Research Article

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On the mixed fractional quantum and Hadamard derivatives for impulsive boundary value problems

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Abstract: In this work, we initiate the study of a new class of impulsive boundary value problems consisting of mixed type fractional quantum and Hadamard derivatives. We will establish existence and uniqueness results by using tools from the functional analysis. We prove the uniqueness result via Banach's contraction mapping principle, while we will use the Leray-Schauder nonlinear alternative to establish an existence result. We also present examples to illustrate the obtained results.

Keywords: fractional impulsive differential equations, instantaneous impulses, impulses, quantum calculus

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1 Introduction

Fractional calculus is a generalization of classical calculus to an arbitrary real order and has evolved as an interesting and important area of research. Fractional differential equations have gained much attention in the literature because some real-world problems in physics, mechanics, engineering, game theory, stability, optimal control, and other fields can be described better with the help of fractional differential equations. Fractional differential equations constitute a significant branch of nonlinear analysis. The theory and applications of the fractional differential equations have been greatly developed; for more details, one can see the monographs [1–8] and references therein.

Impulsive differential equations arise when at certain moments they change their state rapidly and have many applications in physics, engineering, medicine, population dynamics, pharmacology, biotechnology, and economics. There are two types of impulses: the *instantaneous impulses* in which the duration of these changes is relatively short and the *noninstantaneous impulses* in which an impulsive jump starts abruptly at any fixed point and continues on a finite interval of time. For the results of instantaneous impulses, see, e.g., the monographs [9–11], the papers [12–14], and the references cited therein. Nonin-

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stantaneous impulsive differential equation was introduced by Hernández and O'Regan in [15], who pointed out that the instantaneous impulses cannot characterize some processes such as evolution processes in pharmacotherapy. In noninstantaneous impulses, impulsive action starts at an arbitrary fixed point and remains active on a finite time interval, which is much different from classical instantaneous impulsive that changes are relatively short compared to the overall duration of the whole process. Let us consider the hemodynamic equilibrium of a person. The introduction of the drugs in the bloodstream and the consequent absorption for the body are a gradual and continuous process. In fact, this situation should be characterized by a new case of impulsive action, which starts at an arbitrary fixed point and stays active on a finite time interval. For more details, see the monograph [16].

For some recent works, we refer the reader to [17–20] and references therein.

In [21], the notions of q_k -derivative and q_k -integral were introduced, and their properties were investigated. New concepts of fractional quantum calculus involving a new q_k -shifting operator were introduced in [22]. On the other hand, the analysis of fractional differential equations involving Hadamard fractional derivatives has increased interest in the mathematical analysis; see, for example, the recent monograph [8].

Motivated by the aforementioned papers, in this investigation, we initiate the study of a new class of boundary value problems consisting of impulses and mixed type fractional quantum and Hadamard derivatives. More precisely, we study the following impulsive boundary value problem, in which we combine fractional quantum and Hadamard derivatives of the form

$$\begin{cases} {}_{s_i}D_{q_i}^{\alpha_i}x(t) = f(t, x(t)), & t \in [s_i, t_{i+1}), \quad i = 0, 1, 2, \dots, m, \\ {}_{t_i}^H D^{\beta_i}x(t) = g(t, x(t)), & t \in [t_i, s_i), \quad i = 1, 2, 3, \dots, m, \\ x(t_i^+) = \gamma_i x(t_i^-) + \eta_i, \quad x(s_i^+) = \xi_i x(s_i^-) + \theta_i, & i = 1, 2, 3, \dots, m, \\ \lambda_1 x(0) + \lambda_2 x(T) = \lambda_3, \end{cases} \quad (1)$$

where ${}_{s_i}D_{q_i}^{\alpha_i}$, ${}_{t_i}^H D^{\beta_i}$ are the fractional quantum difference and Hadamard fractional derivative in the sense of Caputo type of orders $\alpha_i, \beta_i \in (0, 1]$ with quantum numbers $q_i \in (0, 1)$, $i = 0, 1, \dots, m$, $j = 1, 2, \dots, m$. Let $J_1 = \bigcup_{i=0}^m [s_i, t_{i+1})$, $J_2 = \bigcup_{j=1}^m [t_j, s_j)$, and $J = J_1 \cup J_2 \cup \{T\}$ be given intervals. The functions $f : J_1 \times \mathbb{R} \rightarrow \mathbb{R}$ and $g : J_2 \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous, and $\lambda_1, \lambda_2, \lambda_3, \gamma_i, \eta_i, \xi_i, \theta_i, i = 1, \dots, m$ are constants. In J , the fixed points $0 = s_0 < t_1 < s_1 < t_2 < \dots < t_m < s_m < t_{m+1} = T$ are given.

We establish existence and uniqueness results for the impulsive mixed fractional quantum and Hadamard boundary value problem (1) by using tools from the functional analysis. The main results are presented in Section 3, where Banach's contraction mapping principle is applied for the uniqueness result, while the Leray-Schauder nonlinear alternative is used to establish an existence result. In Section 2, some basic concepts from quantum calculus and Hadamard derivatives are recalled, and also an auxiliary result concerning a linear variant of the problem (1) is proved. This result is pivotal to transform the problem (1) into a fixed point problem. Illustrative numerical examples are also presented.

2 Preliminaries

In this section, we give some basic concepts of fractional quantum and Hadamard calculus such as derivatives and integrals. For more details, see [2,22–24]. Staring, the q -shifting operator is defined by

$${}_a\Phi_q = qx + (1 - q)a, \quad (2)$$

where $x \in \mathbb{R}$, $a \geq 0$, and $0 < q < 1$. For any positive integer k , we have

$${}_a\Phi_q^k(x) = q^k x + (1 - q^k)a = {}_a\Phi_q^{k-1}({}_a\Phi_q) \quad \text{and} \quad {}_a\Phi_q^0(x) = x.$$

The power function of $x - y$ involving q -shifting operator is given by

$${}_a(x - y)_q^{(0)} = 1, \quad {}_a(x - y)_q^{(k)} = \prod_{i=0}^{k-1} (x - {}_a\Phi_q^i(y)), \quad k \in \mathbb{N} \cup \{\infty\}.$$

In general, if $\alpha \in \mathbb{R}$, then we have

$${}_a(x - y)_q^{(\alpha)} = x^{(\alpha)} \prod_{i=0}^{\infty} \frac{1 - \frac{{}_a\Phi_q^i(y/x)}{x}}{1 - \frac{{}_a\Phi_q^{\alpha+i}(y/x)}{x}}.$$

The q -difference of a function f on interval $[a, b]$ can be defined in the term of q -shifting by

$$({}_aD_q f)(t) = \frac{f(t) - f({}_a\Phi_q(t))}{(1 - q)(t - a)}, \quad t \neq a \quad \text{and} \quad ({}_aD_q f)(a) = \lim_{t \rightarrow a} ({}_aD_q f)(t),$$

while the q -integral of a function f on interval $[a, b]$ is defined as follows:

$$({}_aI_q f)(t) = \int_a^t f(s) {}_a d_q s = (1 - q)(t - a) \sum_{i=0}^{\infty} q^i f({}_a\Phi_q^i(t)), \quad t \in [a, b]. \quad (3)$$

Now, we present the formulas of Riemann-Liouville fractional q -integral and Caputo fractional q -derivative over the interval $[a, b]$ as follows.

Definition 2.1. Let $\alpha \geq 0$ and $f: [a, b] \rightarrow \mathbb{R}$ be a function. The Riemann-Liouville fractional q -integral is defined by

$$({}_aI_q^\alpha f)(t) = \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - {}_a\Phi_q(s))_q^{(\alpha-1)} f(s) {}_a d_q s, \quad \alpha > 0, \quad t \in [a, b],$$

and $({}_aI_q^0 f)(t) = f(t)$

Definition 2.2. Let $f: [a, b] \rightarrow \mathbb{R}$ be the n -times q -differentiable function. The fractional q -derivative of the Caputo type of order $\alpha \geq 0$ on interval $[a, b]$ is defined by

$$({}_aD_q^\alpha f)(t) = ({}_aI_q^{n-\alpha} {}_aD_q^n f)(t) = \frac{1}{\Gamma_q(n-\alpha)} \int_a^t (t - {}_a\Phi_q(s))_q^{(n-\alpha-1)} {}_aD_q^n f(s) {}_a d_q s,$$

for $\alpha > 0$ and $({}_aD_q^0 f)(t) = f(t)$, where n is the smallest integer greater than or equal to α and ${}_aD_q^n = \underbrace{{}_aD_q {}_aD_q \cdots {}_aD_q}_{n \text{ times}}$.

Lemma 2.1. Let $\alpha > 0$ and n be the smallest integer greater than or equal to α . Then, for $t \in [a, b]$, we have

$${}_aI_q^\alpha {}_aD_q^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t - a)^k}{\Gamma_q(k+1)} {}_aD_q^k f(a).$$

Let us introduce some notations and definitions of Hadamard fractional calculus as follows.

Definition 2.3. For at least n -times differentiable function $f: [a, b] \rightarrow \mathbb{R}$, $a > 0$, the Caputo type of Hadamard fractional derivative of order α is defined as follows:

$${}^H_aD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\log \frac{t}{s} \right)^{n-\alpha-1} \delta^n f(s) \frac{ds}{s}, \quad n-1 < \alpha < n, \quad (4)$$

where the operator $\delta = t \frac{d}{dt}$, $n = [\alpha] + 1$, and $[\alpha]$ denotes the integer part of the real number α and $\log(\cdot) = \log_e(\cdot)$.

Definition 2.4. The Hadamard fractional integral of order $\alpha > 0$ is defined as follows:

$${}_a^H J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s} \right)^{\alpha-1} f(s) \frac{ds}{s}, \quad \alpha > 0, \quad (5)$$

provided the right side integral exists on $[a, b]$.

Lemma 2.2. [23] Let $x \in C_\delta^n[a, b]$ and $\alpha \in \mathbb{C}$. Then, for $a > 0$, we have

$${}_a^H J^\alpha ({}_a^H D^\alpha x)(t) = x(t) - \sum_{k=0}^{n-1} \frac{\delta^k x(a)}{k!} \log\left(\frac{t}{a}\right)^k.$$

Next, let us introduce the future used notation as follows:

$$\bigotimes_k^i(\xi, \gamma) = \left(\prod_{r=k}^i \xi_r \right) \left(\prod_{k < j \leq i} \gamma_j \right).$$

For examples, $\bigotimes_3^5(\xi, \gamma) = \xi_3 \xi_4 \xi_5 \gamma_4 \gamma_5$, $\bigotimes_4^5(\xi, \gamma) = \xi_4 \xi_5 \gamma_5$, and $\bigotimes_5^5(\xi, \gamma) = \xi_5$, with $\prod_{5 < j \leq 5} \gamma_j = 1$, since the inequality in the subscript is not true. In addition, we assume the constants

$$\Lambda_1(i) := \sum_{k=1}^i \eta_k \bigotimes_k^i(\xi, \gamma), \quad \Lambda_2(i) := \sum_{k=1}^i \theta_k \left(\prod_{k < j}^i \xi_j \gamma_j \right), \quad \Omega := \lambda_1 + \lambda_2 \prod_{k=1}^m \xi_k \gamma_k.$$

The next lemma deals with a linear variant of the boundary value problem (1).

Lemma 2.3. Let $f^* : J_1 \rightarrow \mathbb{R}$ and $g^* : J_2 \rightarrow \mathbb{R}$ be given functions and let $\Omega \neq 0$. In addition, we let $\alpha_j, q_j, j = 0, 1, \dots, m, \beta_i, \gamma_i, \eta_i, \xi_i, \theta_i, i = 1, 2, \dots, m, \lambda_1, \lambda_2$, and λ_3 be given constants, which satisfy problem (1). Then, the linear boundary value problem of mixed type quantum and Hadamard fractional derivatives of the form:

$$\begin{cases} {}_{s_i} D_{q_i}^{\alpha_i} x(t) = f^*(t), & t \in [s_i, t_{i+1}), \quad i = 0, 1, 2, \dots, m, \\ {}_{t_i}^H D^{\beta_i} x(t) = g^*(t), & t \in [t_i, s_i), \quad i = 1, 2, 3, \dots, m, \\ x(t_i^+) = \gamma_i x(t_i^-) + \eta_i, & x(s_i^+) = \xi_i x(s_i^-) + \theta_i, \quad i = 1, 2, 3, \dots, m, \\ \lambda_1 x(0) + \lambda_2 x(T) = \lambda_3, \end{cases} \quad (6)$$

has a unique solution x on J presented by

$$x(t) = \begin{cases} \frac{1}{\Omega} \left(\prod_{k=1}^i \xi_k \gamma_k \right) \left\{ \lambda_3 - \lambda_2 \sum_{k=1}^m \bigotimes_k^m(\xi, \gamma) ({}_k^H J^{\beta_k} g^*)(s_k^-) - \lambda_2 \Lambda_1(m) - \lambda_2 \sum_{k=1}^{m+1} \left(\prod_{k=1}^m \xi_k \gamma_k \right) ({}_{s_{k-1}} I_{q_{k-1}}^{\alpha_{k-1}} f^*)(t_k^-) - \lambda_2 \Lambda_2(m) \right\} \\ + \sum_{k=1}^i \bigotimes_k^i(\xi, \gamma) ({}_k^H J^{\beta_k} g^*)(s_k^-) + \Lambda_1(i) + \sum_{k=1}^i \left(\prod_{k=1}^i \xi_k \gamma_k \right) ({}_{s_{k-1}} I_{q_{k-1}}^{\alpha_{k-1}} f^*)(t_k^-) + \Lambda_2(i) + ({}_{s_i} I_{q_i}^{\alpha_i} f^*)(t), \quad t \in [s_i, t_{i+1}), \\ i = 0, 1, 2, \dots, m, \\ \frac{\gamma_i}{\Omega} \left(\prod_{k=1}^{i-1} \xi_k \gamma_k \right) \left\{ \lambda_3 - \lambda_2 \sum_{k=1}^m \bigotimes_k^m(\xi, \gamma) ({}_k^H J^{\beta_k} g^*)(s_k^-) - \lambda_2 \Lambda_1(m) - \lambda_2 \sum_{k=1}^{m+1} \left(\prod_{k=1}^m \xi_k \gamma_k \right) ({}_{s_{k-1}} I_{q_{k-1}}^{\alpha_{k-1}} f^*)(t_k^-) - \lambda_2 \Lambda_2(m) \right\} \\ + \gamma_i \left[\sum_{k=1}^{i-1} \bigotimes_k^i(\xi, \gamma) ({}_k^H J^{\beta_k} g^*)(s_k^-) + \Lambda_1(i-1) + \sum_{k=1}^{i-1} \left(\prod_{k=1}^{i-1} \xi_k \gamma_k \right) ({}_{s_{k-1}} I_{q_{k-1}}^{\alpha_{k-1}} f^*)(t_k^-) + \Lambda_2(i-1) \right] + \eta_i \\ + ({}_t^H J^{\beta_i} g^*)(t), \quad t \in [t_i, s_i), \quad i = 1, 2, 3, \dots, m, \end{cases} \quad (7)$$

with $\prod_a^b(\cdot) = 1, \sum_a^b(\cdot) = 0$, if $b < a$.

Proof. In the first equation in (6) by taking the fractional q_0 -integral of order α_0 from s_0 to $t \in [s_0, t_1]$, we obtain

$${}_{s_0}I_{q_0}^{\alpha_0}({}_{s_0}D_{q_0}^{\alpha_0}x)(t) = ({}_{s_0}I_{q_0}^{\alpha_0}f^*)(t),$$

which yields

$$x(t) = c_0 + ({}_{s_0}I_{q_0}^{\alpha_0}f^*)(t), \quad (8)$$

where $c_0 = x(0)$. In the second interval $[t_1, s_1]$, we can get $x(t)$ by Hadamard fractional integration of order $\beta_1 > 0$ as follows:

$$x(t) = x(t_1^+) + ({}_{t_1}^HI_{q_1}^{\beta_1}g^*)(t).$$

Since $x(t_1^+) = \gamma_1 x(t_1^-) + \eta_1$ and $x(t_1^-) = c_0 + ({}_{s_0}I_{q_0}^{\alpha_0}f^*)(t_1^-)$, we have

$$x(t) = \gamma_1 c_0 + \gamma_1 ({}_{s_0}I_{q_0}^{\alpha_0}f^*)(t_1^-) + \eta_1 + ({}_{t_1}^HI_{q_1}^{\beta_1}g^*)(t), \quad (9)$$

for $t \in [t_1, s_1]$.

In the third interval $[s_1, t_2]$, it is the fractional q_1 -difference of an unknown $x(t)$. Then, we apply the fractional q_1 -integral of order α_1 by

$$x(t) = x(s_1^+) + ({}_{s_1}I_{q_1}^{\alpha_1}f^*)(t),$$

which implies, by impulsive condition, that

$$x(t) = \xi_1 \gamma_1 c_0 + \xi_1 \gamma_1 ({}_{s_0}I_{q_0}^{\alpha_0}f^*)(t_1^-) + \xi_1 \eta_1 + \xi_1 ({}_{t_1}^HI_{q_1}^{\beta_1}g^*)(s_1^-) + \theta_1 + ({}_{s_1}I_{q_1}^{\alpha_1}f^*)(t).$$

For the fourth interval $[t_2, s_2]$ we can obtain by directing computation that

$$x(t) = \gamma_2 \xi_1 \gamma_1 c_0 + \gamma_2 \xi_1 \gamma_1 ({}_{s_0}I_{q_0}^{\alpha_0}f^*)(t_1^-) + \gamma_2 \xi_1 \eta_1 + \gamma_2 \xi_1 ({}_{t_1}^HI_{q_1}^{\beta_1}g^*)(s_1^-) + \gamma_2 \theta_1 + \gamma_2 ({}_{s_1}I_{q_1}^{\alpha_1}f^*)(t_2^-) + \eta_2 + ({}_{t_2}^HI_{q_2}^{\beta_2}g^*)(t).$$

Therefore, we can predict the solution $x(t)$ of (6) by

$$x(t) = \begin{cases} c_0 \left(\prod_{k=1}^i \xi_k \gamma_k \right) + \sum_{k=1}^i \bigotimes_{k=1}^i (\xi, \gamma) ({}_{t_k}^HI_{q_k}^{\beta_k}g^*)(s_k^-) + \sum_{k=1}^i \eta_k \bigotimes_{k=1}^i (\xi, \gamma) + \sum_{k=1}^i \left(\prod_{k=1}^i \xi_k \gamma_k \right) ({}_{s_{k-1}}I_{q_{k-1}}^{\alpha_{k-1}}f^*)(t_k^-) \\ + \sum_{k=1}^i \theta_k \left(\prod_{k < j}^i \xi_j \gamma_j \right) + ({}_{s_i}I_{q_i}^{\alpha_i}f^*)(t), & t \in [s_i, t_{i+1}), \quad i = 0, 1, 2, \dots, m, \\ \gamma_i \left[c_0 \left(\prod_{k=1}^{i-1} \xi_k \gamma_k \right) + \sum_{k=1}^{i-1} \bigotimes_{k=1}^{i-1} (\xi, \gamma) ({}_{t_k}^HI_{q_k}^{\beta_k}g^*)(s_k^-) + \sum_{k=1}^{i-1} \eta_k \bigotimes_{k=1}^{i-1} (\xi, \gamma) + \sum_{k=1}^{i-1} \left(\prod_{k=1}^{i-1} \xi_k \gamma_k \right) ({}_{s_{k-1}}I_{q_{k-1}}^{\alpha_{k-1}}f^*)(t_k^-) \right. \\ \left. + \sum_{k=1}^{i-1} \theta_k \left(\prod_{k < j}^{i-1} \xi_j \gamma_j \right) \right] + \eta_i + ({}_{t_i}^HI_{q_i}^{\beta_i}g^*)(t), & t \in [t_i, s_i), \quad i = 1, 2, 3, \dots, m. \end{cases} \quad (10)$$

To claim that our formula (10) is true, we use the mathematical induction by putting $i = 0$ and $i = 1$ in the first and second parts of (10), respectively. Then, the initial step holds by (8) and (9). The inductive step will be proved by assuming that the first part of (10) is true for $i = n$, that is,

$$x(t) = c_0 \left(\prod_{k=1}^n \xi_k \gamma_k \right) + \sum_{k=1}^n \bigotimes_{k=1}^n (\xi, \gamma) ({}_{t_k}^HI_{q_k}^{\beta_k}g^*)(s_k^-) + \sum_{k=1}^n \eta_k \bigotimes_{k=1}^n (\xi, \gamma) \\ + \sum_{k=1}^n \left(\prod_{k=1}^n \xi_k \gamma_k \right) ({}_{s_{k-1}}I_{q_{k-1}}^{\alpha_{k-1}}f^*)(t_k^-) + \sum_{k=1}^n \theta_k \left(\prod_{k < j}^n \xi_j \gamma_j \right) + ({}_{s_n}I_{q_n}^{\alpha_n}f^*)(t), \quad t \in [s_n, t_{n+1}).$$

Then, in the consecutive interval $[t_{n+1}, s_{n+1})$, we have

$$\begin{aligned}
x(t) &= x(t_{n+1}^+) + ({}^H J^{\beta_{n+1}} g^*)(t) \\
&= y_{n+1} x(t_{n+1}^-) + \eta_{n+1} + ({}^H J^{\beta_{n+1}} g^*)(t) \\
&= y_{n+1} \left[c_0 \left(\prod_{k=1}^n \xi_k \gamma_k \right) + \sum_{k=1}^n \bigotimes_k (\xi, \gamma) ({}^H J^{\beta_k} g^*)(s_k^-) + \sum_{k=1}^n \eta_k \bigotimes_k (\xi, \gamma) \right. \\
&\quad \left. + \sum_{k=1}^n \left(\prod_{k=1}^n \xi_k \gamma_k \right) ({}_{s_{k-1}} I_{q_{k-1}}^{\alpha_{k-1}} f^*)(t_k^-) + \sum_{k=1}^n \theta_k \left(\prod_{k < j} \xi_j \gamma_j \right) + ({}_{s_n} I_{q_n}^{\alpha_n} f^*)(t_{n+1}^-) \right] + \eta_{n+1} + ({}^H J^{\beta_{n+1}} g^*)(t) \\
&= y_{n+1} \left[c_0 \left(\prod_{k=1}^n \xi_k \gamma_k \right) + \sum_{k=1}^n \bigotimes_k (\xi, \gamma) ({}^H J^{\beta_k} g^*)(s_k^-) + \sum_{k=1}^n \eta_k \bigotimes_k (\xi, \gamma) \right. \\
&\quad \left. + \sum_{k=1}^{n+1} \left(\prod_{k=1}^n \xi_k \gamma_k \right) ({}_{s_{k-1}} I_{q_{k-1}}^{\alpha_{k-1}} f^*)(t_k^-) + \sum_{k=1}^n \theta_k \left(\prod_{k < j} \xi_j \gamma_j \right) \right] + \eta_{n+1} + ({}^H J^{\beta_{n+1}} g^*)(t),
\end{aligned}$$

which implies that the second part of (10) holds for $t \in [t_{n+1}, s_{n+1})$. In addition, suppose that the second part of (10) is fulfilled when $i = n$, that is, $t \in [t_n, s_n)$. Thus, in the consecutive interval $[s_n, t_{n+1})$, we get

$$\begin{aligned}
x(t) &= x(s_n^+) + ({}_{s_n} I_{q_n}^{\alpha_n} f^*)(t) \\
&= \xi_n x(s_n^-) + \theta_n + ({}_{s_n} I_{q_n}^{\alpha_n} f^*)(t) \\
&= \xi_n \left\{ y_n \left[c_0 \left(\prod_{k=1}^{n-1} \xi_k \gamma_k \right) + \sum_{k=1}^{n-1} \bigotimes_k (\xi, \gamma) ({}^H J^{\beta_k} g^*)(s_k^-) + \sum_{k=1}^{n-1} \eta_k \bigotimes_k (\xi, \gamma) \right. \right. \\
&\quad \left. \left. + \sum_{k=1}^n \left(\prod_{k=1}^{n-1} \xi_k \gamma_k \right) ({}_{s_{k-1}} I_{q_{k-1}}^{\alpha_{k-1}} f^*)(t_k^-) + \sum_{k=1}^{n-1} \theta_k \left(\prod_{k < j} \xi_j \gamma_j \right) \right] \right. \\
&\quad \left. + \eta_n + ({}^H J^{\beta_n} g^*)(s_n^-) \right\} + \theta_n + ({}_{s_n} I_{q_n}^{\alpha_n} f^*)(t) \\
&= \xi_n y_n \left[c_0 \left(\prod_{k=1}^{n-1} \xi_k \gamma_k \right) + \sum_{k=1}^{n-1} \bigotimes_k (\xi, \gamma) ({}^H J^{\beta_k} g^*)(s_k^-) + \sum_{k=1}^{n-1} \eta_k \bigotimes_k (\xi, \gamma) \right. \\
&\quad \left. + \sum_{k=1}^n \left(\prod_{k=1}^{n-1} \xi_k \gamma_k \right) ({}_{s_{k-1}} I_{q_{k-1}}^{\alpha_{k-1}} f^*)(t_k^-) + \sum_{k=1}^{n-1} \theta_k \left(\prod_{k < j} \xi_j \gamma_j \right) \right] \\
&\quad + \xi_n \eta_n + \xi_n ({}^H J^{\beta_n} g^*)(s_n^-) + \theta_n + ({}_{s_n} I_{q_n}^{\alpha_n} f^*)(t) \\
&= c_0 \left(\prod_{k=1}^n \xi_k \gamma_k \right) + \sum_{k=1}^n \bigotimes_k (\xi, \gamma) ({}^H J^{\beta_k} g^*)(s_k^-) + \sum_{k=1}^n \eta_k \bigotimes_k (\xi, \gamma) \\
&\quad + \sum_{k=1}^n \left(\prod_{k=1}^n \xi_k \gamma_k \right) ({}_{s_{k-1}} I_{q_{k-1}}^{\alpha_{k-1}} f^*)(t_k^-) + \sum_{k=1}^n \theta_k \left(\prod_{k < j} \xi_j \gamma_j \right) + ({}_{s_n} I_{q_n}^{\alpha_n} f^*)(t).
\end{aligned}$$

Therefore, the first part of (10) is true. Hence, the formula (10) holds for all $t \in [0, T]$.

To take the boundary condition in (6), we put $t = T$ in (10) with $i = m$. Then, we obtain

$$\begin{aligned}
x(T) &= c_0 \left(\prod_{k=1}^m \xi_k \gamma_k \right) + \sum_{k=1}^m \bigotimes_k (\xi, \gamma) ({}^H J^{\beta_k} g^*)(s_k^-) + \sum_{k=1}^m \eta_k \bigotimes_k (\xi, \gamma) \\
&\quad + \sum_{k=1}^m \left(\prod_{k=1}^m \xi_k \gamma_k \right) ({}_{s_{k-1}} I_{q_{k-1}}^{\alpha_{k-1}} f^*)(t_k^-) + \sum_{k=1}^m \theta_k \left(\prod_{k < j} \xi_j \gamma_j \right) + ({}_{s_m} I_{q_m}^{\alpha_m} f^*)(T) \\
&= c_0 \left(\prod_{k=1}^m \xi_k \gamma_k \right) + \sum_{k=1}^m \bigotimes_k (\xi, \gamma) ({}^H J^{\beta_k} g^*)(s_k^-) + \Lambda_1(m) \\
&\quad + \sum_{k=1}^{m+1} \left(\prod_{k=1}^m \xi_k \gamma_k \right) ({}_{s_{k-1}} I_{q_{k-1}}^{\alpha_{k-1}} f^*)(t_k^-) + \Lambda_2(m),
\end{aligned} \tag{11}$$

which leads to

$$c_0 = \frac{1}{\Omega} \left\{ \lambda_3 - \lambda_2 \sum_{k=1}^m \sum_k^m (\xi, \gamma) {}^H J^{\beta_k} g^*(s_k^-) - \lambda_2 \Lambda_1(m) - \lambda_2 \sum_{k=1}^{m+1} \left(\prod_k^m \xi_k \gamma_k \right) ({}_{s_{k-1}} I_{q_{k-1}}^{\alpha_{k-1}} f^*)(t_k^-) - \lambda_2 \Lambda_2(m) \right\}. \quad (12)$$

By substituting the constant c_0 , (12), into (10), we obtain a unique solution of linear mixed fractional quantum and Hadamard impulsive boundary value problem (6).

Conversely, for $t \in [s_i, t_{i+1})$, we take the operator ${}_{s_i} D_{q_i}^{\alpha_i}$ on (10), and then we obtain ${}_{s_i} D_{q_i}^{\alpha_i} x(t) = f^*(t)$. Applying the Hadamard fractional derivative ${}^H D_t^{\beta_i}$ to the second equation of (10), we have ${}^H D_t^{\beta_i} x(t) = g^*(t)$. Replacing i by $i - 1$ and putting t by t_i^- in the first equation of (10), we get $x(t_i^+)$ after multiplying γ_i and adding η_i , respectively. Then, $x(s_i^+)$ is obtained by putting $t = s_i^-$, and respectively, multiplying ξ_i , adding θ_i , in the second equation of (10). The value of λ_3 can be computed by putting $i = 0$ with multiplying λ_1 and $i = m$, $t = T$ with multiplying λ_2 , in the first equation of (10). Therefore, the proof is completed. $\square$

3 Existence and uniqueness results

In the first step of this section, we define the spaces of functions and the operator, which are related to the investigated problem (1). Let J be an interval defined in (1) and let $PC(J, \mathbb{R})$ and $PC^1(J, \mathbb{R})$ be the spaces of piecewise continuous functions defined by $PC(J, \mathbb{R}) = \{x : J \rightarrow \mathbb{R}; x(t) \text{ is continuous everywhere except for some } t_k, \text{ at which } x(t_k^+) \text{ and } x(t_k^-) \text{ exist and } x(t_k^-) = x(t_k), k = 1, 2, \dots, m\}$, and $PC^1(J, \mathbb{R}) = \{x \in PC(J, \mathbb{R}); x'(t) \text{ is continuous everywhere except for some } t_k \text{ at which } x'(t_k^+) \text{ and } x'(t_k^-) \text{ exist and } x'(t_k^-) = x'(t_k), k = 1, 2, \dots, m\}$. Let $E = PC(J, \mathbb{R}) \cap PC^1(J, \mathbb{R})$ be the Banach space with norm $\|x\| = \sup\{|x(t)|, t \in J\}$ and now we define the operator on E by

$$Qx(t) = \begin{cases} \frac{1}{\Omega} \left(\prod_{k=1}^i \xi_k \gamma_k \right) \left\{ \lambda_3 - \lambda_2 \sum_{k=1}^m \sum_k^m (\xi, \gamma) {}^H J^{\beta_k} g_x(s_k^-) - \lambda_2 \Lambda_1(m) - \lambda_2 \sum_{k=1}^{m+1} \left(\prod_k^m \xi_k \gamma_k \right) ({}_{s_{k-1}} I_{q_{k-1}}^{\alpha_{k-1}} f_x)(t_k^-) - \lambda_2 \Lambda_2(m) \right\} \\ + \sum_{k=1}^i \sum_k^i (\xi, \gamma) {}^H J^{\beta_k} g_x(s_k^-) + \Lambda_1(i) + \sum_{k=1}^i \left(\prod_k^i \xi_k \gamma_k \right) ({}_{s_{k-1}} I_{q_{k-1}}^{\alpha_{k-1}} f_x)(t_k^-) + \Lambda_2(i) + ({}_{s_i} I_{q_i}^{\alpha_i} f_x)(t), \\ t \in [s_i, t_{i+1}), i = 0, 1, 2, \dots, m, \\ \frac{\gamma_i}{\Omega} \left(\prod_{k=1}^{i-1} \xi_k \gamma_k \right) \left\{ \lambda_3 - \lambda_2 \sum_{k=1}^m \sum_k^m (\xi, \gamma) {}^H J^{\beta_k} g_x(s_k^-) - \lambda_2 \Lambda_1(m) - \lambda_2 \sum_{k=1}^{m+1} \left(\prod_k^m \xi_k \gamma_k \right) ({}_{s_{k-1}} I_{q_{k-1}}^{\alpha_{k-1}} f_x)(t_k^-) - \lambda_2 \Lambda_2(m) \right\} \\ + \gamma_i \left[\sum_{k=1}^{i-1} \sum_k^{i-1} (\xi, \gamma) {}^H J^{\beta_k} g_x(s_k^-) + \Lambda_1(i-1) + \sum_{k=1}^{i-1} \left(\prod_k^{i-1} \xi_k \gamma_k \right) ({}_{s_{k-1}} I_{q_{k-1}}^{\alpha_{k-1}} f_x)(t_k^-) + \Lambda_2(i-1) \right] \\ + \eta_i + ({}^H J_t^{\beta_i} g_x)(t), \quad t \in [t_i, s_i), i = 1, 2, 3, \dots, m, \end{cases}$$

where abbreviations $f_x(t)$, $g_x(t)$ mean $f_x(t) = f(t, x(t))$ and $g_x(t) = g(t, x(t))$, respectively. By applying the Banach contraction mapping principle, and Leray-Schauder nonlinear alternative, we are in the position to prove the existence and uniqueness of solutions of the problem (1). The following constants

$$\begin{aligned} \Phi_1 &= \frac{|\lambda_3|}{|\Omega|} \left(\prod_{k=1}^m |\xi_k \gamma_k| \right), \quad \Phi_2 = \frac{|\lambda_2|}{|\Omega|} \left(\prod_{k=1}^m |\xi_k \gamma_k| \right) + 1, \\ \Phi_3 &= \sum_{k=1}^m \left| \sum_k^m (\xi, \gamma) \right| \frac{|\log(s_k) - \log(t_k)|^{\beta_k}}{\Gamma(\beta_k + 1)}, \\ \Phi_4 &= \sum_{k=1}^{m+1} \left(\prod_k^m |\xi_k \gamma_k| \right) \frac{(t_k - s_{k-1})^{\alpha_{k-1}}}{\Gamma_{q_{k-1}}(\alpha_{k-1} + 1)}, \end{aligned}$$

will be used in our proofs.

Theorem 3.1. Suppose that the two nonlinear functions $f: J_1 \times \mathbb{R} \rightarrow \mathbb{R}$ and $g: J_2 \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy the following condition:

(H1) There exist constants $L_1, L_2 > 0$ such that for all $x, y \in \mathbb{R}$,

$$|f(t, x) - f(t, y)| \leq L_1|x - y|, \quad t \in J_1 \quad \text{and} \quad |g(t, x) - g(t, y)| \leq L_2|x - y|, \quad t \in J_2.$$

If $(L_1\Phi_4 + L_2\Phi_3)\Phi_2 < 1$, then the mixed fractional quantum and Hadamard derivatives impulsive boundary value problem (1) has a unique solution on J .

Proof. Let B_r be a ball of radius $r > 0$, subset of E , defined by $B_r = \{x \in E : \|x\| \leq r\}$, where r satisfies

$$r \geq \frac{\Phi_1 + (|\Lambda_1(m)| + |\Lambda_2(m)| + M_1\Phi_4 + M_2\Phi_3)\Phi_2}{1 - (L_1\Phi_4 + L_2\Phi_3)\Phi_2}. \quad (13)$$

We will prove that $QB_r \subset B_r$. We set $M_1 = \sup\{|f(t, 0)|, t \in J_1\}$ and $M_2 = \sup\{|g(t, 0)|, t \in J_2\}$. For $t \in J_1$ and $t \in J_2$, respectively, we have

$$\begin{aligned} |Qx(t)| \leq & \frac{1}{|\Omega|} \left(\prod_{k=1}^i |\xi_k \gamma_k| \right) \left\{ |\lambda_3| + |\lambda_2| \sum_{k=1}^m \left| \bigotimes_k (\xi, \gamma) \right| \left({}^H J^{\beta_k} |g_x| \right)(s_k^-) + |\lambda_2| |\Lambda_1(m)| \right. \\ & + |\lambda_2| \sum_{k=1}^{m+1} \left(\prod_k |\xi_k \gamma_k| \right) (s_{k-1} I_{q_{k-1}}^{\alpha_{k-1}} |f_x|)(t_k^-) + |\lambda_2| |\Lambda_2(m)| \left. \right\} + \sum_{k=1}^i \left| \bigotimes_k (\xi, \gamma) \right| \left({}^H J^{\beta_k} |g_x| \right)(s_k^-) + |\Lambda_1(i)| \\ & + \sum_{k=1}^i \left(\prod_k |\xi_k \gamma_k| \right) (s_{k-1} I_{q_{k-1}}^{\alpha_{k-1}} |f_x|)(t_k^-) + |\Lambda_2(i)| + (s_i I_{q_i}^{\alpha_i} |f_x|)(t), \end{aligned}$$

for $t \in [s_i, t_{i+1})$, $i = 0, 1, 2, \dots, m$, and

$$\begin{aligned} |Qx(t)| \leq & \frac{|\gamma_i|}{|\Omega|} \left(\prod_{k=1}^{i-1} |\xi_k \gamma_k| \right) \left\{ |\lambda_3| + |\lambda_2| \sum_{k=1}^m \left| \bigotimes_k (\xi, \gamma) \right| \left({}^H J^{\beta_k} |g_x| \right)(s_k^-) + |\lambda_2| |\Lambda_1(m)| \right. \\ & + |\lambda_2| \sum_{k=1}^{m+1} \left(\prod_k |\xi_k \gamma_k| \right) (s_{k-1} I_{q_{k-1}}^{\alpha_{k-1}} |f_x|)(t_k^-) + |\lambda_2| |\Lambda_2(m)| \left. \right\} + |\gamma_i| \left[\sum_{k=1}^{i-1} \left| \bigotimes_k (\xi, \gamma) \right| \left({}^H J^{\beta_k} |g_x| \right)(s_k^-) + |\Lambda_1(i-1)| \right. \\ & + \sum_{k=1}^i \left(\prod_k |\xi_k \gamma_k| \right) (s_{k-1} I_{q_{k-1}}^{\alpha_{k-1}} |f_x|)(t_k^-) + |\Lambda_2(i-1)| \left. \right] + |\eta_i| + ({}^H J^{\beta_i} |g_x|)(t) \end{aligned}$$

for $t \in [t_i, s_i)$, $i = 1, 2, 3, \dots, m$. From triangle inequality and (H1), we obtain $|f_x(t)| = |f(t, x)| \leq |f(t, x) - f(t, 0)| + |f(t, 0)| \leq L_1 r + M_1$ and $|g_x(t)| = |g(t, x)| \leq |g(t, x) - g(t, 0)| + |g(t, 0)| \leq L_2 r + M_2$. Consequently, we have

$$\begin{aligned} \sup_{t \in J} |Qx(t)| \leq & \frac{1}{|\Omega|} \left(\prod_{k=1}^m |\xi_k \gamma_k| \right) \left\{ |\lambda_3| + |\lambda_2| |\Lambda_1(m)| + |\lambda_2| |\Lambda_2(m)| + (L_2 r + M_2) |\lambda_2| \sum_{k=1}^m \left| \bigotimes_k (\xi, \gamma) \right| \left({}^H J^{\beta_k} (1) \right)(s_k^-) \right. \\ & + (L_1 r + M_1) |\lambda_2| \sum_{k=1}^{m+1} \left(\prod_k |\xi_k \gamma_k| \right) (s_{k-1} I_{q_{k-1}}^{\alpha_{k-1}} (1))(t_k^-) \left. \right\} + (L_2 r + M_2) \sum_{k=1}^m \left| \bigotimes_k (\xi, \gamma) \right| \left({}^H J^{\beta_k} (1) \right)(s_k^-) \\ & + |\Lambda_1(m)| + (L_1 r + M_1) \sum_{k=1}^m \left(\prod_k |\xi_k \gamma_k| \right) (s_{k-1} I_{q_{k-1}}^{\alpha_{k-1}} (1))(t_k^-) + |\Lambda_2(m)| + (L_1 r + M_1) (s_m I_{q_m}^{\alpha_m} (1))(t_{m+1}) \\ = & \Phi_1 + |\Lambda_1(m)| \Phi_2 + |\Lambda_2(m)| \Phi_2 + (L_1 r + M_1) \Phi_2 \Phi_4 + (L_2 r + M_2) \Phi_2 \Phi_3 \\ = & \Phi_1 + |\Lambda_1(m)| \Phi_2 + |\Lambda_2(m)| \Phi_2 + r \Phi_2 (L_1 \Phi_4 + L_2 \Phi_3) + \Phi_2 (M_1 \Phi_4 + M_2 \Phi_3) \\ \leq & r. \end{aligned}$$

Thus, $\|Qx\| \leq r$, where r satisfies (13). Therefore, $QB_r \subset B_r$ holds. In the next step, we will show that Q is a contraction operator. For any $x, y \in B_r$, we have

$$\begin{aligned} |Qx(t) - Qy(t)| &\leq \frac{1}{|\Omega|} \left(\prod_{k=1}^i |\xi_k \gamma_k| \right) \left\{ |\lambda_2| \sum_{k=1}^m \left| \bigotimes_k (\xi, \gamma) \right| \left({}^H J^{\beta_k} |g_x - g_y| \right)(s_k^-) + |\lambda_2| \sum_{k=1}^{m+1} \left(\prod_k |\xi_k \gamma_k| \right) (s_{k-1} I_{q_{k-1}}^{\alpha_{k-1}} |f_x - f_y|)(t_k^-) \right\} \\ &\quad + \sum_{k=1}^i \left| \bigotimes_k (\xi, \gamma) \right| \left({}^H J^{\beta_k} |g_x - g_y| \right)(s_k^-) + \sum_{k=1}^i \left(\prod_k |\xi_k \gamma_k| \right) (s_{k-1} I_{q_{k-1}}^{\alpha_{k-1}} |f_x - f_y|)(t_k^-) + (s_i I_{q_i}^{\alpha_i} |f_x - f_y|)(t), \end{aligned}$$

for $t \in [s_i, t_{i+1})$, $i = 0, 1, 2, \dots, m$, and

$$\begin{aligned} |Qx(t) - Qy(t)| &\leq \frac{|y_i|}{|\Omega|} \left(\prod_{k=1}^{i-1} |\xi_k \gamma_k| \right) \left\{ |\lambda_2| \sum_{k=1}^m \left| \bigotimes_k (\xi, \gamma) \right| \left({}^H J^{\beta_k} |g_x - g_y| \right)(s_k^-) + |\lambda_2| \sum_{k=1}^{m+1} \left(\prod_k |\xi_k \gamma_k| \right) (s_{k-1} I_{q_{k-1}}^{\alpha_{k-1}} |f_x - f_y|)(t_k^-) \right\} \\ &\quad + |y_i| \left[\sum_{k=1}^{i-1} \left| \bigotimes_k (\xi, \gamma) \right| \left({}^H J^{\beta_k} |g_x - g_y| \right)(s_k^-) + \sum_{k=1}^i \left(\prod_k |\xi_k \gamma_k| \right) (s_{k-1} I_{q_{k-1}}^{\alpha_{k-1}} |f_x - f_y|)(t_k^-) \right] \\ &\quad + ({}^H J^{\beta_i} |g_x - g_y|)(t), \end{aligned}$$

for $t \in [t_i, s_i)$, $i = 1, 2, 3, \dots, m$. Then, we have

$$\begin{aligned} \sup_{t \in J} |Qx(t) - Qy(t)| &\leq \frac{1}{|\Omega|} \left(\prod_{k=1}^m |\xi_k \gamma_k| \right) \left\{ (L_2 \|x - y\|) |\lambda_2| \sum_{k=1}^m \left| \bigotimes_k (\xi, \gamma) \right| \left({}^H J^{\beta_k}(1) \right)(s_k^-) \right. \\ &\quad \left. + (L_1 \|x - y\|) |\lambda_2| \sum_{k=1}^{m+1} \left(\prod_k |\xi_k \gamma_k| \right) (s_{k-1} I_{q_{k-1}}^{\alpha_{k-1}}(1))(t_k^-) \right\} \\ &\quad + (L_2 \|x - y\|) \sum_{k=1}^i \left| \bigotimes_k (\xi, \gamma) \right| \left({}^H J^{\beta_k}(1) \right)(s_k^-) + (L_1 \|x - y\|) \sum_{k=1}^i \left(\prod_k |\xi_k \gamma_k| \right) (s_{k-1} I_{q_{k-1}}^{\alpha_{k-1}}(1))(t_k^-) \\ &= (L_1 \Phi_4 + L_2 \Phi_3) \Phi_2 \|x - y\|, \end{aligned}$$

which leads to $\|Qx - Qy\| \leq (L_1 \Phi_4 + L_2 \Phi_3) \Phi_2 \|x - y\|$. Since $(L_1 \Phi_4 + L_2 \Phi_3) \Phi_2 < 1$, then, Q is a contraction. Hence, we can conclude, by using Banach's contraction mapping principle, that the operator Q has a fixed point, which is the solution of the mixed fractional quantum and Hadamard derivatives impulsive boundary value problem (1). This completes the proof. $\square$

The next theorem of Leray-Schauders nonlinear alternative will be used to prove our existence result.

Theorem 3.2. *Given E is a Banach space, and B is a closed, convex subset of E . In addition, let G be an open subset of B such that $0 \in G$. Suppose that $Q : \bar{G} \rightarrow B$ is a continuous, compact (i.e., $Q(\bar{G})$ is a relatively compact subset of B) map. Then either*

- (i) Q has a fixed point in $\bar{G}$, or
- (ii) there is a $x \in \partial G$ (the boundary of G in B) and $\kappa \in (0, 1)$ with $x = \kappa Q(x)$.

Theorem 3.3. *Assume that $f : J_1 \times \mathbb{R} \rightarrow \mathbb{R}$ and $g : J_2 \times \mathbb{R} \rightarrow \mathbb{R}$ are given functions. Further, we suppose that:*
(H2) *There exist two continuous nondecreasing functions $\Psi, \Theta : [0, \infty) \rightarrow (0, \infty)$ and continuous functions $u, v : J \rightarrow \mathbb{R}^+$, such that*

$$|f(t, x)| \leq u(t) \Psi(|x|), \quad \text{for each } (t, x) \in J_1 \times \mathbb{R},$$

and

$$|g(t, x)| \leq v(t) \Theta(|x|), \quad \text{for each } (t, x) \in J_2 \times \mathbb{R}.$$

(H3) There exists a positive constant N , such that

$$\frac{N}{\Phi_1 + |\Lambda_1(m)|\Phi_2 + |\Lambda_2(m)|\Phi_2 + \|v\|\Theta(N)\Phi_2\Phi_3 + \|u\|\Psi(N)\Phi_2\Phi_4} > 1.$$

Then, the mixed fractional quantum and Hadamard derivatives impulsive boundary value problem (1) has at least one solution on J .

Proof. Define a ball $B_\rho = \{x \in E : \|x\| \leq \rho\}$. It is easy to see that the ball B_ρ is a closed, convex subset of E . To use the result in Theorem 3.2, we first will claim that Q is a continuous operator. Define a convergent sequence $\{x_n\}$ such that $\{x_n\} \rightarrow x$. Then, we obtain

$$\begin{aligned} |Qx_n(t) - Qx(t)| &\leq \frac{1}{|\Omega|} \left(\prod_{k=1}^i |\xi_k \gamma_k| \right) \left\{ |\lambda_2| \sum_{k=1}^m \left| \bigotimes_k (\xi, \gamma) \right| \left({}^H J^{\beta_k} |g_{x_n} - g_x| \right)(s_k^-) + |\lambda_2| \sum_{k=1}^{m+1} \left(\prod_k |\xi_k \gamma_k| \right) (s_{k-1} I_{q_{k-1}}^{\alpha_{k-1}} |f_{x_n} - f_x|)(t_k^-) \right. \\ &\quad + \sum_{k=1}^i \left| \bigotimes_k (\xi, \gamma) \right| \left({}^H J^{\beta_k} |g_{x_n} - g_x| \right)(s_k^-) + \sum_{k=1}^i \left(\prod_k |\xi_k \gamma_k| \right) (s_{k-1} I_{q_{k-1}}^{\alpha_{k-1}} |f_{x_n} - f_x|)(t_k^-) \\ &\quad \left. + (s_i I_{q_i}^{\alpha_i} |f_{x_n} - f_x|)(t) \right\} \rightarrow 0, \quad \text{as } n \rightarrow \infty, \end{aligned}$$

for $t \in [s_i, t_{i+1})$, $i = 0, 1, 2, \dots, m$, and

$$\begin{aligned} |Qx_n(t) - Qx(t)| &\leq \frac{|y_i|}{|\Omega|} \left(\prod_{k=1}^{i-1} |\xi_k \gamma_k| \right) \left\{ |\lambda_2| \sum_{k=1}^m \left| \bigotimes_k (\xi, \gamma) \right| \left({}^H J^{\beta_k} |g_{x_n} - g_x| \right)(s_k^-) + |\lambda_2| \sum_{k=1}^{m+1} \left(\prod_k |\xi_k \gamma_k| \right) (s_{k-1} I_{q_{k-1}}^{\alpha_{k-1}} |f_{x_n} - f_x|)(t_k^-) \right\} \\ &\quad + |y_i| \left[\sum_{k=1}^{i-1} \left| \bigotimes_k (\xi, \gamma) \right| \left({}^H J^{\beta_k} |g_{x_n} - g_x| \right)(s_k^-) + \sum_{k=1}^i \left(\prod_k |\xi_k \gamma_k| \right) (s_{k-1} I_{q_{k-1}}^{\alpha_{k-1}} |f_{x_n} - f_x|)(t_k^-) \right] \\ &\quad + ({}^H J^{\beta_i} |g_{x_n} - g_x|)(t) \rightarrow 0, \quad \text{as } n \rightarrow \infty, \end{aligned}$$

for $t \in [t_i, s_i)$, $i = 1, 2, 3, \dots, m$. Then, from the aforementioned two inequalities, the operator Q is continuous.

To prove the compactness of the operator Q , we suppose that $x \in B_\rho$ and then

$$\begin{aligned} |Qx(t)| &\leq \frac{1}{|\Omega|} \left(\prod_{k=1}^m |\xi_k \gamma_k| \right) \left\{ |\lambda_3| + |\lambda_2| |\Lambda_1(m)| + |\lambda_2| |\Lambda_2(m)| + \|v\| \Theta(\rho) |\lambda_2| \sum_{k=1}^m \left| \bigotimes_k (\xi, \gamma) \right| \left({}^H J^{\beta_k}(1) \right)(s_k^-) \right. \\ &\quad \left. + \|u\| \Psi(\rho) |\lambda_2| \sum_{k=1}^{m+1} \left(\prod_k |\xi_k \gamma_k| \right) (s_{k-1} I_{q_{k-1}}^{\alpha_{k-1}}(1))(t_k^-) \right\} + \|v\| \Theta(\rho) \sum_{k=1}^m \left| \bigotimes_k (\xi, \gamma) \right| \left({}^H J^{\beta_k}(1) \right)(s_k^-) + |\Lambda_1(m)| \\ &\quad + \|u\| \Psi(\rho) \sum_{k=1}^m \left(\prod_k |\xi_k \gamma_k| \right) (s_{k-1} I_{q_{k-1}}^{\alpha_{k-1}}(1))(t_k^-) + |\Lambda_2(m)| + \|u\| \Psi(\rho) (s_m I_{q_m}^{\alpha_m}(1))(t_{m+1}) \\ &= \Phi_1 + |\Lambda_1(m)|\Phi_2 + |\Lambda_2(m)|\Phi_2 + \|v\|\Theta(\rho)\Phi_2\Phi_3 + \|u\|\Psi(\rho)\Phi_2\Phi_4 := \Phi_5, \end{aligned} \quad (14)$$

which implies that $\|Qx\| \leq \Phi_5$. Hence, $Q(B_\rho)$ is a uniformly bounded set. To prove the equicontinuity of the set $Q(B_\rho)$, let $t_1, t_2 \in [0, T]$ such that $t_1 < t_2$. Then, for any $x \in B_\rho$, we obtain

$$\begin{aligned} |Qx(t_2) - Qx(t_1)| &= |(s_i I_{q_i}^{\alpha_i} f)(t_2) - (s_i I_{q_i}^{\alpha_i} f)(t_1)| \\ &\leq \|u\| \Psi(\rho) |(s_i I_{q_i}^{\alpha_i}(1))(t_2) - (s_i I_{q_i}^{\alpha_i}(1))(t_1)| \\ &= \frac{\|u\| \Psi(\rho)}{\Gamma_{q_i}(\alpha_i + 1)} [(t_2 - s_i)^{\alpha_i} - (t_1 - s_i)^{\alpha_i}] \rightarrow 0 \end{aligned}$$

as $t_1 \rightarrow t_2$ when $t_1, t_2 \in [s_i, t_{i+1})$, $i = 0, 1, 2, \dots, m$, and

$$\begin{aligned}
|Qx(t_2) - Qx(t_1)| &= |({}^H J^{\beta_i} g)(t_2) - ({}^H J^{\beta_i} g)(t_1)| \\
&\leq \|v\| \Theta(\rho) |({}^H J^{\beta_i}(1))(t_2) - ({}^H J^{\beta_i}(1))(t_1)| \\
&= \frac{\|v\| \Theta(\rho)}{\Gamma(\beta_i + 1)} [|\log(t_2) - \log(t_1)|^{\beta_i} - |\log(t_1) - \log(t_1)|^{\beta_i}] \rightarrow 0
\end{aligned}$$

as $t_1 \rightarrow t_2$, when $t_1, t_2 \in [t_i, s_i]$, $i = 1, 2, 3, \dots, m$. Since the aforementioned two inequalities are convergent to zero independently of x , we can conclude that $Q(B_\rho)$ is an equicontinuous set. Then, we get that QB_ρ is relatively compact. By applying the Arzelà-Ascoli theorem, the operator Q is completely continuous.

In the final step, we will show that the second condition of Theorem 3.2 does not hold. Assume that x is a solution of the problem (1). Then, for fixed $\kappa \in (0, 1)$, we have $x = \kappa Qx$, which implies

$$\frac{\|x\|}{\Phi_1 + |\Lambda_1(m)|\Phi_2 + |\Lambda_2(m)|\Phi_2 + \|v\|\Theta(\|x\|)\Phi_2\Phi_3 + \|u\|\Psi(\|x\|)\Phi_2\Phi_4} \leq 1,$$

by the same method to get (14). From the hypothesis (H_3) , there exists a positive constant N , which satisfies $\|x\| \neq N$. Now, we define the open subset of B_ρ by $G = \{x \in B_\rho : \|x\| < N\}$. Note that $0 \in G$ and $Q(\bar{G})$ is a relatively compact subset of B_ρ . In addition, it is impossible that $x \in \partial G$ such that $x = \kappa Qx$ for some $\kappa \in (0, 1)$. Therefore, by applying the result in (i) of Theorem 3.2, the operator Q has a fixed point $x \in \bar{G}$, which is a solution of the problem (1) on J . The proof is completed. $\square$

If condition (H_2) in Theorem 3.3 is replaced by

$$|f(t, x)| \leq u(t)(a_1|x| + b_1) \quad \text{and} \quad |g(t, x)| \leq v(t)(a_2|x| + b_2), \quad (15)$$

where $a_1, a_2 \geq 0$ and $b_1, b_2 > 0$ for all $(t, x) \in J \times \mathbb{R}$, then we get the following Corollary.

Corollary 3.1. *If (15) holds and if $(\|v\|a_2\Phi_3 + \|u\|a_1\Phi_4)\Phi_2 < 1$, then the problem (1) has at least one solution on J .*

In addition, if $a_1 = a_2 = 0$, that is, f and g are bounded functions, then the problem (1) has at least one solution on J .

Finally, we consider the existence criteria for the initial value problem. If we replace $\lambda_1 = 1$ and $\lambda_2 = 0$ in problem (1), that is,

$$x(0) = \lambda_3, \quad (16)$$

then we obtain an impulsive initial value problem and we also get constants $\Omega = 1$, $\Phi_1 = |\lambda_3|(\prod_{k=1}^m |\xi_k \gamma_k|) := \Phi_1^*$, and $\Phi_2 = 1$.

Corollary 3.2. *If (H_1) holds and $L_1\Phi_4 + L_2\Phi_3 < 1$, then the initial value problem (1)–(16) has a unique solution on J .*

Corollary 3.3. *If condition (H_2) is satisfied and if there exists a constant $N > 0$ such that*

$$\frac{N}{\Phi_1^* + |\Lambda_1(m)| + |\Lambda_2(m)| + \|v\|\Theta(N)\Phi_3 + \|u\|\Psi(N)\Phi_4} > 1,$$

then the initial value problem (1)–(16) has at least one solution on J .

4 Examples

In this section, we give some examples to illustrate the usefulness of our main results.

Example 4.1. Consider the following mixed fractional quantum and Hadamard derivatives impulsive boundary value problem of the form:

$$\begin{cases} {}_{(2i)}D^{\left(\frac{i+2}{i+3}\right)}_{\left(\frac{3i+2}{3i+3}\right)}x(t) = f(t, x(t)), & t \in [2i, 2i+1), \quad i = 0, 1, 2, 3, \\ {}_{(2i-1)}^H D^{\left(\frac{2i+2}{2i+3}\right)}x(t) = g(t, x(t)), & t \in [2i-1, 2i), \quad i = 1, 2, 3, \\ x((2i-1)^+) = \frac{i+1}{i+2}x((2i-1)^-) + \frac{2i+1}{2i+2}, \\ x((2i)^+) = \frac{i+3}{i+4}x((2i)^-) + \frac{2i+3}{2i+4}, \quad i = 1, 2, 3, \\ \frac{1}{7}x(0) + \frac{2}{11}x(7) = \frac{3}{13}. \end{cases} \quad (17)$$

Here, we set $\alpha_i = (i+2)/(i+3)$, $q_i = (3i+2)/(3i+3)$, $i = 0, 1, 2, 3$, and $\beta_i = (2i+2)/(2i+3)$, $i = 1, 2, 3$. Since $[2i, 2i+1) \cup [2i-1, 2i) \cup \{7\}$, for $i = 0, 1, 2, 3$, then we have $J = [0, 7]$. Further, we let $\gamma_i = (i+1)/(i+2)$, $\eta_i = (2i+1)/(2i+2)$, $\xi_i = (i+3)/(i+4)$, $\theta_i = (2i+3)/(2i+4)$, $i = 1, 2, 3$. Then, previous constants can be used to compute that $\Omega \approx 0.1844155844$, $\Lambda_1(3) \approx 1.4833333333$, $\Lambda_2(3) \approx 1.8571428571$, $\Phi_1 \approx 0.7436619719$, $\Phi_2 \approx 0.4957746479$, $\Phi_3 \approx 0.5631729043$, and $\Phi_4 \approx 2.1950314210$.

(i) If the two functions f, g are given, respectively, for $t \in [0, 7]$, by

$$f(t, x) = \frac{e^{-t}}{t+4} \left(\frac{x^2 + 2|x|}{(1+|x|)} \right) + \frac{3}{5}, \quad (18)$$

$$g(t, x) = \frac{1 + 2\cos\pi t}{2} \sin|x| + \frac{1}{3}, \quad (19)$$

then, from (18)–(19), we obtain $|f(t, x) - f(t, y)| \leq (1/2)|x - y|$ and $|g(t, x) - g(t, y)| \leq (3/2)|x - y|$, for $x, y \in \mathbb{R}$. By choosing $L_1 = 1/2$ and $L_2 = 3/2$, we get the relation $(L_1\Phi_4 + L_2\Phi_3)\Phi_2 \approx 0.9629307370 < 1$. Theorem 3.1 can be applied to conclude that the mixed fractional quantum and Hadamard derivatives impulsive boundary value problem (17) with f and g given by (18) and (19) has a unique solution on $[0, 7]$.

(ii) Let the two nonlinear functions f, g are defined on $[0, 7]$ as follows:

$$f(t, x) = \frac{1 + \cos\pi t}{t+30} \left(\frac{x^{26}}{1+x^{24}} + 1 \right), \quad (20)$$

$$g(t, x) = \frac{e^{-t^2}}{12+t} \left(\frac{|x|^{15}}{1+|x|^{13}} + 2 \right). \quad (21)$$

It is obvious to see that

$$|f(t, x)| \leq \frac{1 + \cos\pi t}{t+30} (x^2 + 1)$$

and

$$|g(t, x)| \leq \frac{e^{-t^2}}{12+t} (x^2 + 2).$$

Then, we choose for $u(t) = (1 + \cos\pi t)/(t+30)$, $v(t) = e^{-t^2}/(12+t)$, $\Psi(x) = x^2 + 1$ and $\Theta(x) = x^2 + 2$. Thus, we can find that $\|u\| = 1/15$ and $\|v\| = 1/12$ and there exists a constant $N \in (4.247616499, 6.188984869)$ satisfying condition (H_3) . Therefore, applying the result in Theorem 3.3, the impulsive boundary value problem (17) with f and g given by (20) and (21) has at least solution on $[0, 7]$.

(iii) If the terms x^{26} and $|x|^{15}$ in (20) and (21) are replaced by $|x|^{25}$ and x^{14} , then these are as follows:

$$f(t, x) = \frac{1 + \cos \pi t}{t + 30} \left(\frac{|x|^{25}}{1 + x^{24}} + 1 \right), \quad (22)$$

$$g(t, x) = \frac{e^{-t^2}}{12 + t} \left(\frac{x^{14}}{1 + |x|^{13}} + 2 \right). \quad (23)$$

Therefore, we have

$$|f(t, x)| \leq \frac{1 + \cos \pi t}{t + 30} (|x| + 1) \quad \text{and} \quad |g(t, x)| \leq \frac{e^{-t^2}}{12 + t} (|x| + 2).$$

Since $\|v\|_{\mathcal{A}_2\Phi_3} + \|u\|_{\mathcal{A}_1\Phi_4} \approx 0.1932665035 < 1$, then the problem (17) with f and g given by (22) and (23) has at least one solution on $[0, 7]$, by using Corollary 3.1.

5 Conclusion

In this paper, we initiated the study of a new class of impulsive boundary value problems involving mixed fractional quantum and fractional Hadamard derivatives. Existence and uniqueness results are established by using tools from the fixed point theory. The uniqueness result is proved via Banach's contraction mapping principle, while the Leray-Schauder nonlinear alternative is used to establish an existence result. Examples illustrating the obtained results are also included. The results of the paper are new and enrich the existing literature on impulsive boundary value problems.

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