore, by (14), (15) and (17), we obtain the following theorem. In particular, it shows that the rising factorial moments of the Poisson random variable with parameter \( \alpha \) are given by the Lah-Bell polynomials evaluated at \( a \). This fact seems to be new.

\[ \textbf{Theorem 2.} \text{ Let } X \text{ be the Poisson random variable with parameter } \alpha > 0. \text{ Then we have } \]
\[ E(X_n) = a^n \]
\[ \text{and} \]
\[ E\langle X \rangle_n = B_n^L(\alpha), \quad (n \geq 0). \]

Let \( X \) be the Poisson random variable with parameter \( \alpha > 0 \). From (7) and (8), we have
\[
E\left[ \left( \frac{1}{1-t} \right)^X \right] = \sum_{k=0}^{\infty} \left( \frac{1}{1-t} \right)^k p(k) = \sum_{k=0}^{\infty} \left( \frac{1}{1-t} \right)^k e^{-a} \frac{a^k}{k!} = e^{-a} e^{a(\frac{1}{1-t} - 1)}, \tag{18} \]

Now, by (3) and (18), we obtain the following theorem which says that the probability-generating function of \( X \) is equal to the generating function of the Lah-Bell polynomials.

\[ \textbf{Theorem 3.} \text{ Let } X \text{ be the Poisson random variable with parameter } \alpha > 0. \text{ Then we have } \]
\[ E\left[ \left( \frac{1}{1-t} \right)^X \right] = e^{a(\frac{1}{1-t} - 1)} = \sum_{n=0}^{\infty} B_n^L(\alpha) \frac{t^n}{n!}. \]

From Theorem 3 and (10), we note that
\[
\sum_{n=0}^{\infty} B_n^k(a) \frac{t^n}{n!} = \sum_{k=0}^{\infty} E[X_k] \left(-\log(1 - t)\right)^k \frac{k!}{k!} \\
= \sum_{k=0}^{\infty} (-1)^k B_k(a) \sum_{n=k}^{\infty} (-1)^n S(n, k) \frac{t^n}{n!} \\
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} (-1)^{n-k} S(n, k) B_k(a) \right) \frac{t^n}{n!}.
\]

Therefore, by Theorem 2, (4) and (19), we obtain the following theorem.

**Theorem 4.** Let \( X \) be the Poisson random variable with parameter \( \alpha > 0 \). Then we have

\[
E[X] = B_n^k(a) = \sum_{k=0}^{n} L(n, k) \alpha^k = \sum_{k=0}^{n} (-1)^{n-k} S(n, k) B_k(a).
\]

### 3 Degenerate binomial and degenerate Poisson random variables

In this section, we assume that \( \lambda \in (0, 1) \), and \( p \) is the probability of success of an experiment. For \( \lambda \in (0, 1) \), \( X_{\lambda} \) is the **degenerate binomial random variable with parameter** \( n, p \), denoted by \( X_{\lambda} \sim B(n, p) \), if the probability mass function of \( X_{\lambda} \) is given by

\[
p_{\lambda}(i) = P[X_{\lambda} = i] = \binom{n}{i} \left(p\right)_{i, \lambda} (1 - p)_{n-i, \lambda} \frac{1}{(1)_{n, \lambda}},
\]

where \( i = 0, 1, 2, \ldots, n \).

From (20), we note that

\[
\sum_{i=0}^{\infty} p_{\lambda}(i) = \frac{1}{(1)_{n, \lambda}} \sum_{i=0}^{n} \binom{n}{i} \left(p\right)_{i, \lambda} (1 - p)_{n-i, \lambda} = 1.
\]

For \( X_{\lambda} \sim B(n, k) \), we have

\[
E[X_{\lambda}] = \sum_{i=0}^{\infty} i p_{\lambda}(i) \\
= \frac{1}{(1)_{n, \lambda}} \sum_{i=0}^{\infty} \binom{n}{i} \left(p\right)_{i, \lambda} (1 - p)_{n-i, \lambda} (i) \\
= \frac{n}{(1)_{n, \lambda}} \sum_{i=0}^{\infty} \binom{n-1}{i} \left(p\right)_{i+1, \lambda} (1 - p)_{n-1-i, \lambda} \\
= \frac{np}{(1)_{n, \lambda}} \sum_{i=0}^{n-1} \binom{n-1}{i} \left(p - \lambda\right)_{i, \lambda} (1 - p)_{n-1-i, \lambda} \\
= \frac{np}{(1)_{n, \lambda}} (p - \lambda + 1 - p)_{n-1, \lambda} \\
= \frac{np}{(1)_{n, \lambda}} (1 - \lambda)_{n-1, \lambda}.
\]

Therefore, we obtain the following theorem.
Theorem 5. For $X_i \sim B_i(n, p)$, $(n \geq 0)$, we have

$$E[X_i] = \frac{np}{(1)_{n-\lambda}}(1 - \lambda)_{n-1,\lambda}.$$  

Note that

$$\lim_{\lambda \to 0} E[X_i] = np = E[X],$$

where $X$ is the binomial random variable with parameter $n, p$.

For $X_i \sim B_i(n, p)$, we observe that

$$E[X_i] = \sum_{i=0}^{\infty} i^2 p_i(i)$$

$$= \frac{1}{(1)_{n-\lambda}} \sum_{i=0}^{\infty} \binom{n}{i} p_{i,\lambda}(1 - p)_{n-i,\lambda}$$

$$= \frac{1}{(1)_{n-\lambda}} \sum_{i=0}^{\infty} i(i - 1 + 1) \binom{n}{i} p_{i,\lambda}(1 - p)_{n-i,\lambda}$$

$$= \frac{1}{(1)_{n-\lambda}} \sum_{i=0}^{\infty} i(i - 1) \binom{n}{i} p_{i,\lambda}(1 - p)_{n-i,\lambda} + E[X_i]$$

$$= n(n - 1) \sum_{i=0}^{\infty} \binom{n - 2}{i - 2} p_{i,\lambda}(1 - p)_{n-i,\lambda} + E[X_i]$$

$$= n(n - 1) \sum_{i=0}^{\infty} \binom{n - 2}{i} p_{i,\lambda} n_{-2,\lambda} + E[X_i]$$

$$= n(n - 1) p(p - \lambda) \sum_{i=0}^{\infty} \binom{n - 2}{i} (p - 2\lambda)_{i,\lambda}(1 - p)_{n-i,\lambda} + E[X_i]$$

$$= n(n - 1) p(p - \lambda) (p - 2\lambda + 1 - p)_{n-2,\lambda} + E[X_i]$$

$$= n(n - 1) p(p - \lambda) (p - 2\lambda + 1 - p)_{n-2,\lambda} + \frac{np}{(1)_{n-\lambda}} (1 - \lambda)_{n-1,\lambda}$$

$$= n p(1 - 2\lambda)_{n-2,\lambda} ((p - \lambda)(n - 1) + (1 - \lambda))$$

By using Theorem 5 and (22), the variance $\text{Var}(X_i)$ of the random variable $X_i$ is given by

$$\text{Var}(X_i) = E[X_i^2] - (E[X_i])^2$$

$$= \frac{np}{(1)_{n-\lambda}} (1 - 2\lambda)_{n-2,\lambda} (p(n - 1) + 1 - n\lambda) - \left( \frac{np}{(1)_{n-\lambda}} (1 - \lambda)_{n-1,\lambda} \right)^2$$

$$= \frac{np}{(1)_{n-\lambda}} (1 - 2\lambda)_{n-2,\lambda} (p(n - 1) + 1 + n\lambda) - \left( \frac{np}{(1)_{n-\lambda}} (1 - \lambda)(1 - 2\lambda)_{n-2,\lambda} \right)^2$$

$$= \frac{np}{(1)_{n-\lambda}} (1 - 2\lambda)_{n-2,\lambda} (p(n - 1) + 1 - n\lambda - \frac{np}{(1)_{n-\lambda}} (1 - \lambda)^2 (1 - 2\lambda)_{n-2,\lambda})$$

Therefore, we obtain the following theorem.
Theorem 6. For $X_{\lambda} \sim B(n, p)$, we have

$$\text{Var}(X_{\lambda}) = \frac{np}{(1-n)^2}((n-1)p + 1 - n\lambda - E[X_{\lambda}](1 - \lambda)).$$

Note that

$$\lim_{\lambda \to 0} \text{Var}(X_{\lambda}) = np(1 - p) = \text{Var}(X),$$

where $X$ is the binomial random variable with parameters $n, p$.

The generating function of the moments of $X_{\lambda} \sim B(n, p)$ is given by

$$E[X_{\lambda}^n] = \frac{1}{(1-n)^2} \sum_{i=0}^{n} e^{i\lambda}(1-p)^{n-i}p^i.$$

Thus, we have

$$E[X_{\lambda}^n] = \frac{\frac{d^n}{dn}E[e^{X_{\lambda}}]_{x=0}}{n!} = \frac{1}{(1-n)^2} \sum_{i=0}^{n} \binom{n}{i}i^n(1-p)^{n-i}p^i.$$

For $\lambda \in (0, 1)$, $X_{\lambda}$ is the degenerate Poisson random variable with parameter $\alpha(>0)$, if the probability mass function of $X_{\lambda}$ is given by

$$p_i(\lambda) = P[X_{\lambda} = i] = e^{i\lambda}(\alpha)\frac{\alpha^i}{i!}(1)_{\lambda, 1},$$

where $i = 0, 1, 2, \ldots$.

By (24), we get

$$\sum_{i=0}^{\infty} p_i(\lambda) = e^{\lambda}(\alpha)\frac{\alpha^0}{0!}(1)_{\lambda, 1} = 1.$$

It is easy to show that

$$E[X_{\lambda}] = \frac{\alpha}{1 + \alpha\lambda}$$

and

$$E[X_{\lambda}^2] = \frac{\alpha + \alpha^2}{(1 + \alpha\lambda)^2}.$$

Thus, we have

$$\text{Var}(X_{\lambda}) = E[X_{\lambda}^2] - (E[X_{\lambda}])^2 = \frac{\alpha}{(1 + \alpha\lambda)^2}.$$

Let $X_{\lambda}$ be the degenerate Poisson random variable with parameter $\alpha(>0)$. Then we have

$$E[\langle X_{\lambda}\rangle_n] = \sum_{i=0}^{\infty} \langle i \rangle_n p_i(\lambda) = \sum_{i=0}^{\infty} \langle i \rangle_n e^{i\lambda}(\alpha)\frac{(1)_{\lambda, 1}}{i!} \alpha^i = e^{i\lambda}(\alpha)\sum_{i=0}^{\infty} \langle i \rangle_n e^{i\lambda}(\alpha)\frac{(1)_{\lambda, 1}}{i!} \alpha^i. (25)$$

In view of (3), we may consider the degenerate Lah-Bell polynomials given by

$$e^{i\lambda}(x)e_x(x)\left(1 - \frac{1}{1-t}\right) = \sum_{n=0}^{\infty} B_{\lambda, n}(x)\frac{t^n}{n!}.$$ (26)

Note that

$$\sum_{n=0}^{\infty} \lim_{t \to 0} B_{\lambda, n}(x)\frac{t^n}{n!} = e^{i\lambda}(1-t) = \sum_{n=0}^{\infty} B_{n}(x)\frac{t^n}{n!}.$$
Thus, we have
\[ \lim_{\lambda \to 0} B_{n,\lambda}^L(x) = B_n^L(x), \quad (n \geq 0). \]

Now, we observe that
\[
e^{-t}(1 - t)^{\frac{x}{1-t}} = e^{-t}(1 - t)^{\frac{x}{1-t}} = e^{-t}(1 - t)^{\frac{x}{1-t}} = \sum_{n=0}^{\infty} \left( \frac{1}{1-t} \right)^n\frac{(k)_n}{n!} t^n = \sum_{n=0}^{\infty} \left( \frac{1}{1-t} \right)^n\frac{(k)_n}{n!} t^n.
\]

From (25), (26) and (27), we obtain the next result. In particular, it says that the rising factorial moments of the degenerate Poisson random variable with parameter \( \alpha \) are given by the degenerate Lah-Bell polynomials evaluated at \( \alpha \).

**Theorem 7.** For \( n \geq 0 \), we have
\[
B_{n,\lambda}^L(x) = e^{-t}(1 - t)^{\frac{x}{1-t}} = \sum_{n=0}^{\infty} \left( \frac{1}{1-t} \right)^n\frac{(k)_n}{n!} t^n = \sum_{n=0}^{\infty} \left( \frac{1}{1-t} \right)^n\frac{(k)_n}{n!} t^n.
\]

For the degenerate Poisson random variable \( X_\lambda \) with parameter \( \alpha > 0 \), we have
\[
\mathbb{E}(X_\alpha)_n = B_{n,\alpha}^L(x), \quad (n \geq 0).
\]

The degenerate Bell polynomials are defined in [3] as
\[
e^{-t}(1 - t)^{\frac{x}{1-t}} = \sum_{n=0}^{\infty} \left( \frac{1}{1-t} \right)^n\frac{(k)_n}{n!} t^n = \sum_{n=0}^{\infty} \left( \frac{1}{1-t} \right)^n\frac{(k)_n}{n!} t^n.
\]

Note that
\[
\sum_{n=0}^{\infty} B_{n,\lambda}(x) t^n = e^{e^t-1} = \sum_{n=0}^{\infty} B_n(x) t^n / n!,
\]

where \( B_n(x) \) are the ordinary Bell polynomials. Thus, we have
\[
\lim_{\lambda \to 0} B_{n,\lambda}(x) = B_n(x), \quad (n \geq 0).
\]

Replacing \( t \) by \(-\log(1 - t)\) in (28), we get
\[
e^{-t}(1 - t)^{\frac{x}{1-t}} = \sum_{n=0}^{\infty} \left( \frac{1}{1-t} \right)^n\frac{(k)_n}{n!} t^n = \sum_{n=0}^{\infty} \left( \frac{1}{1-t} \right)^n\frac{(k)_n}{n!} t^n.
\]

Therefore, by (26), (28) and (29), we obtain the following theorem.

**Theorem 8.** For \( n \geq 0 \), we have
\[
B_{n,\lambda}^L(x) = \sum_{k=0}^{n} (-1)^{n-k} S_2(n, k) B_{k,\lambda}(x)
\]

and
\[
B_{n,\lambda}(x) = \sum_{k=0}^{n} (-1)^{n-k} S_2(n, k) B_{k,\lambda}(x),
\]

where \( S_2(n, k), \quad (n, k \geq 0) \), are the Stirling numbers of the second kind defined by
\[
x^n = \sum_{k=0}^{n} S_2(n, k)x^k.
\]

From Theorem 11 of [3], we recall that

\[
B_{n,\lambda}(x) = \sum_{k=0}^{\infty} (1)_{1,\lambda} \left( \frac{x}{1 + \lambda x} \right)^k S_2(n, k).
\]

Combining (30) and (31), we have another expression for \( B_{n,\lambda}(x) \) as follows:

\[
B_{n,\lambda}^L(x) = \sum_{\lambda=0}^{n} \sum_{k=1}^{\infty} (-1)^n k^2 S_2(n, k) S_2(k, l) (1)_{1,\lambda} \left( \frac{x}{1 + \lambda x} \right)^l.
\]

Let \( X_\lambda \) be the degenerate Poisson random variable with parameter \( \alpha > 0 \). Then we have

\[
E\left( \left( \frac{1}{1 - t} \right)^{X_\lambda} \right) = \sum_{i=0}^{\infty} \left( \frac{1}{1 - t} \right)^i p(i) = e_\lambda^{-i}(\alpha) \sum_{i=0}^{\infty} (1)_{1,\lambda} \frac{\alpha^i t^i}{i!} \left( \frac{1}{1 - t} \right)^i = e_\lambda^{-i}(\alpha) e_\lambda \left( \frac{1}{1 - t} \right).
\]

Therefore, we obtain the following theorem from Theorem 7, (32) and (33). In particular, it states that the probability-generating function of \( X_\lambda \) is equal to the generating function of the degenerate Lah-Bell polynomials.

**Theorem 9.** Let \( X_\lambda \) be the degenerate Poisson random variable with parameter \( \alpha > 0 \). Then we have

\[
E\left( \left( \frac{1}{1 - t} \right)^{X_\lambda} \right) = e_\lambda^{-i}(\alpha) e_\lambda \left( \frac{1}{1 - t} \right)
\]

and

\[
E \langle X_\lambda \rangle_n = \sum_{l=0}^{\infty} \left( \sum_{i=0}^{\infty} (-1)^n k^2 S_2(n, k) S_2(k, l) (1)_{1,\lambda} \left( \frac{\alpha}{1 + \lambda \alpha} \right)^l \right), \quad (n \geq 0).
\]

### 4 Conclusion

In this paper, we introduced the degenerate Lah-Bell polynomials which are degenerate versions of the recently introduced Lah-Bell polynomials. As stated in the Section 1, the novelty of this paper is that it reveals the connection between the rising factorial moments of the Poisson random variable and the Lah-Bell polynomials and that between the rising factorial moments of the degenerate Poisson random variable and the degenerate Lah-Bell polynomials.

The details of the results obtained are as follows. For Poisson random variable \( X \) with parameter \( \alpha > 0 \), we showed that the rising factorial moments of \( X \) are given by the Lah-Bell polynomials evaluated at \( \alpha \) (Theorem 2) and also by an expression involving Bell polynomials evaluated at \( \alpha \) (Theorem 4). We also showed that the probability-generating function of \( X \) is equal to the generating function of the Lah-Bell polynomials (Theorem 3). Let \( X_\lambda \sim B_\lambda(n, p) \) be the degenerate binomial random variable with parameter \( n, p \). Then we derived the expectation of \( X_\lambda \) (Theorem 5) and the variance of \( X_\lambda \) (Theorem 6). Now, let \( X_\lambda \) denote the degenerate Poisson random variable with parameter \( \alpha \). Then we showed that the rising factorial moments of \( X_\lambda \) are given by the degenerate Lah-Bell polynomials evaluated at \( \alpha \) (Theorem 7) and also by another expression involving the Stirling numbers of both kinds (Theorem 9). We also showed that the probability-generating function of \( X_\lambda \) is equal to the generating function of the degenerate Lah-Bell
polynomials (Theorem 9). Furthermore, we obtained relations between the degenerate Lah-Bell polynomials and the degenerate Bell polynomials (Theorem 8).

Here, we would like to mention that studying various degenerate versions of some special numbers and polynomials, which was initiated by Carlitz when he investigated the degenerate Bernoulli and Euler polynomials and numbers, regained interests of some mathematicians in recent years. They have been studied by using several different tools like generating functions, combinatorial methods, $p$-adic analysis, umbral calculus, special functions, differential equations, and probability theory as we did in the present paper.

It is one of our future projects to continue to study various degenerate versions of some special polynomials and numbers and to find their applications in physics, science, and engineering as well as in mathematics.

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**References**


