Research Article

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On the convergence of two-step modulus-based matrix splitting iteration method

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Abstract: In this paper, based on the relationship between the linear complementarity problem and its reformulated fixed-point equation, we discuss the conditions of the modulus-based type iteration methods. Moreover, we present some convergence results on the two-step modulus-based matrix splitting iteration method with an H_+ -matrix. Finally, we give the numerical experiments.

Keywords: linear complementarity problem, modulus-based type iteration method, convergence condition

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1 Introduction

The linear complementarity problem is to solve $z \in R^n$ such that

$$z^T r = 0 \text{ with } z \geq 0, \quad r = Az + q \geq 0, \quad (1)$$

where $A = (a_{ij}) \in R^{n \times n}$, $q \in R^n$ are given and the symbol “T” denotes the transpose operation. This problem is usually abbreviated as $LCP(A, q)$, which has many applications, such as the optimal stopping in Markov chain and the free boundary problems of journal bearings. For the detailed materials, see [1–8] and references therein.

In order to compute the numerical solution of the $LCP(A, q)$ with a large and sparse system matrix A , many kinds of modulus-based type iteration methods are presented recently, such as the modulus-based iteration method [8], the nonstationary extrapolated modulus method [9], the modified modulus-based iteration method [10], the modulus-based matrix splitting iteration method [3], the general modulus-based matrix splitting method [6], the two-step modulus-based matrix splitting iteration method [11], the two-sweep modulus-based matrix splitting iteration method [12], the preconditioned general modulus-based matrix splitting method [13], the accelerated modulus-based matrix splitting iteration methods [14,15], and so on. Most of these modulus-based type iteration methods are very practical and efficient. The common character of these methods is that the $LCP(A, q)$ is reformulated as a fixed-point equation through introducing the parameter matrices, and the original $LCP(A, q)$ is solved by solving such an equation with iteration approaches. Besides the modulus-based type iteration methods mentioned above, there are other iteration methods as well as the direct methods. For the detailed materials, readers can refer to [2,3,8–10,16–28] and references therein.

We further study the modulus-based type iteration methods for solving the $LCP(A, q)$ in this paper. Although there are many modulus-based type iteration methods, the relationship between the linear

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complementarity problem and the reformulated equation is rarely discussed. Meanwhile, the involved system matrices in these modulus-based type iteration methods are mainly the positive definite matrix and the H_+ -matrix, which are two types of P -matrix, and these methods and the theories have not been extended to the general P -matrix. By studying the connection between the $LCP(A, q)$ and the fixed-point equation, we propose the conditions of the modulus-based type iteration method, which extends the application scope of this method. For the two-step modulus-based matrix splitting iteration method [11], the convergence has been further studied in [16]. Since this method is very effective, it has been applied to solve other complementarity problems. For the recent works on the two-step modulus-based methods for solving other complementarity problems, readers can refer to [29–35]. As the main part of this paper, we discuss the convergence problem of this method when the system matrix A is an H_+ -matrix and present some convergence conclusions from the matrix spectral radius. In the end, we provide the numerical experiments to verify the proposed results and illustrate the special cases of this method.

2 Preliminaries

In this section, we review some necessary preliminaries in brief, including the concepts, the notations, the reformulation of $LCP(A, q)$, and the two-step modulus-based matrix splitting iteration method. We first introduce the definition of P -matrix, and for this definition and its equivalent expressions as well as the properties of P -matrix, readers can refer to [8, 36–38].

Definition 2.1. A matrix $A \in R^{n \times n}$ is called a P -matrix if all of its principal minors are positive.

$A \in R^{n \times n}$ is called a Z -matrix if $a_{ij} \leq 0$, $i \neq j$, $i, j = 1, 2, \dots, n$, and $A \in R^{n \times n}$ is called an M -matrix if A is a Z -matrix with $A^{-1} \geq 0$. The comparison matrix of A is denoted by $\langle A \rangle = (\langle a_{ij} \rangle)$ with

$$\langle a_{ij} \rangle = \begin{cases} -|a_{ij}|, & i \neq j, \\ |a_{ij}|, & i = j, \end{cases}$$

where “ $|\cdot|$ ” denotes the absolute value function. $A \in R^{n \times n}$ is called an H -matrix if $\langle A \rangle$ is an M -matrix, and the H -matrix is called an H_+ -matrix if all of its diagonal elements are positive [3]. The absolute value matrix of a real matrix $A \in R^{n \times n}$ is denoted by $|A| = (|a_{ij}|)$. The positive vector and the nonnegative vector are denoted by $v > 0$ and $v \geq 0$, respectively. For details, readers can refer to [3, 6] and references therein.

By setting

$$z = |x| + x \quad \text{and} \quad r = \Omega(|x| - x), \quad (2)$$

the $LCP(A, q)$ can be transformed into a fixed-point equation

$$(\Omega + A)x = (\Omega - A)|x| - q, \quad (3)$$

where Ω is a positive diagonal matrix. Equation (3) is a particular case of the original form (5) in [3], where γ is set to be 1. All the modulus-based type iteration methods introduced before are based on (3) or its changed forms.

From (2), we know that if z^* is a solution of (1), then we can obtain a solution x^* of (3), that is,

$$x^* = \frac{z^* - \Omega^{-1}(Az^* + q)}{2}. \quad (4)$$

On the contrary, if x^* is a solution of (3), then we can obtain a solution z^* of (1), that is,

$$z^* = |x^*| + x^*. \quad (5)$$

In other words, we can solve (1) by solving (3) from [3].

For all of the modulus-based type iteration methods mentioned before, the involved initial iteration vector $x^{(0)}$ is arbitrary, which means that equation (3) should have a unique solution if the modulus-based type iteration methods are convergent. Thus, we first discuss the unique solution problem of equation (3) in Section 3.

The two-step modulus-based matrix splitting iteration method and its particular case, that is, the two-step modulus-based accelerated overrelaxation (MBAOR) iteration method are briefly reviewed in the following, and readers can refer to [11] for details.

2.1 The two-step modulus-based matrix splitting iteration method

Let $A = M_i - N_i$, $i = 1, 2$ be two splittings of the matrix A . Given an initial vector $x^{(0)}$, compute $x^{(k+1)} \in \mathbb{R}^n$ by solving the linear systems

$$\begin{cases} (M_1 + \Omega)x^{(k+\frac{1}{2})} = N_1x^{(k)} + (\Omega - A)|x^{(k)}| - q, \\ (M_2 + \Omega)x^{(k+1)} = N_2x^{(k+\frac{1}{2})} + (\Omega - A)|x^{(k+\frac{1}{2})}| - q. \end{cases} \quad (6)$$

Then set

$$z^{(k+1)} = |x^{(k+1)}| + x^{(k+1)} \quad (7)$$

for $k = 0, 1, 2, \dots$ until the iteration sequence $\{z^{(k)}\}_{k=1}^{+\infty} \subset \mathbb{R}^n$ is convergent. Here Ω is a given positive diagonal matrix.

Suppose $A = D - L - U$, where D , $-L$, $-U$ are the diagonal, strictly lower-triangular and strictly upper-triangular matrices of A , respectively. Set

$$\begin{aligned} M_1 &= \frac{1}{\alpha_1}(D - \beta_1 L), \quad N_1 = \frac{1}{\alpha_1}((1 - \alpha_1)D + (\alpha_1 - \beta_1)L + \alpha_1 U), \\ M_2 &= \frac{1}{\alpha_2}(D - \beta_2 U), \quad N_2 = \frac{1}{\alpha_2}((1 - \alpha_2)D + (\alpha_2 - \beta_2)U + \alpha_2 L), \end{aligned} \quad (8)$$

where $\alpha_1 > 0$, $\beta_1 \geq 0$, $\alpha_2 > 0$, $\beta_2 \geq 0$, then we have the particular two-step modulus-based matrix splitting iteration method based on the accelerate overrelaxation (AOR) splittings (8), that is, the two-step MBAOR iteration method below.

2.2 The two-step MBAOR iteration method

$$\begin{cases} \left[\frac{1}{\alpha_1}(D - \beta_1 L) + \Omega \right] x^{(k+\frac{1}{2})} = \left[\frac{1}{\alpha_1}((1 - \alpha_1)D + (\alpha_1 - \beta_1)L + \alpha_1 U) \right] x^{(k)} + (\Omega - A)|x^{(k)}| - q, \\ \left[\frac{1}{\alpha_2}(D - \beta_2 U) + \Omega \right] x^{(k+1)} = \left[\frac{1}{\alpha_2}((1 - \alpha_2)D + (\alpha_2 - \beta_2)U + \alpha_2 L) \right] x^{(k+\frac{1}{2})} + (\Omega - A)|x^{(k+\frac{1}{2})}| - q, \end{cases} \quad (9)$$

with $z^{(k+1)} = |x^{(k+1)}| + x^{(k+1)}$ and $\alpha_1 > 0$, $\beta_1 \geq 0$, $\alpha_2 > 0$, $\beta_2 \geq 0$.

3 Main results

In this section, we discuss the conditions of the modulus-based type iteration methods and the convergence of the two-step modulus-based matrix splitting iteration method.

3.1 The conditions of the modulus-based type iteration method

From the relation between the linear complementarity problem (1) and equation (3), we have the following conclusion.

Theorem 3.1.1. *If the linear complementarity problem (1) has a unique solution for any $q \in R^n$, i.e., A is a P -matrix, then equation (3) has a unique solution for any positive diagonal matrix Ω .*

Proof. From (4), we know that equation (3) has solutions for any $q \in R^n$ and any positive diagonal matrix Ω . Let x^* and x^{**} be two solutions of (3), that is,

$$\begin{aligned}(\Omega + A)x^* &= (\Omega - A)|x^*| - q, \\ (\Omega + A)x^{**} &= (\Omega - A)|x^{**}| - q.\end{aligned}\tag{10}$$

On one hand, from (5), we know that both $|x^*| + x^*$ and $|x^{**}| + x^{**}$ are the solutions of (1). Since the linear complementarity problem (1) has a unique solution, we have

$$|x^*| + x^* = |x^{**}| + x^{**}.\tag{11}$$

Then x^* and x^{**} have the same positive elements with the same positions.

On the other hand, from (10), we have

$$(\Omega + A)(x^* - x^{**}) = (\Omega - A)(|x^*| - |x^{**}|).\tag{12}$$

By reformulating, we have

$$\Omega(x^* - x^{**} - |x^*| + |x^{**}|) + A(x^* - x^{**} + |x^*| - |x^{**}|) = 0.\tag{13}$$

Then, from (11),

$$\Omega(x^* - x^{**} - |x^*| + |x^{**}|) = 0\tag{14}$$

holds. So, we have

$$|x^*| - x^* = |x^{**}| - x^{**}.\tag{15}$$

Thus, x^* and x^{**} have the same negative elements with the same positions. Therefore, we have

$$x^* = x^{**}.\quad \square$$

In Theorem 2.1 of [3], the author has proved the equivalence between the $LCP(A, q)$ and the fixed-point equation in terms of solutions. Here, we emphasize the case that the linear complementarity problem has a unique solution, which is the basis of exploring the convergence of modulus-based type iteration methods for solving the linear complementarity problems.

Corollary 3.1.1. *If (1) has a unique solution for some $q \in R^n$, then (3) has a unique solution for any positive diagonal matrix Ω .*

Proof. Suppose z^* is the unique solution of (1). From (4), we know that (3) has a solution x_Ω^* for any positive diagonal matrix Ω . Let x_Ω^{**} be another solution of (3), similar to the proof of Theorem 3.1.1, we can prove that

$$\begin{aligned}|x_\Omega^*| + x_\Omega^* &= |x_\Omega^{**}| + x_\Omega^{**}, \\ |x_\Omega^*| - x_\Omega^* &= |x_\Omega^{**}| - x_\Omega^{**}.\end{aligned}\tag{16}$$

Thus, $x_\Omega^* = x_\Omega^{**}$ holds and the conclusion is proved. $\square$

3.2 The convergence of the two-step modulus-based matrix splitting iteration method

Now, we discuss a concrete modulus-based type iteration method, that is, the two-step modulus-based matrix splitting iteration method.

Theorem 3.2.1. Let $A \in R^{n \times n}$ be an H_+ -matrix and $A = M_i - N_i$, $i = 1, 2$, be two splittings of A with $\langle M_1 \rangle$ and $\langle M_2 \rangle$ being two M -matrices. If

$$\rho((\langle M_2 \rangle + \Omega)^{-1}(|N_2| + |\Omega - A|)(\langle M_1 \rangle + \Omega)^{-1}(|N_1| + |\Omega - A|)) < 1 \quad (17)$$

for some positive diagonal matrix Ω , then $\{z^{(k)}\}_{k=1}^{+\infty} \subset R_+^n$ generated by (6) with (7) converges to the unique solution of the LCP(A, q) for any initial vector $x^{(0)} \in R^n$.

Proof. From the assumption that A is an H_+ -matrix, we know that the LCP(A, q) has a unique solution z^* and equation (3) also has a unique solution x^* for any positive diagonal matrix Ω from Theorem 3.1.1. Since $\langle M_1 \rangle$, $\langle M_2 \rangle$ are two M -matrices, we have $\langle A \rangle$, $\langle M_1 \rangle$, $\langle M_1 \rangle + \Omega$, $\langle M_2 \rangle$, and $\langle M_2 \rangle + \Omega$ are M -matrices. Then

$$\begin{cases} (M_1 + \Omega)x^* = N_1x^* + (\Omega - A)|x^*| - q, \\ (M_2 + \Omega)x^* = N_2x^* + (\Omega - A)|x^*| - q. \end{cases} \quad (18)$$

Therefore, from (6) and (18), we have

$$\begin{cases} x^{(k+\frac{1}{2})} - x^* = (M_1 + \Omega)^{-1}(N_1(x^{(k)} - x^*) + (\Omega - A)(|x^{(k)}| - |x^*|)), \\ x^{(k+1)} - x^* = (M_2 + \Omega)^{-1}\left(N_2\left(x^{(k+\frac{1}{2})} - x^*\right) + (\Omega - A)\left(|x^{(k+\frac{1}{2})}| - |x^*|\right)\right). \end{cases} \quad (19)$$

By taking absolute value on both sides of each equation in (19), combining with the property of M -matrix, we have

$$|x^{(k+1)} - x^*| \leq (\langle M_2 \rangle + \Omega)^{-1}(|N_2| + |\Omega - A|)(\langle M_1 \rangle + \Omega)^{-1}(|N_1| + |\Omega - A|)(|x^{(k)} - x^*|). \quad (20)$$

Then, if the spectral radius

$$\rho((\langle M_2 \rangle + \Omega)^{-1}(|N_2| + |\Omega - A|)(\langle M_1 \rangle + \Omega)^{-1}(|N_1| + |\Omega - A|)) < 1,$$

the two-step modulus-based matrix splitting iteration method is convergent. $\square$

For an H_+ -matrix, both the H -compatible splitting and the H -splitting satisfy the first condition in Theorem 3.2.1, that is, $\langle M_i \rangle$, $i = 1, 2$, are M -matrices. Therefore, for these two special matrix splittings, we can adopt the two-step modulus-based matrix splitting iteration method by checking condition (17).

For the H -compatible splitting, there is a convergence condition in [11], that is,

$$\rho((\langle M_2 \rangle + \Omega)^{-1}(2|N_2| + |\Omega - M_2|)(\langle M_1 \rangle + \Omega)^{-1}(2|N_1| + |\Omega - M_1|)) < 1. \quad (21)$$

Since

$$\begin{aligned} 0 &\leq (\langle M_2 \rangle + \Omega)^{-1}(|N_2| + |\Omega - A|)(\langle M_1 \rangle + \Omega)^{-1}(|N_1| + |\Omega - A|) \\ &\leq (\langle M_2 \rangle + \Omega)^{-1}(2|N_2| + |\Omega - M_2|)(\langle M_1 \rangle + \Omega)^{-1}(2|N_1| + |\Omega - M_1|), \end{aligned}$$

we have

$$\begin{aligned} &\rho((\langle M_2 \rangle + \Omega)^{-1}(|N_2| + |\Omega - A|)(\langle M_1 \rangle + \Omega)^{-1}(|N_1| + |\Omega - A|)) \\ &\leq \rho((\langle M_2 \rangle + \Omega)^{-1}(2|N_2| + |\Omega - M_2|)(\langle M_1 \rangle + \Omega)^{-1}(2|N_1| + |\Omega - M_1|)) \end{aligned}$$

holds. Therefore, (17) in Theorem 3.2.1 is better.

In the following, we mainly discuss the H -compatible splitting for the two-step modulus-based matrix splitting iteration method and obtain the concrete convergence results.

Theorem 3.2.2. Let $A \in R^{n \times n}$ be an H_+ -matrix and $A = M_i - N_i$, $i = 1, 2$, be two H -compatible splittings of A , i.e., $\langle A \rangle = \langle M_i \rangle - |N_i|$, $i = 1, 2$. If

$$\Omega \geq \text{diag}(A), \quad (22)$$

then $\{z^{(k)}\}_{k=1}^{+\infty} \subset R_+^n$ generated by (6) with (7) is convergent for any initial vector $x^{(0)} \in R^n$.

Proof. From Theorem 3.2.1, we only need to prove that inequality (17) holds when $\Omega \geq \text{diag}(A)$.

Since A is an H_+ -matrix, $\langle A \rangle$ is an M -matrix. Thus, there exists a positive vector $v > 0$ such that

$$\langle A \rangle v > 0. \quad (23)$$

If $\Omega \geq \text{diag}(A)$, then both $\Omega + \langle M_1 \rangle$ and $\Omega + \langle M_2 \rangle$ are M -matrices. For the matrix appeared in (17), we have

$$\begin{aligned} 0 &\leq (\Omega + \langle M_1 \rangle)^{-1}(|N_1| + |\Omega - A|) = I - (\Omega + \langle M_1 \rangle)^{-1}(\Omega - |\Omega - A| + \langle A \rangle) = I - (\Omega + \langle M_1 \rangle)^{-1}2\langle A \rangle, \\ 0 &\leq (\Omega + \langle M_2 \rangle)^{-1}(|N_2| + |\Omega - A|) = I - (\Omega + \langle M_2 \rangle)^{-1}(\Omega - |\Omega - A| + \langle A \rangle) = I - (\Omega + \langle M_2 \rangle)^{-1}2\langle A \rangle. \end{aligned} \quad (24)$$

It follows that

$$\begin{aligned} &(\langle M_2 \rangle + \Omega)^{-1}(|N_2| + |\Omega - A|)(\langle M_1 \rangle + \Omega)^{-1}(|N_1| + |\Omega - A|)v \\ &= (I - 2(\Omega + \langle M_2 \rangle)^{-1}\langle A \rangle)(I - 2(\Omega + \langle M_1 \rangle)^{-1}\langle A \rangle)v \\ &< (I - 2(\Omega + \langle M_2 \rangle)^{-1}\langle A \rangle)v \\ &< v. \end{aligned} \quad (25)$$

Therefore, according to the properties of the nonnegative matrix, we have

$$\rho((\langle M_2 \rangle + \Omega)^{-1}(|N_2| + |\Omega - A|)(\langle M_1 \rangle + \Omega)^{-1}(|N_1| + |\Omega - A|)) < 1,$$

and then this theorem is proved. $\square$

Theorem 3.2.3. Let $A \in R^{n \times n}$ be an H_+ -matrix with $\rho(D^{-1}(|L| + |U|)) < \frac{1}{2}$, where D , $-L$, $-U$ are the diagonal, the strictly lower-triangular, and the strictly upper-triangular matrices of A , respectively. Let $A = M_i - N_i$, $i = 1, 2$, be two H -compatible splittings of A . If

$$\Omega \geq \frac{1}{2}D, \quad (26)$$

then $\{z^{(k)}\}_{k=1}^{+\infty} \subset R_+^n$ generated by (6) with (7) is convergent for any initial vector $x^{(0)} \in R^n$.

Proof. Similar to the proof of Theorem 3.2.2, we only prove (17) holds here. Since A is an H_+ matrix, the equalities (24) can be concretely reformulated as

$$\begin{aligned} 0 &\leq (\Omega + \langle M_1 \rangle)^{-1}(|N_1| + |\Omega - A|) = I - (\Omega + \langle M_1 \rangle)^{-1}(\Omega - |\Omega - A| + \langle A \rangle) \\ &= I - (\Omega + \langle M_1 \rangle)^{-1}(\Omega - |\Omega - D| + D - 2(|L| + |U|)) \\ &= I - (\Omega + \langle M_1 \rangle)^{-1}(\Omega - |\Omega - D| + D(I - 2D^{-1}(|L| + |U|))), \\ 0 &\leq (\Omega + \langle M_2 \rangle)^{-1}(|N_2| + |\Omega - A|) = I - (\Omega + \langle M_2 \rangle)^{-1}(\Omega - |\Omega - A| + \langle A \rangle) \\ &= I - (\Omega + \langle M_2 \rangle)^{-1}(\Omega - |\Omega - D| + D - 2(|L| + |U|)) \\ &= I - (\Omega + \langle M_2 \rangle)^{-1}(\Omega - |\Omega - D| + D(I - 2D^{-1}(|L| + |U|))). \end{aligned} \quad (27)$$

If $\Omega \geq \frac{1}{2}D$, then $\Omega - |\Omega - D| \geq 0$. Moreover, $I - 2D^{-1}(|L| + |U|)$, $D(I - 2D^{-1}(|L| + |U|))$, and $\Omega - |\Omega - D| + D(I - 2D^{-1}(|L| + |U|))$ are three M -matrices. Thus, there exists a positive vector v such that

$$(\Omega - |\Omega - D| + D(I - 2D^{-1}(|L| + |U|)))v > 0$$

and

$$\begin{aligned}
 & (\langle M_2 \rangle + \Omega)^{-1}(|N_2| + |\Omega - A|)(\langle M_1 \rangle + \Omega)^{-1}(|N_1| + |\Omega - A|)v \\
 &= (I - (\Omega + \langle M_2 \rangle)^{-1}(\Omega - |\Omega - D| + D(I - 2D^{-1}(|L| + |U|)))) \\
 &\quad \times (I - (\Omega + \langle M_1 \rangle)^{-1}(\Omega - |\Omega - D| + D(I - 2D^{-1}(|L| + |U|))))v \\
 &< (I - (\Omega + \langle M_2 \rangle)^{-1}(\Omega - |\Omega - D| + D(I - 2D^{-1}(|L| + |U|))))v < v.
 \end{aligned} \tag{28}$$

Thus, from the property of the nonnegative matrix, we have

$$\rho((\langle M_2 \rangle + \Omega)^{-1}(|N_2| + |\Omega - A|)(\langle M_1 \rangle + \Omega)^{-1}(|N_1| + |\Omega - A|)) < 1,$$

and then the conclusion is proved. $\square$

Lemma 3.2.1. Let $A \in R^{n \times n}$ be an H_+ -matrix with $A = D - L - U$, where $D, -L, -U$ are the same as that of Theorem 3.2.3. Let $A = M_i - N_i, i = 1, 2$, be two AOR splittings, that is, $M_i, N_i, i = 1, 2$, satisfy (8). Then the two splittings are H -compatible splittings if and only if

$$0 \leq \beta_i \leq \alpha_i \leq 1, \text{ with } \alpha_i \neq 0, i = 1, 2. \tag{29}$$

Proof. We only prove the case $i = 1$.

The sufficiency: According to the definition of the H -compatible splitting, we only need to prove that the equality $\langle A \rangle = \langle M_1 \rangle - |N_1|$ holds when $0 \leq \beta_1 \leq \alpha_1 \leq 1$ with $\alpha_1 \neq 0$.

If $0 \leq \beta_1 \leq \alpha_1 \leq 1$ with $\alpha_1 \neq 0$, then

$$\begin{aligned}
 \langle M_1 \rangle - |N_1| &= \left\langle \frac{1}{\alpha_1}(D - \beta_1 L) \right\rangle - \left| \frac{1}{\alpha_1}((1 - \alpha_1)D + (\alpha_1 - \beta_1)L + \alpha_1 U) \right| \\
 &= \frac{1}{\alpha_1}D - \frac{\beta_1}{\alpha_1}|L| - \left(\frac{1 - \alpha_1}{\alpha_1}D + \frac{\alpha_1 - \beta_1}{\alpha_1}|L| + |U| \right) \\
 &= D - |L| - |U| = \langle A \rangle.
 \end{aligned}$$

The necessity: From (8), we have the splitting $A = M_1 - N_1$ is an H -compatible splitting with $\alpha_1 > 0, \beta_1 \geq 0$ and A is an H_+ -matrix. Then

$$\begin{aligned}
 \langle A \rangle &= \langle M_1 \rangle - |N_1| \\
 &= \left\langle \frac{1}{\alpha_1}(D - \beta_1 L) \right\rangle - \left| \frac{1}{\alpha_1}((1 - \alpha_1)D + (\alpha_1 - \beta_1)L + \alpha_1 U) \right| \\
 &= \frac{1}{\alpha_1}D - \frac{\beta_1}{\alpha_1}|L| - \left(\left| \frac{1 - \alpha_1}{\alpha_1} \right| D + \left| \frac{\alpha_1 - \beta_1}{\alpha_1} \right| |L| + |U| \right) \\
 &= \left(\frac{1}{\alpha_1} - \left| \frac{1 - \alpha_1}{\alpha_1} \right| \right) D - \left(\frac{\beta_1}{\alpha_1} + \left| \frac{\alpha_1 - \beta_1}{\alpha_1} \right| \right) |L| - |U| \\
 &= D - |L| - |U|.
 \end{aligned}$$

By comparing both sides of the last equality, we can obtain $0 \leq \beta_1 \leq \alpha_1 \leq 1$ with $\alpha_1 \neq 0$. $\square$

Theorem 3.2.4. Let A be an H_+ -matrix with $A = D - L - U$, where $D, -L, -U$ are the same as that of Theorem 3.2.3. Then the two-step MBAOR iteration method (9) is convergent if

$$0 \leq \beta_i \leq \alpha_i \leq 1, \text{ with } \alpha_i \neq 0, i = 1, 2, \text{ and } \Omega \geq D. \tag{30}$$

Proof. From $0 \leq \beta_i \leq \alpha_i \leq 1$ with $\alpha_i \neq 0, i = 1, 2$, and Lemma 3.2.1, we know that the splittings in (8) are two H -compatible splittings. It follows from Theorem 3.2.2 that if $\Omega \geq D$, the two-step MBAOR iteration method (9) is convergent. $\square$

From Theorem 3.2.3 and Lemma 3.2.1, we have the following conclusion on the two-step MBAOR iteration method, whose proof is similar to that of Theorem 3.2.4 and is omitted here.

Theorem 3.2.5. Let A be an H_+ -matrix with $A = D - L - U$, where $D, -L, -U$ are the same as that of Theorem 3.2.3. If $\rho(D^{-1}(|L| + |U|)) < \frac{1}{2}$, then the two-step MBAOR iteration method (9) is convergent if

$$0 \leq \beta_i \leq \alpha_i \leq 1, \text{ with } \alpha_i \neq 0, i = 1, 2, \text{ and } \Omega \geq \frac{1}{2}D. \quad (31)$$

Theorem 3.2.6. Let A be an H_+ -matrix with $A = D - L - U$, where $D, -L, -U$ are the same as that of Theorem 3.2.3. Then the two-step MBAOR iteration method (9) is convergent if either of the following conditions holds:

$$\begin{aligned} 0 \leq \beta_i \leq \alpha_i \leq 1, \alpha_i \neq 0, \text{ with } \Omega \geq D, \\ 0 \leq \beta_i \leq \alpha_i, 1 \leq \alpha_i < \frac{1}{\rho(D^{-1}(|L| + |U|))}, \alpha_1 = \alpha_2 \neq 0, \text{ with } \Omega \geq D. \end{aligned} \quad (32)$$

Moreover, if $\rho(D^{-1}(|L| + |U|)) < \frac{1}{2}$, then the two-step MBAOR iteration method (9) is convergent if either of the following conditions holds:

$$\begin{aligned} 0 \leq \beta_i \leq \alpha_i \leq 1, \alpha_i \neq 0, \text{ with } \Omega \geq \frac{1}{2}D, \\ 0 \leq \beta_i \leq \alpha_i, 1 \leq \alpha_i \leq 2, \alpha_1 = \alpha_2 \neq 0, \text{ with } \frac{1}{2}\left(3 - \frac{2}{\alpha_i}\right)D \leq \Omega \leq D. \end{aligned} \quad (33)$$

Proof. From $\alpha_i > 0, \beta_i \geq 0, i = 1, 2$, we have $\langle M_i \rangle, i = 1, 2$, in (9) are M -matrices. From Theorem 3.2.1, we need to only prove that (17) holds. For the two-step MBAOR iteration method, (27) in Theorem 3.2.3 can turn to

$$\begin{aligned} 0 \leq (\Omega + \langle M_1 \rangle)^{-1}(|N_1| + |\Omega - A|) &= I - (\Omega + \langle M_1 \rangle)^{-1}\left(\Omega + \frac{1 - |1 - \alpha_1|}{\alpha_1}D - |\Omega - D| - 2(|L| + |U|)\right), \\ 0 \leq (\Omega + \langle M_2 \rangle)^{-1}(|N_2| + |\Omega - A|) &= I - (\Omega + \langle M_2 \rangle)^{-1}\left(\Omega + \frac{1 - |1 - \alpha_2|}{\alpha_2}D - |\Omega - D| - 2(|L| + |U|)\right). \end{aligned} \quad (34)$$

Then

$$\Omega + \frac{1 - |1 - \alpha_i|}{\alpha_i}D - |\Omega - D| - 2(|L| + |U|) = \begin{cases} \Omega + D - |\Omega - D| - 2(|L| + |U|) & \text{when } \alpha_i \leq 1, \\ \Omega + \frac{2 - \alpha_i}{\alpha_i}D - |\Omega - D| - 2(|L| + |U|) & \text{when } \alpha_i \geq 1. \end{cases} \quad (35)$$

Next we discuss the two cases of (35) separately.

(i) When $0 < \alpha_i \leq 1$, we have

$$\Omega + D - |\Omega - D| - 2(|L| + |U|) = \begin{cases} 2\langle A \rangle & \text{when } \Omega \geq D, \\ \Omega - |\Omega - D| + D(I - 2D^{-1}(|L| + |U|)) & \text{when } \Omega \geq \frac{1}{2}D. \end{cases} \quad (36)$$

If $\Omega \geq D$ or $\Omega \geq \frac{1}{2}D$ with $\rho(D^{-1}(|L| + |U|)) < \frac{1}{2}$, then the matrix (36) is an M -matrix, and the two-step MBAOR iteration method is convergent from the proofs of Theorems 3.2.2 and 3.2.3, respectively.

(ii) When $\alpha_i \geq 1$, we have

$$\Omega + \frac{2 - \alpha_i}{\alpha_i}D - |\Omega - D| - 2(|L| + |U|) = \begin{cases} 2D\left(\frac{1}{\alpha_i}I - D^{-1}(|L| + |U|)\right) & \text{when } \Omega \geq D, \\ 2\Omega + \left(\frac{2}{\alpha_i} - 3\right)D + D(I - 2D^{-1}(|L| + |U|)) & \text{when } \Omega \leq D. \end{cases} \quad (37)$$

If $\Omega \geq D$ with $1 \leq \alpha_i < \frac{1}{\rho(D^{-1}(|L| + |U|))}$ or $\frac{1}{2}\left(3 - \frac{2}{\alpha_i}\right)D \leq \Omega \leq D$ with $1 \leq \alpha_i \leq 2$ and $\rho(D^{-1}(|L| + |U|)) < \frac{1}{2}$, then the matrix (37) is an M -matrix. Set $\alpha_1 = \alpha_2$, for the two cases, there exist positive vectors u, v satisfying

$$2D\left(\frac{1}{\alpha_i}I - D^{-1}(|L| + |U|)\right)u > 0, \left(2\Omega + \left(\frac{2}{\alpha_i} - 3\right)D + D(I - 2D^{-1}(|L| + |U|))\right)v > 0, \quad i = 1, 2, \quad (38)$$

respectively. Thus, we have

$$\begin{aligned} & (\Omega + \langle M_2 \rangle)^{-1}(|N_2| + |\Omega - A|)(\Omega + \langle M_1 \rangle)^{-1}(|N_1| + |\Omega - A|)u \\ &= \left(I - (\Omega + \langle M_2 \rangle)^{-1} \left(2D\left(\frac{1}{\alpha_2}I - D^{-1}(|L| + |U|)\right) \right) \right) \left(I - (\Omega + \langle M_1 \rangle)^{-1} \left(2D\left(\frac{1}{\alpha_1}I - D^{-1}(|L| + |U|)\right) \right) \right) u \\ &< \left(I - (\Omega + \langle M_2 \rangle)^{-1} \left(2D\left(\frac{1}{\alpha_2}I - D^{-1}(|L| + |U|)\right) \right) \right) u < u \end{aligned} \quad (39)$$

and

$$\begin{aligned} & (\Omega + \langle M_2 \rangle)^{-1}(|N_2| + |\Omega - A|)(\Omega + \langle M_1 \rangle)^{-1}(|N_1| + |\Omega - A|)v \\ &= \left(I - (\Omega + \langle M_2 \rangle)^{-1} \left(2\Omega + \left(\frac{2}{\alpha_2} - 3\right)D + D(I - 2D^{-1}(|L| + |U|)) \right) \right) \\ &\quad \times \left(I - (\Omega + \langle M_1 \rangle)^{-1} \left(2\Omega + \left(\frac{2}{\alpha_1} - 3\right)D + D(I - 2D^{-1}(|L| + |U|)) \right) \right) v \\ &< \left(I - (\Omega + \langle M_2 \rangle)^{-1} \left(2\Omega + \left(\frac{2}{\alpha_2} - 3\right)D + D(I - 2D^{-1}(|L| + |U|)) \right) \right) v < v. \end{aligned} \quad (40)$$

From the properties of nonnegative matrix, we always have

$$\rho((\Omega + \langle M_2 \rangle)^{-1}(|N_2| + |\Omega - A|)(\Omega + \langle M_1 \rangle)^{-1}(|N_1| + |\Omega - A|)) < 1.$$

Therefore, the two-step MBAOR iteration is convergent for the case $\alpha_i \geq 1$ with $\alpha_1 = \alpha_2$.

Collecting (i) and (ii), that is, $0 < \alpha_i \leq 1$ and $\alpha_i \geq 1$, (32) and (33) can be proved. $\square$

From (35), (37), and the proof of Theorem 3.2.6, we can find that if the nonnegative matrix $D^{-1}(|L| + |U|)$ has a positive eigenvector $w > 0$, which satisfies

$$D^{-1}(|L| + |U|)w = \rho(D^{-1}(|L| + |U|))w,$$

then replacing u and v as w in (38) we can prove (17) holds even if $\alpha_1 \neq \alpha_2$. Thus, the condition $\alpha_1 = \alpha_2$ can be deleted in Theorem 3.2.6. We know that when A is an irreducible H_+ -matrix, such positive eigenvector w exists. Hence, the conclusion of Theorem 3.2.6 can be further improved in this case and the result is omitted here.

It is well known that the AOR splitting (8) includes some particular cases, for example: (I) Jacobian(J) splitting when $\alpha_i = 1, \beta_i = 0$; (II) Gauss-Seidel splitting when $\alpha_i = \beta_i = 1$; (III) Successive overrelaxation splitting when $\alpha_i = \beta_i > 0$. For these particular matrix splittings, the conclusions in Theorems 3.2.5 and 3.2.6 also hold.

4 Numerical experiments

In this section, we supply some experiments. We consider the two-step MBAOR iteration method and its particular case, that is, the two-step modulus-based Gauss-Seidel (MBGS) iteration method. The iteration

steps, the elapsed time, the norm of the residual vector, and the spectral radius function in (17) are denoted by IT, CPU, RES, and $\rho(\alpha, \beta)$, respectively. RES and $\rho(\alpha, \beta, \omega)$ are defined as:

$$\text{RES}(z^{(k)}) = \|\min(z^{(k)}, Az^{(k)} + q)\|_2, \rho(\alpha, \beta, \omega) = \rho(\mathcal{L}_{2,\alpha,\beta,\omega}\mathcal{L}_{1,\alpha,\beta,\omega}).$$

Here, $\|\cdot\|$ is a vector norm, $z^{(k)}$ is the k th approximate solution, and

$$\begin{aligned}\mathcal{L}_{1,\alpha_1,\beta_1,\omega} &= \left(\left\langle \frac{1}{\alpha_1}(D - \beta_1 L) \right\rangle + \omega D \right)^{-1} \left(\left| \frac{1}{\alpha_1}((1 - \alpha_1)D + (\alpha_1 - \beta_1)L + \alpha_1 U) \right| + |\omega D - A| \right), \\ \mathcal{L}_{2,\alpha_2,\beta_2,\omega} &= \left(\left\langle \frac{1}{\alpha_2}(D - \beta_2 U) \right\rangle + \omega D \right)^{-1} \left(\left| \frac{1}{\alpha_2}((1 - \alpha_2)D + (\alpha_2 - \beta_2)U + \alpha_2 L) \right| + |\omega D - A| \right).\end{aligned}$$

In addition, we denote

$$\rho = \rho(D^{-1}(|L| + |U|))$$

and

$$\rho(\omega) = \rho(\mathcal{L}_{2,1,1,\omega}\mathcal{L}_{1,1,1,\omega}),$$

respectively. The matrix A in $\text{LCP}(A, q)$ is generated by $A(\mu, \eta, \zeta) = \hat{A} + \mu I + \eta B + \zeta C$, where μ, η , and ζ are three given constants, which guarantee the matrix $A(\mu, \eta, \zeta)$ to be a P -matrix. Specifically,

$$\hat{A} = \text{Tridiag}(-I, S, -I) \in R^{n \times n}$$

is a block-tridiagonal matrix,

$$B = \text{Tridiag}(0, 0, 1) \in R^{n \times n} \quad \text{and} \quad S = \text{Tridiag}(-1, 4, -1) \in R^{m \times m}$$

are two tridiagonal matrices, and

$$C = \text{diag}([1, 2, 1, 2, \dots]) \in R^{n \times n}$$

is a diagonal matrix, where m and n are two positive integers satisfying $n = m^2$. For convenience, we set

$$q = (1, -1, 1, -1, \dots, 1, -1, \dots)^T \in R^n,$$

then the $\text{LCP}(A(\mu, \eta, \zeta), q)$ has a unique solution when $A(\mu, \eta, \zeta)$ is a P -matrix. All initial vectors in our experiments are selected to be

$$x^{(0)} = (1, 0, 1, 0, \dots, 1, 0, \dots)^T \in R^n.$$

The iteration process stops when $\text{RES}(z^{(k)}) < 10^{-5}$ or the number of iteration steps reaches $10m$. We consider six cases in our experiments, that is, $A(0, 0, 1)$, $A(0, 1, 1)$, $A(1, 0, 0)$, $A(1, 1, -1)$, $A(1, 0, 1)$, and $A(1, 1, 1)$, all of which are H_+ -matrices. Under the same conditions, we also test other cases, that is, $\hat{A} = \text{Tridiag}(-0.5I, S, -0.5I) \in R^{n \times n}$ and $\hat{A} = \text{Tridiag}(-S, S, -S) \in R^{n \times n}$. Since the numerical results are similar, we omit them in this paper. All computations are run by using Matlab version 2016 on a Dell Laptop (Intel(R) Core(TM) i5-7200U CPU @ 2.50 GHz 4.00 GB RAM).

Example 4.1. In this example, we test the convergence domain and the parameter ω for the two-step MBGS iteration method with $\Omega = \omega D$, $\omega > 0$. Set $m = 30$, then we obtain Table 1 as follows.

From Table 1, we have the following observations:

- (i) For $A(0, 0, 1)$, $A(1, 0, 0)$, and $A(1, 1, -1)$, all of which satisfy $\rho = 0.7296 > \frac{1}{2}$. Thus, for the three cases, the convergence domain satisfies $\Omega \geq D$ from Theorem 3.2.4. Although the two-step MBGS iteration method is convergent for the three cases when $\Omega = \frac{1}{2}D$ and the initial vector is $x^{(0)}$, the convergence domain cannot be expanded to $\Omega \geq \frac{1}{2}D$, which cannot be guaranteed by Theorem 3.2.5. However, for $A(0, 1, 1)$, since $\rho = 0.4325 < \frac{1}{2}$, we have a larger convergence domain $\Omega \geq \frac{1}{2}D$ from Theorem 3.2.5.
- (ii) The numerical results also show that the two-step MBGS iteration method with $\Omega = D$ is the best case in our experiments.

Table 1: Numerical results of the two-step MBGS iteration method

ρ	$A(0, 0, 1)$				$A(0, 1, 1)$			
	0.7296				0.4255			
	$\Omega \geq D$				$\Omega \geq \frac{1}{2}D$			
	$\rho(\omega)$	IT	CPU	RES	$\rho(\omega)$	IT	CPU	RES
$\Omega = \frac{1}{2}D$	1.9720	11	0.051	3.5726×10^{-6}	0.7759	11	0.049	5.4198×10^{-6}
$\Omega = D$	0.4480	7	0.040	6.4079×10^{-7}	0.1262	6	0.027	8.3381×10^{-6}
$\Omega = \frac{3}{2}D$	0.5576	11	0.064	2.2155×10^{-6}	0.2463	10	0.048	2.8405×10^{-6}
$\Omega = 2D$	0.6317	14	0.065	6.0896×10^{-6}	0.3458	13	0.067	2.9037×10^{-6}

ρ	$A(1, 0, 0)$				$A(1, 1, -1)$			
	0.7959				0.6951			
	$\Omega \geq D$				$\Omega \geq D$			
	$\rho(\omega)$	IT	CPU	RES	$\rho(\omega)$	IT	CPU	RES
$\Omega = \frac{1}{2}D$	2.3615	12	0.054	5.3077×10^{-6}	1.7950	24	0.124	8.9534×10^{-6}
$\Omega = D$	0.5555	8	0.035	2.6641×10^{-6}	0.3961	32	0.151	5.6348×10^{-6}
$\Omega = \frac{3}{2}D$	0.6494	13	0.056	2.1278×10^{-6}	0.5091	43	0.198	8.0561×10^{-6}
$\Omega = 2D$	0.7109	16	0.065	9.0763×10^{-6}	0.5941	54	0.256	9.3205×10^{-6}

Table 2: Numerical results of the two-step MBAOR iteration method for $A(1, 0, 1)$

$\rho(\alpha, \beta, 1)$	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	$\alpha_{10} = 1$
β_1	$\frac{2157}{2504}$	$\frac{3781}{5012}$	$\frac{509}{760}$	$\frac{919}{1528}$	$\frac{877}{1608}$	$\frac{791}{1586}$	$\frac{198}{431}$	$\frac{112}{263}$	$\frac{289}{728}$	$\frac{1199}{3224}$
β_2	$\frac{595}{694}$	$\frac{481}{643}$	$\frac{492}{743}$	$\frac{194}{327}$	$\frac{1147}{2136}$	$\frac{809}{1650}$	$\frac{862}{1911}$	$\frac{8938}{21397}$	$\frac{677}{1740}$	$\frac{1115}{3061}$
β_3	$\frac{2140}{2509}$	$\frac{593}{800}$	$\frac{991}{1515}$	$\frac{1171}{2003}$	$\frac{1089}{2062}$	$\frac{1091}{2266}$	$\frac{956}{2161}$	$\frac{3489}{8525}$	$\frac{323}{848}$	$\frac{361}{1013}$
β_4	$\frac{4260}{5023}$	$\frac{957}{1304}$	$\frac{5195}{8048}$	$\frac{1026}{1783}$	$\frac{594}{1145}$	$\frac{348}{737}$	$\frac{601}{1387}$	$\frac{173}{432}$	$\frac{556}{1493}$	$\frac{1249}{3587}$
β_5	$\frac{729}{865}$	$\frac{1393}{1919}$	$\frac{1247}{1960}$	$\frac{1004}{1775}$	$\frac{1921}{3775}$	$\frac{935}{2022}$	$\frac{1535}{3622}$	$\frac{718}{1835}$	$\frac{589}{1620}$	$\frac{4116}{12115}$
β_6	$\frac{1708}{2041}$	$\frac{791}{1103}$	$\frac{397}{634}$	$\frac{1163}{2095}$	$\frac{603}{1210}$	$\frac{151}{334}$	$\frac{737}{1781}$	$\frac{671}{1758}$	$\frac{1133}{3197}$	$\frac{327}{988}$
β_7	$\frac{391}{471}$	$\frac{723}{1022}$	$\frac{961}{1562}$	$\frac{211}{388}$	$\frac{869}{1784}$	$\frac{956}{2167}$	$\frac{367}{910}$	$\frac{262}{705}$	$\frac{1031}{2990}$	$\frac{1150}{3573}$
β_8	$\frac{491}{597}$	$\frac{1724}{2475}$	$\frac{491}{814}$	$\frac{1104}{2077}$	$\frac{257}{541}$	$\frac{2045}{4761}$	$\frac{1486}{3789}$	$\frac{269}{745}$	$\frac{303}{905}$	$\frac{313}{1002}$
β_9	$\frac{1939}{2384}$	$\frac{873}{1276}$	$\frac{1051}{1782}$	$\frac{415}{801}$	$\frac{590}{1277}$	$\frac{1371}{3287}$	$\frac{800}{2103}$	$\frac{765}{2186}$	$\frac{3203}{9876}$	$\frac{389}{1286}$
$\beta_{10} = 1$	$\frac{535}{667}$	$\frac{519}{775}$	$\frac{1105}{1923}$	$\frac{388}{771}$	$\frac{1318}{2943}$	$\frac{2556}{6331}$	$\frac{625}{1699}$	$\frac{579}{1712}$	$\frac{339}{1082}$	$\frac{149}{510}$

IT	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	$\alpha_{10} = 1$
β_1	59	31	21	17	14	12	10	9	8	8
β_2	58	30	21	16	13	11	10	9	8	7
β_3	56	29	20	16	13	11	10	9	8	7
β_4	54	28	20	15	13	11	9	8	8	7
β_5	51	27	19	15	12	10	9	8	7	7
β_6	49	26	18	14	12	10	9	8	7	7
β_7	46	25	18	14	11	10	9	8	7	6
β_8	42	23	17	13	11	9	8	7	7	6
β_9	44	22	16	13	11	9	8	7	7	6
$\beta_{10} = 1$	45	22	15	12	10	9	8	7	6	6

Example 4.2. In this example, we test the parameters α and β in the two-step MBAOR iteration method. In the two-step MBAOR iteration method (9), we set Ω to be a fixed positive diagonal matrix, that is, $\Omega = D$, and set $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta \in [0, 1]$ with $\alpha \neq 0$. We divide the interval $[0, 1]$ into ten equal parts and set α, β to be $\frac{i}{10}$, $i = 1, 2, \dots, 10$, and $m = 30$. The numerical results on the spectral radius $\rho(\alpha, \beta)$ and the iteration steps IT are shown in Tables 2 and 3 and Figure 1 as follows.

Figure 1 corresponds to Tables 2 and 3. We can see:

- (i) For $\Omega = D$, when $0 \leq \beta \leq \alpha \leq 1$ with $\alpha \neq 0$, the two splittings in (8) are H -compatible splittings of an H_+ -matrix. When $\alpha = \beta = 1$, that is, the two-step MBGS iteration method is the best case in the experiments.
- (ii) For $\Omega = D$, when $0 < \alpha \leq \beta \leq 1$, the splittings in (8) are not H -compatible splittings. However, the two-step MBAOR iteration method is also convergent based on Theorem 3.2.2. In addition, the two-step MBGS iteration method is still the best case in the larger convergence domain.

From Examples 4.1 and 4.2, we can find that for the two-step MBAOR iteration method, when $0 \leq \beta_i \leq \alpha_i \leq 1$ with $\alpha_i \neq 0$, if $\Omega = \omega D$ with $\omega \in [1, +\infty)$ or $\omega \in \left[\frac{1}{2}, +\infty\right)$ and $\rho(D^{-1}(|L| + |U|)) < \frac{1}{2}$, then the two-step MBGS iteration method with $\Omega = D$, i.e., $\alpha_i = \beta_i = 1$, $\omega = 1$ in (9) is the best case in our experiments.

Table 3: Numerical results of the two-step MBAOR iteration method for $A(1, 1, 1)$

$\rho(\alpha, \beta, 1)$	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	$\alpha_{10} = 1$
β_1	$\frac{1807}{2324}$	$\frac{630}{1027}$	$\frac{621}{1268}$	$\frac{37}{94}$	$\frac{1201}{3726}$	$\frac{646}{2457}$	$\frac{418}{1923}$	$\frac{348}{1951}$	$\frac{641}{4317}$	$\frac{235}{1951}$
β_2	$\frac{477}{616}$	$\frac{1443}{2374}$	$\frac{435}{896}$	$\frac{1174}{3009}$	$\frac{613}{1943}$	$\frac{1583}{6137}$	$\frac{52}{245}$	$\frac{475}{2707}$	$\frac{3926}{27125}$	$\frac{665}{5556}$
β_3	$\frac{791}{1025}$	$\frac{355}{587}$	$\frac{233}{485}$	$\frac{1099}{2860}$	$\frac{285}{917}$	$\frac{746}{2935}$	$\frac{556}{2691}$	$\frac{467}{2748}$	$\frac{149}{1060}$	$\frac{992}{8667}$
β_4	$\frac{669}{871}$	$\frac{398}{663}$	$\frac{973}{2049}$	$\frac{991}{2619}$	$\frac{1093}{3584}$	$\frac{1072}{4317}$	$\frac{1683}{8378}$	$\frac{196}{1181}$	$\frac{243}{1775}$	$\frac{475}{4251}$
β_5	$\frac{811}{1060}$	$\frac{400}{673}$	$\frac{569}{1212}$	$\frac{9935}{26748}$	$\frac{427}{1427}$	$\frac{319}{1319}$	$\frac{298}{1509}$	$\frac{309}{1916}$	$\frac{853}{6563}$	$\frac{219}{2026}$
β_6	$\frac{763}{1003}$	$\frac{685}{1162}$	$\frac{770}{1663}$	$\frac{427}{1160}$	$\frac{600}{2039}$	$\frac{618}{2599}$	$\frac{577}{3051}$	$\frac{641}{4134}$	$\frac{946}{7529}$	$\frac{811}{7818}$
β_7	$\frac{1028}{1361}$	$\frac{591}{1013}$	$\frac{833}{1835}$	$\frac{237}{661}$	$\frac{515}{1796}$	$\frac{481}{2070}$	$\frac{662}{3533}$	$\frac{862}{5689}$	$\frac{226}{1835}$	$\frac{403}{4064}$
β_8	$\frac{745}{989}$	$\frac{947}{1640}$	$\frac{799}{1776}$	$\frac{884}{2489}$	$\frac{320}{1147}$	$\frac{610}{2703}$	$\frac{407}{2245}$	$\frac{2319}{15628}$	$\frac{91}{762}$	$\frac{377}{4025}$
β_9	$\frac{2931}{3916}$	$\frac{1523}{2670}$	$\frac{796}{1799}$	$\frac{551}{1586}$	$\frac{617}{2247}$	$\frac{1545}{7036}$	$\frac{193}{1103}$	$\frac{347}{2432}$	$\frac{496}{4329}$	$\frac{99}{1066}$
$\beta_{10} = 1$	$\frac{1033}{1391}$	$\frac{1647}{2918}$	$\frac{2027}{4651}$	$\frac{813}{2396}$	$\frac{970}{3603}$	$\frac{213}{1010}$	$\frac{367}{2143}$	$\frac{119}{874}$	$\frac{485}{4451}$	$\frac{367}{4197}$
IT	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	$\alpha_{10} = 1$
β_1	55	29	20	15	13	11	10	9	8	7
β_2	54	28	20	15	12	11	9	8	8	7
β_3	52	27	19	15	12	10	9	8	7	7
β_4	51	27	19	14	12	10	9	8	7	7
β_5	49	26	18	14	12	10	9	8	7	7
β_6	46	25	18	14	11	10	9	8	7	6
β_7	43	24	17	13	11	10	8	8	7	6
β_8	43	22	16	13	11	9	8	7	7	6
β_9	45	21	16	13	11	9	8	7	7	6
$\beta_{10} = 1$	45	22	15	12	10	9	8	7	6	6

Figure 1: The spectral radius function $\rho(\alpha, \beta, 1)$ (for (a) $A(1,0,1)$ and (c) $A(1,1,1)$) and the iteration steps IT (for (b) $A(1,0,1)$ and (d) $A(1,1,1)$) for the two-step MBAOR iteration method.

5 Concluding remarks

In this paper, we discuss the condition of the modulus-based type iteration methods for solving the LCP (A, q) . Then, for the two-step modulus-based matrix splitting iteration method, we propose some convergence conditions when the system matrix A is an H_+ -matrix. In addition, we give the numerical experiments to show the presented results and investigate some particular cases of the two-step MBAOR iteration method.

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