Research Article

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Euler-type sums involving multiple harmonic sums and binomial coefficients

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Abstract: In this paper, we mainly show that generalized Euler-type sums of multiple harmonic sums with reciprocal binomial coefficients can be expressed in terms of rational linear combinations of products of classical multiple zeta values (MZVs) and multiple harmonic star sums (MHSSs). Furthermore, applying the stuffle relations, we prove that the Euler-type sums involving products of generalized harmonic numbers and reciprocal binomial coefficients can be evaluated by MZVs and MHSSs.

Keywords: Euler-type sum, multiple zeta value, multiple harmonic (star) sum, binomial coefficient

MSC 2020: 11M32, 11M99

1 Introduction

We begin with some basic notations. Let $\mathbb{N}$ (resp. $\mathbb{Z}$) be the set of positive integers (resp. integers) and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. A finite sequence $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{N}^r$ is called a *composition*. We define its *weight* and the *depth* of $\mathbf{k}$, respectively, by

$$|\mathbf{k}| := k_1 + \dots + k_r \quad \text{and} \quad \text{dep}(\mathbf{k}) := r.$$

If $k_1 > 1$, $\mathbf{k}$ is called *admissible*.

For a composition $\mathbf{k} = (k_1, \dots, k_r)$ and positive integer n , the classical *multiple harmonic sums* (MHSs) and the classical *multiple harmonic star sums* (MHSSs) are defined by

$$\zeta_n(\mathbf{k}) \equiv \zeta_n(k_1, k_2, \dots, k_r) := \sum_{n \geq n_1 > n_2 > \dots > n_r \geq 1} \frac{1}{n_1^{k_1} n_2^{k_2} \dots n_r^{k_r}}, \quad (1.1)$$

$$\zeta_n^*(\mathbf{k}) \equiv \zeta_n^*(k_1, k_2, \dots, k_r) := \sum_{n \geq n_1 \geq n_2 \geq \dots \geq n_r \geq 1} \frac{1}{n_1^{k_1} n_2^{k_2} \dots n_r^{k_r}}, \quad (1.2)$$

when $n < k$, then $\zeta_n(\mathbf{k}) := 0$, and $\zeta_n(\emptyset) = \zeta_n^*(\emptyset) := 1$. For $\mathbf{k} = (k) \in \mathbb{N}$,

$$\zeta_n(k) \equiv H_n^{(k)} = \sum_{j=1}^n \frac{1}{j^k}$$

is the generalized harmonic number of order k , and furthermore, if $k = 1$, then $H_n \equiv H_n^{(1)}$ is the classical harmonic number. When taking the limit $n \rightarrow \infty$ in (1.1) and (1.2), we get the so-called classical *multiple zeta values* (MZVs) and the classical *multiple zeta star values* (MZSVs), respectively

$$\zeta(\mathbf{k}) := \lim_{n \rightarrow \infty} \zeta_n(\mathbf{k}), \quad (1.3)$$

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$$\zeta^*(\mathbf{k}) := \lim_{n \rightarrow \infty} \zeta_n^*(\mathbf{k}), \quad (1.4)$$

defined for an admissible composition $\mathbf{k}$ to ensure convergence of the series. The study of multiple zeta (star) values began in the early 1990s with the works of Hoffman [1] and Zagier [2]. For an admissible composition $\mathbf{k}$, Hoffman [1] called (1.3) multiple harmonic series. Zagier [2] called (1.3) MZVs since for $r = 1$ they generalize the usual Riemann zeta values $\zeta(k)$. Identities involving multiple harmonic (star) sums and multiple zeta (star) values have been extensively studied in the literature in the last three decades (see, e.g., [3,4] and references therein), since they play an essential role in number theory, combinatorics, analysis of algorithms and many other areas.

There are also some studies on the sums involving harmonic numbers and binomial coefficients. The readers may consult the works presented in [5–10]. Motivated by Wang-Xu's paper [8] and Xu et al. paper [10], they discussed the evaluations of some Euler-type sums involving harmonic numbers and binomial coefficients, such as

$$S_{\pi_1, q}(\mathbf{k}) = \sum_{n=1}^{\infty} \frac{H_n^{(\pi_1)}}{n^q \prod_{i=1}^r \binom{n+k_i}{k_i}}, \quad S_{\pi_1}^q(\mathbf{k}) = \sum_{n=1}^{\infty} \frac{n^q H_n^{(\pi_1)}}{\prod_{i=1}^r \binom{n+k_i}{k_i}},$$

and some other forms. In particular, they proved that the sums $S_{\pi_1, q}(\mathbf{k})$ and $S_{\pi_1}^q(\mathbf{k})$ can be expressed as rational linear combinations of products of zeta values, linear Euler sums (or double zeta values), harmonic numbers and double harmonic star sums.

In the present paper, we mainly show that generalized MHSs with reciprocal binomial coefficients of types

$$\sum_{m=1}^{\infty} \frac{\zeta_{m-1}(k_2, \dots, k_r)}{m^{k_1} \binom{m+n}{m}} \quad (1.5)$$

can be expressed in terms of linear combinations of classical MHSs and classical MZVs with depth less than or equal to r . Then applying the shuffle relations, also called quasi-shuffle relations (see [11]), we know that for any composition $\mathbf{k} = (k_1, \dots, k_r)$, the product $H_n^{(k_1)} \dots H_n^{(k_r)}$ can be expressed in terms of linear combinations of MHSs (for the explicit formula, see [12, equation (2.4)]). Hence, we can prove that the Euler-type sums

$$\sum_{m=1}^{\infty} \frac{H_{n-1}^{(k_2)} \dots H_{n-1}^{(k_r)}}{m^{k_1} \binom{m+n}{m}} \quad (1.6)$$

can be evaluated by a linear combinations of products of the classical MHVs and classical MZVs with depth less than or equal to r . Some illustrative special cases as well as immediate consequences of the main results are also considered. Here $(k_1, \dots, k_r) \in \mathbb{N}^r$ and $n \in \mathbb{N}$.

2 Explicit evaluations of Euler-type sums

In this section, we first develop closed form representations for the following integral involving multiple polylogarithm function

$$\int_0^1 x^{n-1} \text{Li}_{k_1, k_2, \dots, k_r}(x) dx$$

by using the iterated integral. Then, using the integral, we give some explicit evaluations for the Euler-type sums of MHSs with reciprocal binomial coefficients (1.5). For convenience, for a composition $\mathbf{k} \equiv \mathbf{k}_r =$

$(k_1, k_2, \dots, k_r)$, we let $\vec{k}_l := (k_1, k_2, \dots, k_l)$, $\vec{k}_0 = \emptyset$, $\overleftarrow{k}_l := (k_{l+1}, k_{l+2}, \dots, k_r)$, $\overleftarrow{k}_l(j) = (k_{l+1} + 1 - j, k_{l+2}, \dots, k_r)$, $\overleftarrow{k}_r = \overleftarrow{k}_r(j) := \emptyset$ and

$$H_n(k_1, k_2, \dots, k_r) := \frac{1}{n} \zeta_n^*(k_2, \dots, k_r) \quad \text{and} \quad H_n(k_1) := \frac{1}{n^{k_1}}.$$

The theory of iterated integrals was developed first by Chen in the 1960s [13,14]. It has played important roles in the study of algebraic topology and algebraic geometry in the past half century. Its simplest form over $\mathbb{R}$ is

$$\int_a^b f_p(t) dt f_{p-1}(t) dt \cdots f_1(t) dt := \int_{a < t_p < \cdots < t_1 < b} f_p(t_p) f_{p-1}(t_{p-1}) \cdots f_1(t_1) dt_1 dt_2 \cdots dt_p,$$

which can be easily extended to iterated path integrals over $\mathbb{C}$.

Theorem 2.1. Let $r, n \in \mathbb{N}$ and $\mathbf{k} := (k_1, \dots, k_r) \in \mathbb{N}^r$. Then

$$\int_0^1 x^{n-1} \text{Li}_{k_1, k_2, \dots, k_r}(x) dx = \sum_{l=0}^r (-1)^{|\vec{k}_l|-l} \sum_{j=1}^{k_{l+1}-1} (-1)^{j-1} H_n(\vec{k}_l, j) \zeta(\overleftarrow{k}_l(j)), \quad (2.1)$$

where $k_{r+1} := 2$ and $\text{Li}_{k_1, \dots, k_r}(z)$ is the single-variable multiple polylogarithm function defined by

$$\text{Li}_{k_1, \dots, k_r}(z) := \sum_{n_1 > \dots > n_r > 0} \frac{z^{n_1}}{n_1^{k_1} \cdots n_r^{k_r}}, \quad z \in [-1, 1] \text{ and } (k_1, z) \neq (1, 1). \quad (2.2)$$

Proof. According to the definition of multiple polylogarithm function, we have

$$\frac{d}{dx} \text{Li}_{k_1, k_2, \dots, k_r}(x) = \begin{cases} \frac{1}{x} \text{Li}_{k_1-1, k_2, \dots, k_r}(x), & k_1 > 1, \\ \frac{1}{1-x} \text{Li}_{k_2, \dots, k_r}(x), & k_1 = 1. \end{cases} \quad (2.3)$$

Hence, applying (2.3) we obtain the following iterated integral expression:

$$\text{Li}_{\mathbf{k}}(x) = \int_0^x \left(\frac{dt}{1-t} \right) \left(\frac{dt}{t} \right)^{k_r-1} \cdots \left(\frac{dt}{1-t} \right) \left(\frac{dt}{t} \right)^{k_1-1}. \quad (2.4)$$

Using integration by parts, by an elementary calculation, we deduce the recurrence relation that

$$\int_0^1 x^{n-1} \text{Li}_{\mathbf{k}}(x) dx = \sum_{j=1}^{k_r-1} \frac{(-1)^{j-1}}{n^j} \zeta(\mathbf{k}_{r-1}, k_r + 1 - j) + \frac{(-1)^{k_r-1}}{n^{k_r}} \sum_{j=1}^n \int_0^1 x^{j-1} \text{Li}_{\mathbf{k}_{r-1}}(x) dx.$$

Thus, we arrive at the desired formula by a direct calculation. $\square$

Corollary 2.2. For a composition $\mathbf{k} = (k_1, \dots, k_r)$ and $n \in \mathbb{N}$, we have

$$\sum_{m=1}^{\infty} \frac{\zeta_{m-1}(k_2, \dots, k_r)}{m^{k_1}(m+n)} = \sum_{l=0}^r (-1)^{|\vec{k}_l|-l} \sum_{j=1}^{k_{l+1}-1} (-1)^{j-1} H_n(\vec{k}_l, j) \zeta(\overleftarrow{k}_l(j)). \quad (2.5)$$

Proof. According to the definition of multiple polylogarithm function, we have

$$\int_0^1 x^{n-1} \text{Li}_{k_1, k_2, \dots, k_r}(x) dx = \sum_{m=1}^{\infty} \int_0^1 x^{n-1} \frac{\zeta_{m-1}(k_2, \dots, k_r)}{m^{k_1}} x^m dx = \sum_{m=1}^{\infty} \frac{\zeta_{m-1}(k_2, \dots, k_r)}{m^{k_1}(m+n)}.$$

Then, applying (2.1) yields the desired formula. $\square$

Theorem 2.3. For a composition $\mathbf{k} = (k_1, \dots, k_r)$ and $n \in \mathbb{N}$ with $k_{r+1} := 2$, we have

$$\sum_{m=1}^{\infty} \frac{\zeta_{m-1}(k_2, \dots, k_r)}{m^{k_1} \binom{m+n}{n}} = \sum_{i=1}^n (-1)^{i-1} i \binom{n}{i} \left\{ \sum_{l=0}^r (-1)^l \overline{\mathbf{k}}_l \sum_{j=1}^{k_{l+1}-1} (-1)^{j-1} H_i(\overline{\mathbf{k}}_l, j) \zeta(\overline{\mathbf{k}}_l(j)) \right\}. \quad (2.6)$$

Proof. For suitably selected sequences $\{f_m\}$, we remark that [10, p. 951]

$$\sum_{m=1}^{\infty} \frac{f_m}{m^p \binom{m+n}{n}} = \sum_{i=1}^n (-1)^{i-1} i \binom{n}{i} \sum_{m=1}^{\infty} \frac{f_m}{m^p(m+i)}.$$

Hence, we obtain

$$\sum_{m=1}^{\infty} \frac{\zeta_{m-1}(k_2, \dots, k_r)}{m^{k_1} \binom{m+n}{n}} = \sum_{i=1}^n (-1)^{i-1} i \binom{n}{i} \sum_{m=1}^{\infty} \frac{\zeta_{m-1}(k_2, \dots, k_r)}{m^{k_1}(m+i)}.$$

Thus, we finish the proof by using (2.5). $\square$

Let

$$\left\{ \begin{array}{c} a_1 \\ +a_2 \\ \vdots \\ +a_n \end{array} \right\} := a_1 + a_2 + \dots + a_n.$$

From Theorem 2.3, we can get the following examples.

Example 2.4. Setting $r = 1, 2$ and 3 in Theorem 2.3, we have

$$\sum_{m=1}^{\infty} \frac{1}{m^{k_1} \binom{m+n}{n}} = \sum_{i=1}^n (-1)^{i-1} i \binom{n}{i} \left\{ \sum_{j=1}^{k_1-1} (-1)^{j-1} \frac{\zeta(k_1+1-j)}{i^j} + (-1)^{k_1-1} \frac{H_i}{i^{k_1}} \right\}, \quad (2.7)$$

$$\sum_{m=1}^{\infty} \frac{\zeta_{m-1}(k_2)}{m^{k_1} \binom{m+n}{n}} = \sum_{i=1}^n (-1)^{i-1} i \binom{n}{i} \left\{ \sum_{j=1}^{k_1-1} (-1)^{j-1} \frac{\zeta(k_1+1-j, k_2)}{i^j} + (-1)^{k_1+k_2} \frac{\zeta_i^*(k_2, 1)}{i^{k_1}} \right. \\ \left. + (-1)^{k_1-1} \sum_{j=1}^{k_2-1} (-1)^{j-1} \frac{\zeta_i(j)}{i^{k_1}} \zeta(k_2+1-j) \right\}, \quad (2.8)$$

$$\sum_{m=1}^{\infty} \frac{\zeta_{m-1}(k_2, k_3)}{m^{k_1} \binom{m+n}{n}} = \sum_{i=1}^n (-1)^{i-1} i \binom{n}{i} \left\{ \sum_{j=1}^{k_1-1} \frac{(-1)^{j-1}}{i^j} \zeta(k_1+1-j, k_2, k_3) + (-1)^{k_1-1} \sum_{j=1}^{k_2-1} (-1)^{j-1} \frac{\zeta_i(j)}{i^{k_1}} \zeta(k_2+1-j, k_3) \right. \\ \left. + (-1)^{k_1+k_2} \sum_{j=1}^{k_3-1} (-1)^{j-1} \frac{\zeta_i^*(k_2, j)}{i^{k_1}} \zeta(k_3+1-j) + (-1)^{k_1+k_2+k_3-1} \frac{\zeta_i^*(k_2, k_3, 1)}{i^{k_1}} \right\}. \quad (2.9)$$

Now, we give an evaluation for (1.6) with $r = 3$. The key idea to study (1.6) is to express $H_n^{(k_1)} \dots H_n^{(k_r)}$ by using general MHSs of the form (1.1) by applying the stuffle relations, also called quasi-shuffle relations (see [11]). In fact, we know that for any composition $\mathbf{k} = (k_1, \dots, k_r)$, the product $H_n^{(k_1)} \dots H_n^{(k_r)}$ can be expressed in terms of linear combinations of MHSs (for the explicit formula, see [12, equation (2.4)]), for example

$$H_n^{(m)} H_n^{(p)} = \zeta_n(m) \zeta_n(p) = \zeta_n(m, p) + \zeta_n(p, m) + \zeta_n(p+m). \quad (2.10)$$

Hence, the Euler-type sums (1.6) can be expressed in terms of linear combinations of products of MHSs and MZVs.

Theorem 2.5. For positive integers k_1, k_2, k_3 and n , we have

$$\sum_{m=1}^{\infty} \frac{H_{m-1}^{(k_2)} H_{m-1}^{(k_3)}}{m^{k_1} \binom{m+n}{n}} = \sum_{i=1}^n (-1)^{i-1} i \binom{n}{i} \left\{ \begin{aligned} & \sum_{j=1}^{k_1-1} \frac{(-1)^{j-1}}{i^j} \left[\zeta(k_1+1-j, k_2, k_3) + \zeta(k_1+1-j, k_3, k_2) \right] \\ & + (-1)^{k_1+k_2+k_3} \frac{\zeta_i^*(k_2, k_3, 1) + \zeta_i^*(k_3, k_2, 1) + \zeta_i^*(k_2+k_3, 1)}{i^{k_1}} \\ & + (-1)^{k_1-1} \sum_{j=1}^{k_2-1} (-1)^{j-1} \zeta(k_2+1-j, k_3) \frac{\zeta_i^*(j)}{i^{k_1}} \\ & + (-1)^{k_1-1} \sum_{j=1}^{k_3-1} (-1)^{j-1} \zeta(k_3+1-j, k_2) \frac{\zeta_i^*(j)}{i^{k_1}} \\ & + (-1)^{k_1-1} \sum_{j=1}^{k_2+k_3-1} (-1)^{j-1} \zeta(k_2+k_3+1-j) \frac{\zeta_i^*(j)}{i^{k_1}} \\ & + (-1)^{k_1+k_2} \sum_{j=1}^{k_3-1} (-1)^{j-1} \frac{\zeta_i^*(k_2, j)}{i^{k_1}} \zeta(k_3+1-j) \\ & + (-1)^{k_1+k_3} \sum_{j=1}^{k_2-1} (-1)^{j-1} \frac{\zeta_i^*(k_3, j)}{i^{k_1}} \zeta(k_2+1-j) \end{aligned} \right\}. \quad (2.11)$$

Proof. The theorem follows immediately from (2.9) and (2.10). $\square$

3 More general sums

From [8, Theorem 3.1], we know that the key idea to study the following more general Euler-type sums

$$\sum_{m=1}^{\infty} \frac{\zeta_{m-1}(k_2, \dots, k_r)}{m^{k_1} \prod_{i=1}^p \binom{m+n_i}{n_i}} \quad \text{and} \quad \sum_{m=1}^{\infty} \frac{H_{m-1}^{(k_2)} \dots H_{m-1}^{(k_r)}}{m^{k_1} \prod_{i=1}^p \binom{m+n_i}{n_i}}$$

is to express the sums $\sum_{m=1}^{\infty} \frac{\zeta_{m-1}(k_2, \dots, k_r)}{m^{k_1} \prod_{i=1}^p \binom{m+n_i}{n_i}}$ by using the MZVs and multiple harmonic (star) sums, where $(k_1, \dots, k_r) \in \mathbb{N}^r$, $(n_1, \dots, n_p) \in \mathbb{N}^p$ and $n, p, q \in \mathbb{N}$. Moreover, according to the definition of multiple polylogarithm function, we have the identity

$$\int_0^1 x^{n-1} \log^p(x) \text{Li}_{k_1, k_2, \dots, k_r}(x) dx = p! (-1)^p \sum_{m=1}^{\infty} \frac{\zeta_{m-1}(k_2, \dots, k_r)}{m^{k_1} (m+n)^{p+1}}. \quad (3.1)$$

Hence, we need to evaluate the integral on left-hand side of the aforementioned formula. We can get the following recurrence relation.

Theorem 3.1. For a composition $\mathbf{k} = (k_1, \dots, k_r)$ and $p, n \in \mathbb{N}$, we have

$$\begin{aligned} \int_0^1 x^{n-1} \log^p(x) \text{Li}_{k_1, k_2, \dots, k_r}(x) dx &= \sum_{l=1}^r (-1)^{|\vec{\mathbf{k}}_{l-1}|+l-1} \sum_{j=1}^{k_l} (-1)^j \sum_{n_0 \geq n_1 \geq \dots \geq n_{l-1} \geq 1} \frac{p \int_0^1 x^{n_{l-1}-1} \log^{p-1}(x) \text{Li}_{\overleftarrow{\mathbf{k}}_{l-1}(j)}(x) dx}{n_0^{k_1} n_1^{k_2} \dots n_{l-2}^{k_{l-1}} n_{l-1}^j} \\ &+ p! \sum_{l=1}^r (-1)^{|\vec{\mathbf{k}}_l|+l+p-1} \frac{\zeta(p+1, k_{l+1}, \dots, k_r)}{n^{k_l}} \zeta_n^*(k_2, \dots, k_r) \\ &+ (-1)^{|\vec{\mathbf{k}}_r|+r+p} p! \frac{\zeta_n^*(k_2, \dots, k_r, p+1)}{n^{k_1}}, \end{aligned} \quad (3.2)$$

where $n_0 := n$.

Proof. Using integration by parts, we deduce

$$\begin{aligned}
 & \int_0^1 x^{n-1} \log^p(x) \text{Li}_{k_1, k_2, \dots, k_r}(x) dx \\
 &= \frac{1}{n} \int_0^1 \log^p(x) \text{Li}_{\overrightarrow{k_r}}(x) dx^n \\
 &= -\frac{p}{n} \int_0^1 x^{n-1} \log^{p-1}(x) \text{Li}_{k_1, \dots, k_r}(x) dx - \frac{1}{n} \int_0^1 x^{n-1} \log^p(x) \text{Li}_{k_1-1, k_2, \dots, k_r}(x) dx \\
 &= \dots \\
 &= \sum_{j=1}^{k_1-1} \frac{(-1)^j}{n^j} p \int_0^1 x^{n-1} \log^{p-1}(x) \text{Li}_{k_1+1-j, k_2, \dots, k_r}(x) dx + \frac{(-1)^{k_1-1}}{n^{k_1-1}} \int_0^1 x^{n-1} \log^p(x) \text{Li}_{k_1, k_2, \dots, k_r}(x) dx \\
 &= \sum_{j=1}^{k_1} \frac{(-1)^j}{n^j} p \int_0^1 x^{n-1} \log^{p-1}(x) \text{Li}_{k_1+1-j, k_2, \dots, k_r}(x) dx \\
 &\quad + (-1)^{k_1+p} p! \frac{\zeta(p+1, k_2, \dots, k_r)}{n^{k_1}} - \frac{(-1)^{k_1}}{n^{k_1}} \sum_{n_1=1}^n \int_0^1 x^{n_1-1} \log^p(x) \text{Li}_{k_2, \dots, k_r}(x) dx \\
 &= \dots \\
 &= \sum_{l=1}^r (-1)^{l-1+k_1+\dots+k_{l-1}} \sum_{j=1}^{k_l} (-1)^j \sum_{n=n_0 \geq n_1 \geq \dots \geq n_{l-1} \geq 1} \frac{p}{n_0^{k_1} n_1^{k_2} \dots n_{l-2}^{k_{l-1}} n_{l-1}^j} \int_0^1 x^{n_{l-1}-1} \log^{p-1}(x) \text{Li}_{k_l+1-j, k_{l+1}, \dots, k_r}(x) dx \\
 &\quad + \sum_{l=1}^r (-1)^{l-1} (-1)^{k_1+\dots+k_{l-1}+p} p! \frac{\zeta(p+1, k_{l+1}, \dots, k_r)}{n^{k_1}} \zeta_n^*(k_2, \dots, k_r) \\
 &\quad + \frac{(-1)^{k_1+\dots+k_r+p}}{n^{k_1}} \sum_{n \geq n_1 \geq \dots \geq n_r \geq 1} \frac{1}{n_1^{k_1} \dots n_{r-1}^{k_r}} \int_0^1 x^{n_r-1} \log^p(x) dx \\
 &= \sum_{l=1}^r (-1)^{|\overrightarrow{k_{l-1}}|+l-1} \sum_{j=1}^{k_l} (-1)^j \sum_{n_0 \geq n_1 \geq \dots \geq n_{l-1} \geq 1} \frac{p \int_0^1 x^{n_{l-1}-1} \log^{p-1}(x) \text{Li}_{\overrightarrow{k_{l-1}(j)}}(x) dx}{n_0^{k_1} n_1^{k_2} \dots n_{l-2}^{k_{l-1}} n_{l-1}^j} \\
 &\quad + p! \sum_{l=1}^r (-1)^{|\overrightarrow{k_l}|+l+p-1} \frac{\zeta(p+1, k_{l+1}, \dots, k_r)}{n^{k_1}} \zeta_n^*(k_2, \dots, k_r) \\
 &\quad + (-1)^{|\overrightarrow{k_r}|+r+p} p! \frac{\zeta_n^*(k_2, \dots, k_r, p+1)}{n^{k_1}}.
 \end{aligned}$$

Thus, the desired evaluation is obtained. $\square$

Therefore, from Theorems 2.1 and 3.1, we arrive at the conclusion that the integral

$$\int_0^1 x^{n-1} \log^p(x) \text{Li}_{k_1, k_2, \dots, k_r}(x) dx$$

can be written as linear combinations of products of MZVs and MHSSs. For example, we have

$$\begin{aligned}
 \int_0^1 x^{n-1} \log(x) \text{Li}_{\{1\}_r}(x) dx &= -\frac{\zeta_n^*(\{1\}_r)}{n^2} - \frac{1}{n} \sum_{i=1}^r (\zeta_n^*(\{1\}_{i-1}, 2, \{1\}_{r-i}) - \zeta(i+1) \zeta_n^*(\{1\}_{r-i})) \\
 &= \sum_{m=1}^{\infty} \frac{\zeta_{m-1}(\{1\}_{r-1})}{m(n+m)^2},
 \end{aligned} \tag{3.3}$$

where $\{1\}_d$ is denoted by the sequence of 1's with d repetitions.

Furthermore, we can get the following corollary.

Corollary 3.2. For positive integers r and n , we have

$$\sum_{m=1}^{\infty} \frac{\zeta_{m-1}(\{1\}_{r-1})}{m \binom{m+n}{n}^2} = \sum_{j=1}^n \left(j \binom{n}{j} \right)^2 \left\{ \frac{\zeta_j^*(\{1\}_r)}{j^2} - 2(H_{n-j} - H_{j-1}) \frac{\zeta_j^*(\{1\}_r)}{j} + \frac{1}{j} \sum_{i=1}^r (\zeta_j^*(\{1\}_{i-1}, 2, \{1\}_{r-i}) - \zeta(i+1) \zeta_j^*(\{1\}_{r-i})) \right\}. \quad (3.4)$$

Proof. Applying [8, equation (13)], we can find that

$$\sum_{m=1}^{\infty} \frac{\zeta_{m-1}(\{1\}_{r-1})}{m \binom{m+n}{n}^2} = \sum_{j=1}^n \left(j \binom{n}{j} \right)^2 \left\{ \sum_{m=1}^{\infty} \frac{\zeta_{m-1}(\{1\}_{r-1})}{m(m+j)^2} - 2(H_{n-j} - H_{j-1}) \sum_{m=1}^{\infty} \frac{\zeta_{m-1}(\{1\}_{r-1})}{m(m+j)} \right\}.$$

Using (2.5) and (3.3) yields the desired formula. $\square$

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