Research Article

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Sharp function estimates and boundedness for Toeplitz-type operators associated with general fractional integral operators

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Abstract: In this paper, we establish some sharp maximal function estimates for certain Toeplitz-type operators associated with some fractional integral operators with general kernel. As an application, we obtain the boundedness of the Toeplitz-type operators on the Lebesgue, Morrey and Triebel-Lizorkin spaces. The operators include the Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

Keywords: Toeplitz-type operator, sharp maximal function, Morrey space, Triebel-Lizorkin space, Lipschitz function, Littlewood-Paley operator, Marcinkiewicz operator, Bochner-Riesz operator, integral operators

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1 Introduction and preliminaries

The classical Morrey space was introduced by Morrey in [1] to investigate the local behavior of solutions to second-order elliptic partial differential equations (also see [2]). As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of operator on the Morrey spaces. The boundedness of the maximal operator, the singular integral operator, the fractional integral operator and their commutators on Morrey spaces have been studied by many authors (see [3–6]).

On the other hand, it is very important to study weighted boundedness of these operators in harmonic analysis. The weighted boundedness of these operators on Lebesgue spaces was obtained by Muckenhoupt, Wheeden, Coifman and Fefferman (see [7]). Now, as the development of singular integral operators and their commutators (see [8–24]), some more general operators than commutators are studied. The Toeplitz-type operators are the ones. In [9,12], some Toeplitz-type operators related to the singular integral operators and strongly singular integral operators are introduced, and the boundedness for the operators generated by BMO and Lipschitz functions are obtained. In this paper, we will study the Toeplitz-type operators generated by some fractional integral operators with general kernel. As an application, we obtain the boundedness of the operators on the Lebesgue, Morrey and Triebel-Lizorkin space. The operators include Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

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2 Preliminaries

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f , the sharp maximal function of f is defined by

$$M^\#(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x) dx$. It is well known that (see [7])

$$M^\#(f)(x) \approx \sup_{Q \ni x} \inf_{c \in \mathbb{C}} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

Let

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

For $\eta > 0$, let $M_\eta(f)(x) = M(|f|^\eta)^{1/\eta}(x)$.

For $0 < \eta < 1$ and $1 \leq r < \infty$, set

$$M_{\eta,r}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|^{1-\eta/n}} \int_Q |f(y)|^r dy \right)^{1/r}.$$

The A_p weight is defined by (see [7])

$$A_p = \left\{ w \in L^1_{\text{loc}}(R^n) : \sup_Q \left(\frac{1}{|Q|} \int_Q w(x) dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty \right\}, \quad 1 < p < \infty,$$

and

$$A_1 = \{ w \in L^p_{\text{loc}}(R^n) : M(w)(x) \leq Cw(x), \text{ a.e.} \}.$$

For $\beta > 0$ and $p > 1$, let $\dot{F}^{\beta,\infty}_p(R^n)$ be the homogeneous Triebel-Lizorkin space (see [18]).

For $0 < \beta < 1$, the Lipschitz space $\text{Lip}_\beta(R^n)$ is the space of functions f such that

$$\|f\|_{\text{Lip}_\beta} = \sup_{\substack{x,y \in R^n \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\beta} < \infty.$$

Definition 1. Let φ be a positive, increasing function on R^+ and there exists a constant $D > 0$ such that

$$\varphi(2t) \leq D\varphi(t) \quad \text{for } t \geq 0.$$

Let f be a locally integrable function on R^n . Set, for $0 \leq \eta < n$ and $1 \leq p < n/\eta$,

$$\|f\|_{L^{p,\eta,\varphi}} = \sup_{x \in R^n, d > 0} \left(\frac{1}{\varphi(d)^{1-p\eta/n}} \int_{Q(x,d)} |f(y)|^p dy \right)^{1/p},$$

where $Q(x, d) = \{y \in R^n : |x - y| < d\}$. The generalized Morrey space is defined by

$$L^{p,\eta,\varphi}(R^n) = \{f \in L^1_{\text{loc}}(R^n) : \|f\|_{L^{p,\eta,\varphi}} < \infty\}.$$

We write $L^{p,\eta,\varphi}(R^n) = L^{p,\varphi}(R^n)$ if $\eta = 0$, which is the generalized Morrey space. If $\varphi(d) = d^\delta$, $\delta > 0$, then $L^{p,\varphi}(R^n) = L^{p,\delta}(R^n)$, which is the classical Morrey spaces (see [25,26]). If $\varphi(d) = 1$, then $L^{p,\varphi}(R^n) = L^p(R^n)$, which is the Lebesgue space (see [7]).

As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the operator on the Morrey spaces (see [10–13]).

In this paper, we will study some fractional integral operators as follows (see [27]).

Definition 2. Fixed $0 < \delta < n$. Let $F_t(x, y)$ be defined on $R^n \times R^n \times [0, +\infty)$ and b be a locally integrable function on R^n , set

$$F_{\delta,t}(f)(x) = \int_{R^n} F_t(x, y)f(y)dy$$

and

$$F_{\delta,t}^b(f) = \sum_{k=1}^m F_{\delta,t}^{k,1} M_b F_t^{k,2}(f)$$

for every bounded and compactly supported function f , where $F_{\delta,t}^{k,1}(f)$ are $F_{\delta,t}(f)$ or $\pm I$ (the identity operator), $F_t^{k,2}(f)$ are a linear operator, $k = 1, \dots, m$ and $M_b(f) = bf$ for every bounded and compactly supported function f .

Let H be the Banach space $H = \{h : \|h\| < \infty\}$. For each fixed $x \in R^n$, we view $F_{\delta,t}(f)(x)$ as the mapping from $[0, +\infty)$ to H . And F_t satisfies: there is a sequence of positive constant numbers $\{C_j\}$ such that for any $j \geq 1$,

$$\int_{2^j|y-z| \leq |x-y|} (|F_t(x, y) - F_t(x, z)| + |F_t(y, x) - F_t(z, x)|) dx \leq C$$

and

$$\left(\int_{2^j|z-y| \leq |x-y| < 2^{j+1}|z-y|} (|F_t(x, y) - F_t(x, z)| + |F_t(y, x) - F_t(z, x)|)^q dy \right)^{1/q} \leq C_j (2^j|z-y|)^{-n/q'+\delta}$$

for some $1 < q' < 2$ with $1/q + 1/q' = 1$.

Set

$$T_\delta(f)(x) = |F_{\delta,t}(f)(x)|,$$

which T_δ is bounded on $L^2(R^n)$. Moreover, let b be a locally integrable function on R^n . The Toeplitz-type operator related to T_δ is defined by

$$T_\delta^b(f) = |F_{\delta,t}^b(f)|,$$

where $|F_t^{k,2}(f)| = T^{k,2}(f)$ are the bounded operators on $L^p(R^n)$ for $1 < p < \infty$.

Note that the commutator $[b, T](f) = bT(f) - T(bf)$ is a particular operator of the Toeplitz-type operators T_δ^b . The Toeplitz-type operators T_δ^b are the non-trivial generalizations of the commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [19,20]). The main purpose of this paper is to prove the sharp maximal inequalities for the Toeplitz-type operators T_δ^b . As the application, we obtain the the L^p -norm inequality and Triebel-Lizorkin spaces boundedness for the Toeplitz-type operators T_δ^b .

3 Theorems

We shall prove the following theorems.

Theorem 1. Let T_δ be the integral operator as Definition 2, the sequence $\{C_j\} \in l^1$, $0 < \beta < 1$, $q' \leq s < \infty$ and $b \in \text{Lip}_\beta(R^n)$. If $g \in L^u(R^n)$ ($1 < u < \infty$) and $F_{\delta,t}^1(g) = 0$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,

$$M^\#(T_\delta^b(f))(\tilde{x}) \leq C \|b\|_{\text{Lip}_\beta} \sum_{k=1}^m M_{\beta+\delta,s}(T^{k,2}(f))(\tilde{x}).$$

Theorem 2. Let T_δ be the integral operator as Definition 2, the sequence $\{2^{j\beta}C_j\} \in l^1$, $0 < \beta < 1$, $q' \leq s < \infty$ and $b \in \text{Lip}_\beta(R^n)$. If $g \in L^u(R^n)$ ($1 < u < \infty$) and $F_{\delta,t}^1(g) = 0$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,

$$\sup_{Q \ni \tilde{x}} \inf_{c \in R^n} \frac{1}{|Q|^{1+\beta/n}} \int_Q |T_\delta^b(f)(x) - c| dx \leq C \|b\|_{\text{Lip}_\beta} \sum_{k=1}^m M_{\delta,s}(T^{k,2}(f))(\tilde{x}).$$

Theorem 3. Let T_δ be the integral operator as Definition 2, the sequence $\{jC_j\} \in l^1$, $0 < \beta < 1$, $q' \leq s < \infty$ and $b \in \text{BMO}(R^n)$. If $g \in L^u(R^n)$ ($1 < u < \infty$) and $F_{\delta,t}^1(g) = 0$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,

$$M^\#(T_\delta^b(f))(\tilde{x}) \leq C \|b\|_{\text{BMO}} \sum_{k=1}^m M_{\delta,s}(T^{k,2}(f))(\tilde{x}).$$

Theorem 4. Let T_δ be the integral operator as Definition 2, the sequence $\{C_j\} \in l^1$, $0 < \beta < \min(1, n - \delta)$, $q' < p < n/(\beta + \delta)$, $1/r = 1/p - (\beta + \delta)/n$ and $b \in \text{Lip}_\beta(R^n)$. If $g \in L^u(R^n)$ ($1 < u < \infty$) and $F_{\delta,t}^1(g) = 0$, then T_δ^b is bounded from $L^p(R^n)$ to $L^r(R^n)$.

Theorem 5. Let T_δ be the integral operator as Definition 2, the sequence $\{C_j\} \in l^1$, $0 < \beta < \min(1, n - \delta)$, $q' < p < n/(\beta + \delta)$, $1/r = 1/p - (\beta + \delta)/n$, $0 < D < 2^n$ and $b \in \text{Lip}_\beta(R^n)$. If $g \in L^u(R^n)$ ($1 < u < \infty$) and $F_{\delta,t}^1(g) = 0$, then T_δ^b is bounded from $L^{p,\beta+\delta,\varphi}(R^n)$ to $L^{r,\varphi}(R^n)$.

Theorem 6. Let T_δ be the integral operator as Definition 2, the sequence $\{2^{j\beta}C_j\} \in l^1$, $0 < \beta < 1$, $q' < p < n/\delta$, $1/r = 1/p - \delta/n$ and $b \in \text{Lip}_\beta(R^n)$. If $g \in L^u(R^n)$ ($1 < u < \infty$) and $F_{\delta,t}^1(g) = 0$, then T_δ^b is bounded from $L^p(R^n)$ to $\dot{F}_r^{\beta,\infty}(R^n)$.

Theorem 7. Let T_δ be the integral operator as Definition 2, the sequence $\{jC_j\} \in l^1$, $q' < p < n/\delta$, $1/r = 1/p - \delta/n$ and $b \in \text{BMO}(R^n)$. If $g \in L^u(R^n)$ ($1 < u < \infty$) and $F_{\delta,t}^1(g) = 0$, then T_δ^b is bounded from $L^p(R^n)$ to $L^r(R^n)$.

Theorem 8. Let T_δ be the integral operator as Definition 2, the sequence $\{jC_j\} \in l^1$, $q' < p < n/\delta$, $1/r = 1/p - \delta/n$, $0 < D < 2^n$ and $b \in \text{BMO}(R^n)$. If $g \in L^u(R^n)$ ($1 < u < \infty$) and $F_{\delta,t}^1(g) = 0$, then T_δ^b is bounded from $L^{p,\delta,\varphi}(R^n)$ to $L^{r,\varphi}(R^n)$.

4 Proofs of theorems

To prove the theorems, we need the following lemma.

Lemma 1. (see [27]) Let T_δ be the integral operator as Definition 2 and the sequence $\{C_j\} \in l^1$. Then T_δ is bounded from $L^p(R^n)$ to $L^r(R^n)$ for $1 < p < n/\delta$ and $1/r = 1/p - \delta/n$.

Lemma 2. (see [18]) For $0 < \beta < 1$ and $1 < p < \infty$, we have

$$\|f\|_{F_p^{\beta,\infty}} \approx \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - f_Q| dx \right\|_{L^p} \approx \left\| \supinf_{Q \ni \cdot, c} \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - c| dx \right\|_{L^p}.$$

Lemma 3. (see [7]) Let $0 < p < \infty$ and $w \in \bigcup_{1 \leq r < \infty} A_r$. Then, for any smooth function f for which the left-hand side is finite,

$$\int_{R^n} M(f)(x)^p w(x) dx \leq C \int_{R^n} M^\#(f)(x)^p w(x) dx.$$

Lemma 4. (see [8]) Suppose that $0 < \eta < n$, $1 \leq s < p < n/\eta$ and $1/r = 1/p - \eta/n$. Then

$$\|M_{\eta,s}(f)\|_{L^r} \leq C \|f\|_{L^p}.$$

Lemma 5. Let $1 < p < \infty$, $0 < D < 2^n$. Then, for any smooth function f for which the left-hand side is finite,

$$\|M(f)\|_{L^{p,\varphi}} \leq C \|M^\#(f)\|_{L^{p,\varphi}}.$$

Proof. For any cube $Q = Q(x_0, d)$ in R^n , we know $M(\chi_Q) \in A_1$ for any cube $Q = Q(x, d)$ by [7]. Noticing that $M(\chi_Q) \leq 1$ and $M(\chi_Q)(x) \leq d^n/(|x - x_0| - d)^n$ if $x \in Q^c$, by Lemma 3, we have, for $f \in L^{p,\varphi}(R^n)$,

$$\begin{aligned} \int_Q M(f)(x)^p dx &= \int_{R^n} M(f)(x)^p \chi_Q(x) dx \\ &\leq \int_{R^n} M(f)(x)^p M(\chi_Q)(x) dx \\ &\leq C \int_{R^n} M^\#(f)(x)^p M(\chi_Q)(x) dx \\ &= C \left(\int_Q M^\#(f)(x)^p M(\chi_Q)(x) dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} M^\#(f)(x)^p M(\chi_Q)(x) dx \right) \\ &\leq C \left(\int_Q M^\#(f)(x)^p dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} M^\#(f)(x)^p \frac{|Q|}{|2^{k+1}Q|} dx \right) \\ &\leq C \left(\int_Q M^\#(f)(x)^p dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q} M^\#(f)(x)^p 2^{-kn} dy \right) \\ &\leq C \|M^\#(f)\|_{L^{p,\varphi}}^p \sum_{k=0}^{\infty} 2^{-kn} \varphi(2^{k+1}d) \\ &\leq C \|M^\#(f)\|_{L^{p,\varphi}}^p \sum_{k=0}^{\infty} (2^{-n}D)^k \varphi(d) \\ &\leq C \|M^\#(f)\|_{L^{p,\varphi}}^p \varphi(d), \end{aligned}$$

thus

$$\left(\frac{1}{\varphi(d)} \int_Q M(f)(x)^p dx \right)^{1/p} \leq C \left(\frac{1}{\varphi(d)} \int_Q M^\#(f)(x)^p dx \right)^{1/p}$$

and

$$\|M(f)\|_{L^{p,\varphi}} \leq C \|M^\#(f)\|_{L^{p,\varphi}}.$$

This completes the proof. $\square$

Lemma 6. (see [6]) Let $0 < D < 2^n$, $1 \leq s < p < n/\eta$ and $1/r = 1/p - \eta/n$. Then

$$\|M_{\eta,s}(f)\|_{L^{r,\eta,\varphi}} \leq C\|f\|_{L^{p,\varphi}}.$$

Lemma 7. Let $0 \leq \eta < n$, $0 < D < 2^n$ and T be the bounded linear operators on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Then

$$\|T(f)\|_{L^{p,\eta,\varphi}} \leq C\|f\|_{L^{p,\eta,\varphi}}.$$

The proof of the lemma is similar to that of Lemma 5 by Lemma 4; we omit the details.

Proof of Theorem 1. It suffices to prove for $f \in C_0^\infty(\mathbb{R}^n)$ and some constant C_0 , the following inequality holds:

$$\frac{1}{|Q|} \int_Q |T_\delta^b(f)(x) - C_0| dx \leq C\|b\|_{\text{Lip}_\beta} \sum_{k=1}^m M_{\beta+\delta,s}(T^{k,2}(f))(\tilde{x}).$$

Without loss of generality, we may assume $T_\delta^{k,1}$ are T_δ ($k = 1, \dots, m$). Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Write

$$F_{\delta,t}^b(f)(x) = F_{\delta,t}^{b-b_Q}(f)(x) = F_{\delta,t}^{(b-b_Q)\chi_{2Q}}(f)(x) + F_{\delta,t}^{(b-b_Q)\chi_{(2Q)^c}}(f)(x) = f_1(x) + f_2(x).$$

Then

$$\begin{aligned} \frac{1}{|Q|} \int_Q |T_\delta^b(f)(x) - |f_2(x_0)|| dx &= \frac{1}{|Q|} \int_Q ||F_{\delta,t}^b(f)(x) - |f_2(x_0)|| dx \\ &\leq \frac{1}{|Q|} \int_Q |F_{\delta,t}^b(f)(x) - f_2(x_0)| dx \\ &\leq \frac{1}{|Q|} \int_Q |f_1(x)| dx + \frac{1}{|Q|} \int_Q |f_2(x) - f_2(x_0)| dx = I_1 + I_2. \end{aligned}$$

For I_1 , choose $1 < r < \infty$ such that $1/r = 1/s - \delta/n$, by (L^s, L^r) -boundedness of T_δ (see Lemma 1) and Hölder's inequality, we obtain

$$\begin{aligned} \frac{1}{|Q|} \int_Q ||F_{\delta,t}^{k,1} M_{(b-b_Q)\chi_{2Q}} F_t^{k,2}(f)(x)|| dx &= \frac{1}{|Q|} \int_Q |T_\delta^{k,1} M_{(b-b_Q)\chi_{2Q}} T^{k,2}(f)(x)| dx \\ &\leq \left(\frac{1}{|Q|} \int_{\mathbb{R}^n} |T_\delta^{k,1} M_{(b-b_Q)\chi_{2Q}} T^{k,2}(f)(x)|^r dx \right)^{1/r} \\ &\leq C|Q|^{-1/r} \left(\int_{2Q} |M_{(b-b_Q)\chi_{2Q}} T^{k,2}(f)(x)|^s dx \right)^{1/s} \\ &\leq C|Q|^{-1/r} \left(\int_{2Q} (|b(x) - b_Q| |T^{k,2}(f)(x)|)^s dx \right)^{1/s} \\ &\leq C|Q|^{-1/r} \|b\|_{\text{Lip}_\beta} |2Q|^{\beta/n} |Q|^{1/s - (\beta+\delta)/n} \left(\frac{1}{|Q|^{1-s(\beta+\delta)/n}} \int_Q |T^{k,2}(f)(x)|^s dx \right)^{1/s} \\ &\leq C\|b\|_{\text{Lip}_\beta} M_{\beta+\delta,s}(T^{k,2}(f))(\tilde{x}), \end{aligned}$$

thus

$$I_1 \leq \sum_{k=1}^m \frac{1}{|Q|} \int_Q ||F_{\delta,t}^{k,1} M_{(b-b_Q)\chi_{2Q}} F_t^{k,2}(f)(x)|| dx \leq C\|b\|_{\text{Lip}_\beta} \sum_{k=1}^m M_{\beta+\delta,s}(T^{k,2}(f))(\tilde{x}).$$

For I_2 , by the boundedness of T and recalling that $s > q'$, we obtain, for $x \in Q$,

$$\begin{aligned}
 & \|F_{\delta,t}^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}F_t^{k,2}(f)(x) - F_{\delta,t}^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}F_t^{k,2}(f)(x_0)\| \\
 & \leq \int_{(2Q)^c} |b(y) - b_{2Q}| |F_t(x, y) - F_t(x_0, y)| |T^{k,2}(f)(y)| dy \\
 & \leq \sum_{j=1}^{\infty} \int_{2^j d \leq |y-x_0| < 2^{j+1}d} |F_t(x, y) - F_t(x_0, y)| |b(y) - b_{2Q}| |T^{k,2}(f)(y)| dy \\
 & \leq C \|b\|_{\text{Lip}_\beta} \sum_{j=1}^{\infty} |2^{j+1}Q|^{\beta/n} \left(\int_{2^j d \leq |y-x_0| < 2^{j+1}d} |F_t(x, y) - F_t(x_0, y)|^q dy \right)^{1/q} \left(\int_{2^{j+1}Q} |T^{k,2}(f)(y)|^{q'} dy \right)^{1/q'} \\
 & \leq C \|b\|_{\text{Lip}_\beta} \sum_{j=1}^{\infty} |2^{j+1}Q|^{\beta/n} C_j (2^j d)^{-n/q' + \delta} |2^{j+1}Q|^{1/q' - (\beta + \delta)/n} \left(\frac{1}{|2^{j+1}Q|^{1-s(\beta + \delta)/n}} \int_{2^{j+1}Q} |T^{k,2}(f)(y)|^s dy \right)^{1/s} \\
 & \leq C \|b\|_{\text{Lip}_\beta} M_{\beta + \delta, s}(T^{k,2}(f))(\tilde{x}) \sum_{j=1}^{\infty} C_j \\
 & \leq C \|b\|_{\text{Lip}_\beta} M_{\beta + \delta, s}(T^{k,2}(f))(\tilde{x}),
 \end{aligned}$$

thus

$$\begin{aligned}
 I_2 & \leq \frac{1}{|Q|} \int_Q \sum_{k=1}^m \|F_{\delta,t}^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}F_t^{k,2}(f)(x) - F_{\delta,t}^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}F_t^{k,2}(f)(x_0)\| dx \\
 & \leq C \|b\|_{\text{Lip}_\beta} \sum_{k=1}^m M_{\beta + \delta, s}(T^{k,2}(f))(\tilde{x}).
 \end{aligned}$$

These complete the proof of Theorem 1. $\square$

Proof of Theorem 2. It suffices to prove for $f \in C_0^\infty(\mathbb{R}^n)$ and some constant C_0 , the following inequality holds:

$$\frac{1}{|Q|^{1+\beta/n}} \int_Q |T_\delta^b(f)(x) - C_0| dx \leq C \|b\|_{\text{Lip}_\beta} \sum_{k=1}^m M_{\delta, s}(T^{k,2}(f))(\tilde{x}).$$

Without loss of generality, we may assume $T_\delta^{k,1}$ are T_δ ($k = 1, \dots, m$). Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Write

$$F_{\delta,t}^b(f)(x) = F_{\delta,t}^{b-b_Q}(f)(x) = F_{\delta,t}^{(b-b_Q)\chi_{2Q}}(f)(x) + F_{\delta,t}^{(b-b_Q)\chi_{(2Q)^c}}(f)(x) = f_1(x) + f_2(x)$$

and

$$\begin{aligned}
 \frac{1}{|Q|^{1+\beta/n}} \int_Q |T_\delta^b(f)(x) - f_2(x_0)| dx &= \frac{1}{|Q|^{1+\beta/n}} \int_Q \|F_{\delta,t}^b(f)(x) - f_2(x_0)\| dx \\
 &\leq \frac{1}{|Q|^{1+\beta/n}} \int_Q \|F_{\delta,t}^b(f)(x) - f_2(x_0)\| dx \\
 &\leq \frac{1}{|Q|^{1+\beta/n}} \int_Q \|f_1(x)\| dx + \frac{1}{|Q|^{1+\beta/n}} \int_Q \|f_2(x) - f_2(x_0)\| dx \\
 &= I_3 + I_4.
 \end{aligned}$$

By using the same argument as in the proof of Theorem 1, we obtain, for $1 < r < \infty$ with $1/r = 1/s - \delta/n$,

$$\begin{aligned}
I_3 &\leq \sum_{k=1}^m \frac{C}{|Q|^{1+\beta/n}} |Q|^{1-1/r} \left(\int_{2Q} |T_\delta^{k,1} M_{(b-b_Q)\chi_{2Q}} T^{k,2}(f)(x)|^r dx \right)^{1/r} \\
&\leq \sum_{k=1}^m \frac{C}{|Q|^{\beta/n}} |Q|^{-1/r} \left(\int_{2Q} |M_{(b-b_Q)\chi_{2Q}} T^{k,2}(f)(x)|^s dx \right)^{1/s} \\
&\leq \sum_{k=1}^m \frac{C}{|Q|^{\beta/n}} |Q|^{-1/r} \|b\|_{\text{Lip}_\beta} |2Q|^{\beta/n} \left(\int_{2Q} |T^{k,2}(f)(x)|^s dx \right)^{1/s} \\
&\leq C \|b\|_{\text{Lip}_\beta} \sum_{k=1}^m \left(\frac{1}{|2Q|^{1-s\delta/n}} \int_{2Q} |T^{k,2}(f)(x)|^s dx \right)^{1/s} \\
&\leq C \|b\|_{\text{Lip}_\beta} \sum_{k=1}^m M_{\delta,s}(T^{k,2}(f))(\tilde{x}), \\
I_4 &\leq \sum_{k=1}^m \frac{1}{|Q|^{1+\beta/n}} \int_Q \sum_{j=1}^\infty \int_{2^j d \leq |y-x_0| < 2^{j+1}d} |b(y) - b_{2Q}| |F_t(x, y) - F_t(x_0, y)| |T^{k,2}(f)(y)| dy dx \\
&\leq \sum_{k=1}^m \frac{C}{|Q|^{1+\beta/n}} \int_Q \sum_{j=1}^\infty \|b\|_{\text{Lip}_\beta} |2^{j+1}Q|^{\beta/n} \left(\int_{2^j d \leq |y-x_0| < 2^{j+1}d} |F_t(x, y) - F_t(x_0, y)|^q dy \right)^{1/q} \left(\int_{2^{j+1}Q} |T^{k,2}(f)(y)|^{q'} dy \right)^{1/q'} dx \\
&\leq C \|b\|_{\text{Lip}_\beta} \sum_{k=1}^m |Q|^{-\beta/n} \sum_{j=1}^\infty |2^{j+1}Q|^{\beta/n} C_j (2^j d)^{-n/q'+\delta} |2^{j+1}Q|^{1/q'-\delta/n} \left(\frac{1}{|2^{j+1}Q|^{1-s\delta/n}} \int_{2^{j+1}Q} |T^{k,2}(f)(y)|^s dy \right)^{1/s} \\
&\leq C \|b\|_{\text{Lip}_\beta} \sum_{k=1}^m M_{\delta,s}(T^{k,2}(f))(\tilde{x}) \sum_{j=1}^\infty 2^{j\beta} C_j \\
&\leq C \|b\|_{\text{Lip}_\beta} \sum_{k=1}^m M_{\delta,s}(T^{k,2}(f))(\tilde{x}).
\end{aligned}$$

This completes the proof of Theorem 2. $\square$

Proof of Theorem 3. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0 , the following inequality holds:

$$\frac{1}{|Q|} \int_Q |T_\delta^b(f)(x) - C_0| dx \leq C \|b\|_{\text{BMO}} \sum_{k=1}^m M_{\delta,s}(T^{k,2}(f))(\tilde{x}).$$

Without loss of generality, we may assume $T_\delta^{k,1}$ are $T_\delta(k = 1, \dots, m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Similar to the proof of Theorem 1, we have

$$F_{\delta,t}^b(f)(x) = F_{\delta,t}^{b-b_Q}(f)(x) = F_{\delta,t}^{(b-b_Q)\chi_{2Q}}(f)(x) + F_{\delta,t}^{(b-b_Q)\chi_{(2Q)^c}}(f)(x) = f_1(x) + f_2(x)$$

and

$$\begin{aligned}
\frac{1}{|Q|} \int_Q |T_\delta^b(f)(x) - f_2(x_0)| dx &= \frac{1}{|Q|} \int_Q \|F_{\delta,t}^b(f)(x) - f_2(x_0)\| dx \\
&\leq \frac{1}{|Q|} \int_Q \|F_{\delta,t}^b(f)(x) - f_2(x_0)\| dx \\
&\leq \frac{1}{|Q|} \int_Q \|f_1(x)\| dx + \frac{1}{|Q|} \int_Q \|f_2(x) - f_2(x_0)\| dx = I_5 + I_6.
\end{aligned}$$

For I_5 , choose $1 < t < s$, by Hölder's inequality and the boundedness of T_δ with $1 < r < \infty$ and $1/r = 1/t - \delta/n$, we obtain

$$\begin{aligned} \frac{1}{|Q|} \int_Q \|F_{\delta,t}^{k,1} M_{(b-b_Q)\chi_{2Q}} F_t^{k,2}(f)(x)\| dx &\leq \frac{1}{|Q|} \int_Q |T_\delta^{k,1} M_{(b-b_Q)\chi_{2Q}} T^{k,2}(f)(x)| dx \\ &\leq \left(\frac{1}{|Q|} \int_{R^n} |T_\delta^{k,1} M_{(b-b_Q)\chi_{2Q}} T^{k,2}(f)(x)|^r dx \right)^{1/r} \\ &\leq C|Q|^{-1/r} \left(\int_{R^n} |M_{(b-b_Q)\chi_{2Q}} T^{k,2}(f)(x)|^t dx \right)^{1/t} \\ &\leq C|Q|^{-1/r} \left(\int_{2Q} |T^{k,2}(f)(x)|^s dx \right)^{1/s} \left(\int_{2Q} |b(x) - b_Q|^{st/(s-t)} dx \right)^{(s-t)/st} \\ &\leq C\|b\|_{\text{BMO}} \left(\frac{1}{|2Q|^{1-s\delta/n}} \int_{2Q} |T^{k,2}(f)(x)|^s dx \right)^{1/s} \\ &\leq C\|b\|_{\text{BMO}} M_{\delta,s}(T^{k,2}(f))(\tilde{x}), \end{aligned}$$

thus

$$I_5 \leq \sum_{k=1}^l \frac{1}{|Q|} \int_Q \|F_{\delta,t}^{k,1} M_{(b-b_Q)\chi_{2Q}} F_t^{k,2}(f)(x)\| dx \leq C\|b\|_{\text{BMO}} \sum_{k=1}^m M_{\delta,s}(T^{k,2}(f))(\tilde{x}).$$

For I_6 , recalling that $s > q'$, taking $1 < p < \infty$ with $1/p + 1/q + 1/s = 1$, we obtain, for $x \in Q$,

$$\begin{aligned} &\|F_{\delta,t}^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} F_{\delta,t}^{k,2}(f)(x) - F_t^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} F_t^{k,2}(f)(x_0)\| \\ &\leq \int_{(2Q)^c} |b(y) - b_{2Q}| |F_t(x, y) - F_t(x_0, y)| |T^{k,2}(f)(y)| dy \\ &\leq \sum_{j=1}^{\infty} \int_{2^j d \leq |y-x_0| < 2^{j+1} d} |F_t(x, y) - F_t(x_0, y)| |b(y) - b_{2Q}| |T^{k,2}(f)(y)| dy \\ &\leq \sum_{j=1}^{\infty} \left(\int_{2^j d \leq |y-x_0| < 2^{j+1} d} |F_t(x, y) - F_t(x_0, y)|^q dy \right)^{1/q} \left(\int_{2^{j+1} Q} |b(y) - b_Q|^p dy \right)^{1/p} \left(\int_{2^{j+1} Q} |T^{k,2}(f)(y)|^s dy \right)^{1/s} \\ &\leq C\|b\|_{\text{BMO}} \sum_{j=1}^{\infty} C_j (2^j d)^{-n/q'+\delta} j (2^j d)^{n/p} (2^j d)^{n/s-\delta} \left(\frac{1}{|2^{j+1} Q|^{1-s\delta/n}} \int_{2^{j+1} Q} |T^{k,2}(f)(y)|^s dy \right)^{1/s} \\ &\leq C\|b\|_{\text{BMO}} M_{\delta,s}(T^{k,2}(f))(\tilde{x}) \sum_{j=1}^{\infty} j C_j \\ &\leq C\|b\|_{\text{BMO}} M_{\delta,s}(T^{k,2}(f))(\tilde{x}), \end{aligned}$$

thus

$$I_6 \leq \frac{1}{|Q|} \int_Q \sum_{k=1}^l \|F_{\delta,t}^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} F_t^{k,2}(f)(x) - F_t^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} F_t^{k,2}(f)(x_0)\| dx \leq C\|b\|_{\text{BMO}} \sum_{k=1}^l M_{\delta,s}(T^{k,2}(f))(\tilde{x}).$$

This completes the proof of Theorem 3. $\square$

Proof of Theorem 4. Choose $q' < s < p$ in Theorem 1, we have, by Lemmas 3 and 4,

$$\begin{aligned} \|T_\delta^b(f)\|_{L^{r'}} &\leq \|M(T_\delta^b(f))\|_{L^{r'}} \\ &\leq C\|M^\#(T_\delta^b(f))\|_{L^{r'}} \\ &\leq C\|b\|_{\text{Lip}_\beta} \sum_{k=1}^m \|M_{\beta+\delta,s}(T^{k,2}(f))\|_{L^{r'}} \\ &\leq C\|b\|_{\text{Lip}_\beta} \sum_{k=1}^m \|T^{k,2}(f)\|_{L^p} \\ &\leq C\|b\|_{\text{Lip}_\beta} \|f\|_{L^p}. \end{aligned}$$

This completes the proof. $\square$

Proof of Theorem 5. Choose $q' < s < p$ in Theorem 1, we have, by Lemmas 5, 6 and 7,

$$\begin{aligned} \|T_\delta^b(f)\|_{L^{r,\varphi}} &\leq \|M(T_\delta^b(f))\|_{L^{r,\varphi}} \\ &\leq C\|M^\#(T_\delta^b(f))\|_{L^{r,\varphi}} \\ &\leq C\|b\|_{\text{Lip}_\beta} \sum_{k=1}^m \|M_{\beta+\delta,s}(T^{k,2}(f))\|_{L^{r,\varphi}} \\ &\leq C\|b\|_{\text{Lip}_\beta} \sum_{k=1}^m \|T^{k,2}(f)\|_{L^{p,\beta+\delta,\varphi}} \\ &\leq C\|b\|_{\text{Lip}_\beta} \|f\|_{L^{p,\beta+\delta,\varphi}}. \end{aligned}$$

This completes the proof. $\square$

Proof of Theorem 6. Choose $q' < s < p$ in Theorem 2, we have, by Lemmas 2 and 3,

$$\begin{aligned} \|T_\delta^b(f)\|_{\dot{F}_r^{\beta,\infty}} &\leq C \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q |T_\delta^b(f)(x) - C_0| dx \right\|_{L^{r'}} \\ &\leq C\|b\|_{\text{Lip}_\beta} \sum_{k=1}^m \|M_{\delta,s}(T^{k,2}(f))\|_{L^{r'}} \leq C\|b\|_{\text{Lip}_\beta} \sum_{k=1}^m \|T^{k,2}(f)\|_{L^p} \\ &\leq C\|b\|_{\text{Lip}_\beta} \|f\|_{L^p}. \end{aligned}$$

This completes the proof. $\square$

Proof of Theorem 7. Choose $q' < s < p$ in Theorem 3, we have, by Lemmas 3 and 4,

$$\|T_\delta^b(f)\|_{L^{r'}} \leq \|M(T_\delta^b(f))\|_{L^p} \leq C\|M^\#(T_\delta^b(f))\|_{L^{r'}} \leq C\|b\|_{\text{BMO}} \|f\|_{L^p}.$$

This completes the proof. $\square$

Proof of Theorem 8. Choose $q' < s < p$ in Theorem 3, we have, by Lemmas 5, 6 and 7,

$$\begin{aligned} \|T_\delta^b(f)\|_{L^{r,\varphi}} &\leq \|M(T_\delta^b(f))\|_{L^{r,\varphi}} \\ &\leq C\|M^\#(T_\delta^b(f))\|_{L^{r,\varphi}} \\ &\leq C\|b\|_{\text{BMO}} \sum_{k=1}^m \|M_{\delta,s}(T^{k,2}(f))\|_{L^{r,\varphi}} \\ &\leq C\|b\|_{\text{BMO}} \sum_{k=1}^m \|T^{k,2}(f)\|_{L^{p,\delta,\varphi}} \\ &\leq C\|b\|_{\text{BMO}} \|f\|_{L^{p,\delta,\varphi}}. \end{aligned}$$

This completes the proof. $\square$

5 Applications

In this section, we shall apply Theorems 1–8 of the paper to some particular operators such as the Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

Application 1. Littlewood-Paley operator.

Fixed $0 \leq \delta < n$, $\varepsilon > 0$. Let ψ be a fixed function which satisfies:

- (1) $\int_{\mathbb{R}^n} \psi(x) dx = 0$,
- (2) $|\psi(x)| \leq C(1 + |x|)^{-(n+1-\delta)}$,
- (3) $|\psi(x+y) - \psi(x)| \leq C|y|^\varepsilon(1 + |x|)^{-(n+1+\varepsilon-\delta)}$ when $2|y| < |x|$.

Let $\psi_t(x) = t^{-n+\delta}\psi(x/t)$ for $t > 0$ and $F_t(f)(x) = \int_{\mathbb{R}^n} f(y)\psi_t(x-y)dy$. The Littlewood-Paley operator is defined by (see [28])

$$g_\psi(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t} \right)^{1/2}.$$

Set H as the space

$$H = \left\{ h : \|h\| = \left(\int_0^\infty |h(t)|^2 dt/t \right)^{1/2} < \infty \right\}.$$

Let b be a locally integrable function on $\mathbb{R}^n$. The Toeplitz-type operator related to the Littlewood-Paley operator is defined by

$$g_\psi^b(f)(x) = \left(\int_0^\infty |F_t^b(f)(x)|^2 \frac{dt}{t} \right)^{1/2},$$

where

$$F_t^b = \sum_{k=1}^m F_t^{k,1} M_b F_t^{k,2},$$

$F_t^{k,1}$ are F_t or $\pm I$ (the identity operator), $T^{k,2} = \|F_t^{k,2}\|$ are the bounded linear operators on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ and $k = 1, \dots, m$, $M_b(f) = bf$. Then, for each fixed $x \in \mathbb{R}^n$, $F_t^b(f)(x)$ may be viewed as the mapping from $[0, +\infty)$ to H , and it is clear that

$$g_\psi^b(f)(x) = \|F_t^b(f)(x)\|, \quad g_\psi(f)(x) = \|F_t(f)(x)\|.$$

It is easily shown that g_ψ^b satisfies the conditions of Theorems 1–8 (see [14–16]), thus Theorems 1–8 hold for g_ψ^b .

Application 2. Marcinkiewicz operator.

Fixed $0 \leq \delta < n$, $0 < \gamma \leq 1$. Let Ω be homogeneous of degree zero on $\mathbb{R}^n$ with $\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0$.

Assume that $\Omega \in \text{Lip}_\gamma(S^{n-1})$. Set $F_t(f)(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y) dy$. The Marcinkiewicz operator is defined by (see [29])

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2}.$$

Set H as the space

$$H = \left\{ h : \|h\| = \left(\int_0^\infty |h(t)|^2 dt/t^3 \right)^{1/2} < \infty \right\}.$$

Let b be a locally integrable function on R^n . The Toeplitz-type operator related to the Marcinkiewicz operator is defined by

$$\mu_\Omega^b(f)(x) = \left(\int_0^\infty |F_t^b(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_t^b = \sum_{k=1}^m F_t^{k,1} M_b F_t^{k,2},$$

$F_t^{k,1}$ are F_t or $\pm I$ (the identity operator), $T^{k,2} = \|F_t^{k,2}\|$ are the bounded linear operators on $L^p(R^n)$ for $1 < p < \infty$ and $k = 1, \dots, m$, $M_b(f) = bf$. Then, it is clear that

$$\mu_\Omega^b(f)(x) = \|F_t^b(f)(x)\|, \quad \mu_\Omega(f)(x) = \|F_t(f)(x)\|.$$

It is easily shown that μ_Ω^b satisfies the conditions of Theorems 1–8 (see [14–16, 30]), thus Theorems 1–8 hold for μ_Ω^b .

Application 3. Bochner-Riesz operator.

Let $\delta > (n-1)/2$, $F_t^\delta(\hat{f})(\xi) = (1 - t^2|\xi|^2)_+^\delta \hat{f}(\xi)$ and $B_t^\delta(z) = t^{-n} B^\delta(z/t)$ for $t > 0$. The maximal Bochner-Riesz operator is defined by (see [14])

$$B_{\delta,*}(f)(x) = \sup_{t>0} |F_t^\delta(f)(x)|.$$

Set H as the space $H = \{h : \|h\| = \sup_{t>0} |h(t)| < \infty\}$. Let b be a locally integrable function on R^n . The Toeplitz-type operator related to the maximal Bochner-Riesz operator is defined by

$$B_{\delta,*}^b(f)(x) = \sup_{t>0} |B_{\delta,t}^b(f)(x)|,$$

where

$$B_{\delta,t}^b = \sum_{k=1}^m F_t^{k,1} M_b F_t^{k,2},$$

$F_t^{k,1}$ are F_t or $\pm I$ (the identity operator), $T^{k,2} = \|F_t^{k,2}\|$ are the bounded linear operators on $L^p(R^n)$ for $1 < p < \infty$ and $k = 1, \dots, m$, $M_b(f) = bf$. Then

$$B_{\delta,*}^b(f)(x) = \|B_{\delta,t}^b(f)(x)\|, \quad B_*^\delta(f)(x) = \|B_t^\delta(f)(x)\|.$$

It is easily shown that $B_{\delta,*}^b$ satisfies the conditions of Theorems 1–8 with $\delta = 0$ (see [14]), thus Theorems 1–8 hold for $B_{\delta,*}^b$.

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