Research Article

Yangxin Tang, Lin Zheng*, and Liping Luan

Solutions to a multi-phase model of sea ice growth

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Abstract: The multi-phase systems have found their applications in many fields. We shall apply this approach to investigate the multi-phase dynamics of sea ice growth. In this paper, the weak solution existence and uniqueness of parabolic differential equations are proved. Then large-time behavior of solutions is studied, and also the existence of the global attractor is proved. The key tool in this article is the energy method. Our existence proof is only in one dimension.

Keywords: nonlinear differential equations, existence of solutions, evolution of phase boundaries, global attractor, large time behavior

MSC 2020: 35K51, 74N20

1 Introduction

The formation of sea ice plays an important role in the earth’s climate because it determines the large-scale heat and mass transport delivered to the surface of polar oceans. In the Arctic, temperatures are rising twice as fast as in other parts of the world. As sea ice has begun to melt, see, e.g., [1–3], that exposes more of the ocean’s surface to the air, which will be available to power storms. Storms surge into coastal areas, causing erosion and the increased flooding in low-lying areas. Sea ice is usually sandwiched between water at the bottom and a layer of snow at the top. Thus, we will investigate multi-phase dynamics of the sea ice growth. In this paper, we will apply the phase-field method to model the growth of sea ice. Although this method was developed in the 1980s, it has become a theoretical research tool in many fields, see, e.g., [4–9]. As we all know, the application of the multi-phase model proposed in this paper in the study of sea ice growth is the first phase-field model of sea ice evolution.

The classic Stefan problem has been studied, see, e.g., [10]. They need to add appropriate conditions at the interface of the tracking movement. Theoretical analysis is very difficult. The ice formation in turbulent sea water was studied by Boussinesq approximation, see, e.g., [11]. The ice formation in dry snow has been studied, see, e.g., [12]. The temperature gradient is imposed on the snow, the snow microstructure changes due to heat transport, sublimation and resublimation in [12]. The seawater ice model of sea ice growth has been studied, see, e.g., [7]. Ice formation is a complex phenomenon. When ice forms, the geophysical and biological processes occur in the polar ocean and subsequent salt rejection and turbulent convection are ignored [7]. Its construction is inspired by the phase-field model of alloy solidification (see, e.g., [13]). The heat transfer problem plays an important role in the formation of sea ice [7]. Thermodynamics and...
solidification theory of alloys (see, e.g., [14]) are widely used in metallurgical and geological applications to study the solidification system. In this paper, the boundary evolution of sea water, snow, and ice is studied by using the phase-field theory including heat transfer and microscopic order parameter dynamics. Our construction is inspired by the phase-field model of multi-phase alloy solidification (see, e.g., [14]). When ice forms, the geophysical and biological processes occur in the polar ocean, and subsequent salt rejection and turbulent convection are ignored again. The model is generalization of the one introduced by Steinbach et al. [14], for isothermal solidification/melting process of kinds of alloys. The problem considered in the literature is most closely related to the present work, which is the solidification of alloys, in which the heat transfer needs to be dealt with. Thus, our model is the nonisothermal solidification process in this paper. In this paper, we contribute to only show that in one dimension the initial boundary value problem has solutions and long-time behavior.

According to the standard definition, the order parameter in the phase transition problem represents the non-zero property of the system in a different region of the phase space and 0 otherwise (see, e.g., [14]). The order parameter corresponds to the structural order of the solid

Our model must satisfy the following system of partial differential equations:

\[
\begin{align*}
  u_t - k_1(u) \Delta u - u \Delta w - k_3(v) \Delta u - u \Delta v &= -a_1 u v (w - u) \theta(h_u + h_v(u, v, 0) v_u + h_w(u, 0, w) w_u), \\
  v_t - k_2(u) \Delta v - v \Delta w - k_3(u) \Delta v - v \Delta u &= -a_2 v w (w - v) \theta(h_v + h_u(u, v, 0) v_u + h_w(0, v, w) w_u), \\
  w_t - k_3(u) \Delta w - w \Delta u - k_3(v) \Delta w - w \Delta v &= -a_3 w u (u - w) \theta(h_w + h_u(u, 0, w) u_u + h_v(0, v, w) v_u), \\
  \theta_t - \frac{1}{2} h_u u_t - \frac{1}{2} h_v v_t - \frac{1}{2} h_w w_t &= D \Delta \theta,
\end{align*}
\]

for \((t, x) \in (0, \infty) \times \Omega\). The boundary and initial conditions are as follows:

\[
\begin{align*}
  u(t, x) &= 0, \quad v(t, x) = 0, \quad w(t, x) = 0, \quad (t, x) \in [0, \infty) \times \Omega, \\
  u(0, x) &= u_0, \quad v(0, x) = v_0(x), \quad w(0, x) = w_0(x), \quad x \in \Omega.
\end{align*}
\]

Here, \(\Omega \subset \mathbb{R}^3\) is an open bounded domain. The function \(\theta\) is the temperature, and the phase-field functions \(u, v, w\) are the respective fractions of ice, snow, and water; thus, physically \(u + v + w = 1\). Here, the parameters \(k_1, k_2, k_3, a_1, a_2, a_3, D\) and \(D\) are positive. We allow a temperature to be given \(a \text{ priori}\), but it must also be determined by the physical process that takes place. This means that the model considers the phase changes of sea ice caused by the temperature change. Thus, in order to close the system, it is necessary to include another equation of unknown temperature. The dimensionless temperature \(\theta\) is scaled so that \(\theta = 0\) is the solidification temperature. In the free energy,

\[
F[u, v, w, \theta] = \int_{\Omega} \left\{ \frac{k_1}{2} \nabla u \nabla - w \nabla u + \frac{k_2}{2} \nabla v \nabla - v \nabla w + \frac{k_3}{2} \nabla w \nabla - v \nabla w + \frac{1}{2} \psi'(u, v, w) + \theta^2 \right\} dV,
\]

where

\[
\psi'(u, v, w) = \frac{a_1}{2} u^2 w^2 + \frac{a_2}{2} v^2 w^2 + \frac{a_3}{2} u^2 v^2,
\]

we choose for \(\psi \in C^2(\mathbb{R}, [0, \infty))\), which represents the double-well potential. The function \(\theta\) satisfies

\[
\theta = e + \frac{1}{2} h(u, v, w),
\]

where \(h(u, v, w)\) is nondecreasing smooth function satisfying \(h(0, 0, 0) = 0\) and \(h(1, 1, 1) = 1\), \(e\) is the local enthalpy.

Due to the smear out of the phase-field variables \(p_i\), at the triple point, all three phase-fields have non-zero values, while at the dual-phase lines, only two phase-fields interact, see, e.g., [14]. In the change of
solid-liquid interface, the order parameter changes with the change of material volume fraction. Within a system exhibiting three different phase states, for example, A, B, and liquid phases, the A vanishes at the transition to liquid and it changes to the B at the transition to the B phase. Thereby, we might interpret the B phase and liquid as “non A phase states.” It is easy to see this in the case of dual-phase systems, \( u = 1 - v \), or \( u = 1 - w \), or \( v = 1 - w \) and \( \frac{\partial u}{\partial w} = -1 \), or \( \frac{\partial u}{\partial v} = -1 \), or \( \frac{\partial v}{\partial w} = -1 \).

Let us assume that all functions depend on variables \( \chi_i \) and \( t \). To simplify symbols, denote \( \chi_i \) by \( x \). The set \( \Omega = (a, b) \) is an open bounded interval with constants \( a < b \). We write \( Q_T = (0, T_e) \times \Omega \), where \( T_e \) is a positive constant and define

\[
(u, \varphi)_Z = \int_Z u(y)\varphi(y)dy,
\]

for \( Z = \Omega \) or \( Z = Q_{\Omega} \).

Since \( u + v + w = 1 \) and \( u_t + v_t + w_t = 0 \), the first three equations are not independent, and in the case of one space dimension, equations can be rewritten in the following form:

\[
\begin{align*}
    u_t - (k_1(1 - v) + k_2v)u_{xx} - (k_1 - k_3)uv_{xx} &= -a_1u(1 - u - v)(1 - 2u - v) - a_0v(\nu - u) - \theta(h_v - h(u, v, 0) - h_v(u, 0, w)), \\
    v_t - (k_2(1 - u) + k_3u)v_{xx} - (k_2 - k_1)uv_{xx} &= -a_2v(1 - u - v)(1 - u - 2v) - a_0u(\nu - v) - \theta(h_v - h(u, v, 0) - h_v(u, v, 0, w)), \\
    \theta_t - D\theta_{xx} &= \frac{1}{2}(h_u - h_v)u_t + \frac{1}{2}(h_v - h_w)v_t.
\end{align*}
\]

The boundary and initial conditions therefore are as follows:

\[
\begin{align*}
    u(t, x) &= 0, \quad v(t, x) = 0, \quad \theta(t, x) = 0, \quad (t, x) \in (0, T_e) \times \Omega, \\
    u(0, x) &= u_0(x), \quad v(0, x) = v_0(x), \quad \theta(0, x) = \theta_0(x), \quad x \in \Omega.
\end{align*}
\]

**Definition 1.1.** Let \( (u_0, v_0, \theta_0) \in H^1_0(\Omega) \times H^1_0(\Omega) \times L^2(\Omega) \). A function \((u, v, \theta)\) with

\[
\begin{align*}
    u, v &\in L^\infty(0, T; H^1_0(\Omega)) \cap L^2(0, T; H^2(\Omega)), \\
    \theta &\in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)),
\end{align*}
\]

is a weak solution to problems (1.7)–(1.11), if for all \( \varphi \in C_0^\infty((-\infty, T_e) \times \Omega) \), there holds

\[
\begin{align*}
    0 &= (u, \varphi_t)_{Q_0} - a(u, \varphi_t)_{Q_0} - (u, \varphi)_{Q_0} - (v, \varphi)_{Q_0} - a_2(u, \varphi)_{Q_0} - v_0(u, \varphi)_{Q_0} + (u_0, \varphi(0))_\Omega, \\
    0 &= (v, \varphi_t)_{Q_0} - (v, \varphi)_{Q_0} - (u, \varphi_t)_{Q_0} - (u, \varphi)_{Q_0} - a_0(u, \varphi)_{Q_0} - 0 + (v_0, \varphi(0))_\Omega, \\
    0 &= (\theta, \varphi_t)_{Q_0} - (\theta, \varphi)_{Q_0} - (\theta, \varphi_t)_{Q_0} - (\theta, \varphi)_{Q_0} - a_2(\theta, \varphi)_{Q_0} - (\theta_0, \varphi(0))_\Omega,
\end{align*}
\]

where \( \alpha = (k_1(1 - v) + k_2v) > 0, \beta = k_2 - k_3, \gamma = (k_2(1 - u) + k_3u) > 0, \lambda = k_2 - k_3, \) and \( T_e \) is a positive constant.

The main results of this article are as follows.

**Theorem 1.1.** For all \((u_0, v_0, \theta_0) \in H^1_0(\Omega) \times H^1_0(\Omega) \times L^2(\Omega) \), there exists a unique weak solution \((u, v, \theta)\) to problem (1.7)–(1.11), which in addition to (1.12)–(1.13) satisfies

\[
\begin{align*}
    u_t &\in L^2(0, T_e; L^2(\Omega)), \quad u_t \in L^4(Q_{T_e}), \\
    v_t &\in L^2(0, T_e; L^2(\Omega)), \quad \theta_t \in L^2(0, T_e; H^1(\Omega)),
\end{align*}
\]
Definition 1.2. Let $X$ be a Banach space. A one-parameter family $S(t), 0 \leq t < \infty$, of bounded linear operators from $X$ into $X$ is a semigroup bounded linear operator on $X$ if

(i) $S(0) = I$, ($I$ is identity operator on $X$),
(ii) $S(t + h) = S(t)S(h)$ for every $t, h \geq 0$ (the semigroup property).

Theorem 1.2. Let $\Omega$ denotes an open bounded set of $\mathbb{R}$. The semigroup $S(t)$ associated with the initial boundary value problem (1.7)–(1.11) possesses maximal attractor $A$, which is bounded in $H^2_0(\Omega)$, compact and connected in $L^2(\Omega)$. Its basin of attraction is the whole space $L^2(\Omega)$. Assume that coefficients are suitable large. Then, $\|u\|_{L^\infty(\Omega)}$, $\|v\|_{L^\infty(\Omega)}$, and $\|\theta\|$ decrease exponentially to 0 as $t \to \infty$.

Notation. In the following sections, we employ the letter $C$ to denote any positive constants that can be explicitly computed in terms of known quantities and may change from line to line. The $L^2(\Omega)$-norm is denoted by $\|\cdot\|$.

The paper is organized as follows: we will prove the existence of local solutions for the nonlinear equations (1.7)–(1.11) by using the Banach fixed-point theorem in Section 2. We shall establish a priori estimates for the solution in Section 3. We investigate the long-time behavior of a solution by the a priori estimates in Section 4.

2 Existence of local solutions

In this section, we obtain the existence and uniqueness of solutions by the use of the Banach fixed point theorem. Two theorems easily follow it. Now, we define the operator

$$A : (\mu, v, \omega, \theta) \mapsto (u, v, w, \theta),$$

which is defined by the following problem:

$$u_t - au_{xx} = \beta \nu v_{xx} - a \mu(1 - \mu - v)(1 - 2\mu - v) - \alpha \mu \nu(v - \mu) - \partial(h_u - h_v(\mu, v, 0) - h_w(\mu, 0, \omega)), \quad (2.1)$$

$$v_t - \gamma v_{xx} = \lambda \nu v_{xx} - a \nu(1 - \mu - v)(1 - \mu - 2v) - \alpha \nu \mu(\mu - v) - \partial(h_u - h_v(\mu, v, 0) - h_w(0, v, \omega)), \quad (2.2)$$

$$\theta_t - D\Delta \theta = \frac{1}{2}(h_u - h_w)u_t + \frac{1}{2}(h_v - h_w)v_t, \quad (2.3)$$

with the boundary and initial conditions (1.10)–(1.11) and $\omega = 1 - \mu - v$.

We prove the existence and uniqueness of solutions by using the Banach fixed point theorem.

Lemma 2.1. (Banach’s fixed point theorem) Let $X$ denote a Banach space, assume

$$A : X \to X$$

is a nonlinear mapping, and suppose that

$$\|A[u] - A[v]\| \leq \gamma \|u - v\|, \quad u, v \in X,$$

for some constants $0 < \gamma < 1$. Then, $A$ has a unique fixed point.

Theorem 2.1. If $t > 0$ is small enough and $(u_0, v_0, \theta_0) \in H^2_0(\Omega) \times H^2_0(\Omega) \times L^2(\Omega)$, we have

$$(\mu, v, \omega, \theta) \in X,$$

where

$$X = L^\infty(0, t; H^2_0(\Omega)) \cap L^4(Q_t) \cap L^2(0, t; H^2(\Omega)) \times L^\infty(0, t; L^2(\Omega)) \cap L^2(0, t; H^2_0(\Omega)), \quad (2.5)$$
which satisfies for any \( \tau \in [0, t] \)
\[
\|u(\tau, x)\|_X + \|v(\tau, x)\|_X + \|\theta(\tau, x)\|_X \leq 2(\|u_0\|_{H^1(\Omega)} + \|v_0\|_{H^1(\Omega)} + \|\theta_0\|_{L^2(\Omega)}).
\] (2.6)

Here, we will prove that the solution space of equations (2.1)–(2.3) belongs to \( X \).

**Proof.** Multiplying (2.1) by \( u \) and integrating, we obtain
\[
\frac{1}{2} \frac{d}{dt} \|u\|^2 + \alpha \|u\|^2 = (\beta \mu u_x, u) - a_\mu (\mu(1 - \mu - \nu)(1 - 2\mu - \nu), u) \\
- a_\nu (\mu(1 - \mu - \nu), u) - \mathcal{H}(\mu - \mathcal{H}u, \mu, \nu, 0) - h(u, \mu, \nu, 0, \omega, u) \\
\leq |\mathcal{H}| \|u\|_{L^\infty(\Omega)} \|u_x\| + a_\mu \|u\|_{H^1(\Omega)} \big( \|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} \big) \|u_x\| \\
+ a_\nu \|u\|_{H^1(\Omega)} \big( 1 + \|u\|_{H^1(\Omega)} + \|v\|_{H^1(\Omega)} \big) \big( 1 + 2\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} \big) \|u_x\| \\
+ (\|h_u\|_{L^\infty(\Omega)} + \|h_v(\mu, \nu, 0)\|_{L^\infty(\Omega)} + \|h_w(\mu, \nu, 0)\|_{L^\infty(\Omega)}) \|u_x\| \\
\leq C_\alpha \|u_x\|^2 + \frac{\alpha}{2} \|u_x\|^2 + C + C_\alpha \|u\|^2. 
\] (2.7)

We have
\[
\frac{d}{dt} \|u\| \leq C \|u\|_{H^1(\Omega)} \|u_x\| + a_\mu \|u\|_{H^1(\Omega)} \big( \|u\|_{H^1(\Omega)} + \|v\|_{H^1(\Omega)} \big) \\
+ a_\nu \|u\|_{H^1(\Omega)} \big( 1 + \|u\|_{H^1(\Omega)} + \|v\|_{H^1(\Omega)} \big) \big( 1 + 2\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} \big) + C \|\theta\|. 
\] (2.8)

Thus, we obtain
\[
\|u\| \leq Ct + \|u_0\| \leq 2\|u_0\| 
\] (2.9)
and
\[
\int_0^t \|u\|^2 d\tau \leq 2\|u_0\|^2. 
\] (2.10)

Hereafter, we use \( Ct < 1 \) to denote a small enough positive constant. Multiplying (2.1) by \(- u_{xx}\) and integrating, we formally obtain
\[
\frac{1}{2} \frac{d}{dt} \|u_x\|^2 + \alpha \|u_x\|^2 = (\beta \mu u_x, u) - a_\mu (\mu(1 - \mu - \nu)(1 - 2\mu - \nu), u) \\
- \mathcal{H}(\mu - h_x, \mu, \nu, 0) - h_x(\mu, \mu, \nu, 0, \omega, u) \\
\leq |\mathcal{H}| \|u\|_{L^\infty(\Omega)} \|u_x\| + a_\mu \|u\|_{H^1(\Omega)} \big( \|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} \big) \|u_x\| \\
+ a_\nu \|u\|_{H^1(\Omega)} \big( 1 + \|u\|_{H^1(\Omega)} + \|v\|_{H^1(\Omega)} \big) \big( 1 + 2\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} \big) \|u_x\| \\
+ (\|h_u\|_{L^\infty(\Omega)} + \|h_v(\mu, \nu, 0)\|_{L^\infty(\Omega)} + \|h_w(\mu, \nu, 0)\|_{L^\infty(\Omega)}) \|u_x\| \\
\leq C_\alpha \|u_x\|^2 + \frac{\alpha}{2} \|u_x\|^2 + C + C_\alpha \|u\|^2. 
\] (2.11)

Thus, we obtain
\[
\|u_x\|^2 + \int_0^t \|u_x\|^2 d\tau \leq C_\alpha \int_0^t \|v_x\|^2 d\tau + C_\alpha \|u_0\|^2 \int_0^t \|u_0\|^2 d\tau + C \|u_0\| \|u_0\|^2 \leq 2\|u_0\|^2. 
\] (2.12)

Using the inequalities (2.9) and (2.10) and the Gagliardo-Nirenberg inequality, we have
\[
\int_0^t \int_\Omega u^4 dx \, d\tau \leq \int_0^t \|u\|_{L^2(\Omega)}^2 \|u\|_{L^\infty(\Omega)} d\tau \leq \int_0^t \|u\|_{L^2(\Omega)}^2 d\tau \leq C. 
\] (2.13)

Thus, we have
\[
\|u\| \leq 2(\|u_0\|_{H^1(\Omega)} + \|v_0\|_{H^1(\Omega)} + \|\theta_0\|_{L^2(\Omega)}). 
\] (2.14)
We obtain in a similar way as earlier that
\[ \|v\|_X \leq 2\|u\|_{H^j(\Omega)} + \|v\|_{H^j(\Omega)} + \|\theta\|_{L^2(\Omega)}. \] (2.15)

Using equations (2.1) and (2.2) and the inequalities (2.14)-(2.15), we obtain
\[ \int_0^t \|u_t\|^2 \, dt \leq 2\|u_0\|_{H^j(\Omega)} + \|v_0\|_{H^j(\Omega)} + \|\theta_0\|_{L^2(\Omega)}, \] (2.16)

\[ \int_0^t \|v_t\|^2 \, dt \leq 2\|u_0\|_{H^j(\Omega)} + \|v_0\|_{H^j(\Omega)} + \|\theta_0\|_{L^2(\Omega)}. \] (2.17)

Multiplying (2.3) by \( \theta \) and integrating, we obtain
\[ \left. \frac{1}{2} \frac{d}{dt} \|\theta\|^2 + D\|\theta\|^2 = \left( \frac{1}{2}(h_u - h_w)u_t + \frac{1}{2}(h_v - h_w)v_t, \theta \right) \right| \]
\[ \leq \frac{1}{2} \left( h_u \|u\|_{L^\infty(\Omega)} + h_v \|v\|_{L^\infty(\Omega)} \right) \|u_t\| + \frac{1}{2} \left( \|h_u\|_{L^\infty(\Omega)} + \|h_v\|_{L^\infty(\Omega)} \right) \|v_t\| \|\theta\| \]
\[ \leq C \left( \|u_t\| + \|v_t\| \right) \|\theta\|. \] (2.18)

From this, we yield
\[ \|\theta\| \leq C \int_0^t (\|u_t\| + \|v_t\|) \, dt + \|\theta_0\| \leq 2\|\theta_0\|. \] (2.19)

Using the inequalities (2.18) and (2.19), we obtain
\[ \int_0^t \|\theta_t\|^2 \, dt \leq 2\|\theta_0\|^2. \] (2.20)

Now, we prove that \( A \) is a contraction. We consider arbitrary \((\mu_1, \nu_1, \omega_1, \theta_1) \in X, (\mu_2, \nu_2, \omega_2, \theta_2) \in X\) and denote \( A(\mu, \nu, \omega, \theta) = (u, v, w, \theta) \), for \( i = 1, 2, (\bar{u}, \bar{v}, \bar{w}, \bar{\theta}) = (u_1 - u_2, v_1 - v_2, w_2 - \theta_1 - \theta_2) \). Then, \((\bar{u}, \bar{v}, \bar{w}, \bar{\theta})\) satisfies
\[ \bar{u}_t - \alpha \bar{u}_{xx} = \beta (\mu_1 \nu_{1xx} - \mu_2 \nu_{2xx}) - a_1 (\mu_1 \nu_1 (\omega_1 - \nu_1) - \mu_2 \nu_2 (\omega_2 - \nu_2)) \]
\[ - a_2 \mu_1 \nu_1 (\nu_1 - \mu_1) - \beta (h_u, \mu_1, \nu_1, \omega_1) - h_v (\mu_1, \nu_1, 0) - h_w (\mu_1, 0, \omega_1)) \]
\[ + a_2 \mu_2 \nu_2 (\nu_2 - \mu_2) - \beta (h_u, \mu_2, \nu_2, \omega_2) - h_v (\mu_2, \nu_2, 0) - h_w (\mu_2, 0, \omega_2)) \]
\[ \bar{v}_t - \gamma \bar{v}_{xx} = \lambda (\nu_1 \mu_{1xx} - \nu_2 \mu_{2xx}) - a_2 (\nu_1 \omega_1 (\omega_1 - \nu_1) - \nu_2 \omega_2 (\omega_2 - \nu_2)) \]
\[ - a_3 \nu_1 (\mu_1 - \nu_1) - \beta (h_v, \mu_1, \nu_1, \omega_1) - h_u (\mu_1, \nu_1, 0) - h_w (0, \nu_1, \omega_1)) \]
\[ + a_3 \nu_2 (\mu_2 - \nu_2) - \beta (h_v, \mu_2, \nu_2, \omega_2) - h_u (0, \nu_2, \omega_2), \]
\[ \bar{w}_t - D \bar{w}_{xx} = \frac{1}{2} (h_u (u_1, \nu_1, w_1) - h_w (u_1, \nu_1, w_1)) v_t + \frac{1}{2} (h_v (u_1, \nu_1, w_1) - h_w (u_1, \nu_1, w_1)) v_t \]
\[ - \frac{1}{2} (h_w (u_2, \nu_2, w_2) - h_w (u_2, \nu_2, w_2)) u_t - \frac{1}{2} (h_v (u_2, \nu_2, w_2) - h_w (u_2, \nu_2, w_2)) v_t, \]
\[ \bar{u} = 0, \quad \bar{v} = 0, \quad \bar{w} = 0, \quad (t, x) \in (0, T) \times \partial \Omega, \]
\[ \bar{u} = 0, \quad \bar{v} = 0, \quad \bar{w} = 0, \quad x \in \Omega. \] (2.21) (2.22) (2.23) (2.24) (2.25)

**Theorem 2.2.** Let the assumptions of Theorem 2.1 be fulfilled. Then, \( A \) is a contraction, that is,
\[ \|A(\mu_1, \nu_1, \omega_1, \theta_1) - A(\mu_2, \nu_2, \omega_2, \theta_2)\| \leq \delta \| (\mu_1, \nu_1, \omega_1, \theta_1) - (\mu_2, \nu_2, \omega_2, \theta_2) \| \] (2.26)
for any \( 0 < \delta < 1 \) and all \((\mu_1, \nu_1, \omega_1, \theta_1), (\mu_2, \nu_2, \omega_2, \theta_2) \in X\).
Proof. Multiplying (2.21) by \( \tilde{u} \) and integrating, we obtain
\[
\frac{1}{2} \frac{d}{dt} \| \tilde{u} \|^2 + a \| \tilde{u} \|^2 \leq C \left( \| v_{12x} - v_{22x} \| + \| \mu_1 - \mu_2 \| \right) \| \tilde{u} \| \\
+ C \left( \| v_1 - v_2 \| + \| \omega_1 - \omega_2 \| + \| \theta_1 - \theta_2 \| \right) \| \tilde{u} \| ,
\]
where we have used Hölder’s inequality, we have
\[
\| \tilde{u} \| \leq C \| (\mu_1, v_1, \omega_1, \theta_1) - (\mu_2, v_2, \omega_2, \theta_2) \|_{x}.
\]
Thus, we obtain
\[
\int_0^t \| \tilde{u} \|^2 \, dt \leq C \| (\mu_1, v_1, \omega_1, \theta_1) - (\mu_2, v_2, \omega_2, \theta_2) \|_{x} .
\]

Multiplying (2.21) by \( -\tilde{u}_{xx} \) and integrating by Hölder’s inequalities, we formally get
\[
\frac{1}{2} \frac{d}{dt} \| \tilde{u}_{xx} \|^2 + a \| \tilde{u}_{xx} \|^2 \leq C \left( \| v_{12x} - v_{22x} \| + \| \mu_1 - \mu_2 \| \right) \| \tilde{u}_{xx} \| \\
+ C \left( \| v_1 - v_2 \| + \| \omega_1 - \omega_2 \| + \| \theta_1 - \theta_2 \| \right) \| \tilde{u}_{xx} \| \\
\leq C \left( \| v_{12x} - v_{22x} \| + \| \mu_1 - \mu_2 \| \right)^2 + C \left( \| v_1 - v_2 \| + \| \omega_1 - \omega_2 \| + \| \theta_1 - \theta_2 \| \right)^2 \leq C \left( \| v_{12x} - v_{22x} \|^2 + \| \mu_1 - \mu_2 \|^2 + \| v_1 - v_2 \|^2 + \| \omega_1 - \omega_2 \|^2 + \| \theta_1 - \theta_2 \|^2 \right)
\]
Thus, we get
\[
\| \tilde{u}_{xx} \|^2 + a \int_0^t \| \tilde{u}_{xx} \|^2 \, dt \leq C \| (\mu_1, v_1, \omega_1, \theta_1) - (\mu_2, v_2, \omega_2, \theta_2) \|_{x} .
\]

We obtain in a similar way as earlier that
\[
\int_0^t \| \tilde{v} \|^2 \, dt + a \int_0^t \| \tilde{v} \|^2 \, dt \leq C \| (\mu_1, v_1, \omega_1, \theta_1) - (\mu_2, v_2, \omega_2, \theta_2) \|_{x} .
\]

By using equations (2.21) and (2.22) and the inequalities (2.28)–(2.39), we obtain
\[
\int_0^t \| \tilde{u} \|^2 \, dt \leq C \| (\mu_1, v_1, \omega_1, \theta_1) - (\mu_2, v_2, \omega_2, \theta_2) \|_{x} ,
\]
\[
\int_0^t \| \tilde{v} \|^2 \, dt \leq C \| (\mu_1, v_1, \omega_1, \theta_1) - (\mu_2, v_2, \omega_2, \theta_2) \|_{x} .
\]

Multiplying (2.23) by \( \tilde{\theta} \) and integrating, using Hölder’s inequalities, we get
\[
\frac{1}{2} \frac{d}{dt} \| \tilde{\theta} \|^2 + D \| \tilde{\theta} \|^2 = \left( \frac{1}{2} (h_{uv}(u_1, v_1, w_1) - h_{uv}(u_1, v_1, v_1)) u_{111}, \tilde{\theta} \right) + \left( \frac{1}{2} (h_{uv}(u_1, v_1, v_1) - h_{uv}(u_1, v_1, w_1)) v_{111}, \tilde{\theta} \right) \\
- \frac{1}{2} (h_{uv}(u_2, v_2, w_2) - h_{uv}(u_2, v_2, w_2)) u_{111}, \tilde{\theta} \right) - \left( \frac{1}{2} (h_{uv}(u_2, v_2, w_2) - h_{uv}(u_2, v_2, w_2)) v_{111}, \tilde{\theta} \right) \\
\leq C \left( \| \tilde{u} \| + \| \tilde{v} \| + \| \tilde{\theta} \| \right) \| u_{111} \| + \| \tilde{\theta} \| + C \left( \| \tilde{u} \| + \| \tilde{v} \| + \| \tilde{\theta} \| \right) \| v_{111} \| + \| \tilde{\theta} \|,
and we obtain
\[ \int_0^t \|\tilde{u}_t\|^2 \, dt \leq C t \| (\mu_1, v_1, \omega_1, \vartheta_1) - (\mu_2, v_2, \omega_2, \vartheta_2) \|_X. \] (2.36)

Multiplying (2.21) by \(-\tilde{u}_{xx}\) and integrating by Hölder’s inequalities, we formally get
\[ \int_0^t \|\tilde{u}_t\|^2 \, dt \leq C t \| (\mu_1, v_1, \omega_1, \vartheta_1) - (\mu_2, v_2, \omega_2, \vartheta_2) \|_X. \] (2.37)

Thus, we get
\[ |\tilde{u}|^2 + \alpha \int_0^t \|\tilde{u}_{xx}\|^2 \, dt \leq C t \| (\mu_1, v_1, \omega_1, \vartheta_1) - (\mu_2, v_2, \omega_2, \vartheta_2) \|_X. \] (2.38)

We obtain in a similar way as earlier that
\[ \| \varphi \|_{L^\infty(0, t; H^2(\Omega))} + \alpha \int_0^t \| \varphi \|^2 \, dt \leq C t \| (\mu_1, v_1, \omega_1, \vartheta_1) - (\mu_2, v_2, \omega_2, \vartheta_2) \|_X. \] (2.39)

Using equations (2.21) and (2.22) and inequalities (2.28)–(2.39), we obtain
\[ \int_0^t \|\tilde{u}_t\|^2 \, dt \leq C t \| (\mu_1, v_1, \omega_1, \vartheta_1) - (\mu_2, v_2, \omega_2, \vartheta_2) \|_X, \] (2.40)
\[ \int_0^t \|\tilde{v}_t\|^2 \, dt \leq C t \| (\mu_1, v_1, \omega_1, \vartheta_1) - (\mu_2, v_2, \omega_2, \vartheta_2) \|_X. \] (2.41)

Multiplying (2.23) by \(\tilde{\vartheta}\) and integrating, using Hölder’s inequalities, we get
\[ \frac{1}{2} \frac{d}{dt} \|\tilde{\vartheta}\|^2 + D \|\tilde{\vartheta}\|^2 \]
\[ = \left( \frac{1}{2} \left( \hat{h}_w(u_2, v_2, w_2) - \hat{h}_w(u_1, v_1, w_1) \right) u_1t, \tilde{\vartheta} \right) + \left( \frac{1}{2} \right) \left( \hat{h}_w(u_2, v_2, w_2) - \hat{h}_w(u_1, v_1, w_1) \right) v_1t, \tilde{\vartheta} \]
\[ + \left( \frac{1}{2} \right) \left( \hat{h}_v(u_2, v_2, w_2) - \hat{h}_v(u_1, v_1, w_1) \right) u_2t, \tilde{\vartheta} \]
\[ + \left( \frac{1}{2} \right) \left( \hat{h}_v(u_2, v_2, w_2) - \hat{h}_v(u_1, v_1, w_1) \right) v_2t, \tilde{\vartheta} \]
\[ \leq C (\|\tilde{u}\| + \|\tilde{v}\| + \|\tilde{w}\|) |u_1t| |\tilde{\vartheta}| + C (\|\tilde{u}\| + \|\tilde{v}\| + \|\tilde{w}\|) |v_1t| |\tilde{\vartheta}|, \] (2.42)

we obtain
\[ \| \tilde{\vartheta} \| \leq C t \| (\mu_1, v_1, \omega_1, \vartheta_1) - (\mu_2, v_2, \omega_2, \vartheta_2) \|_X. \] (2.43)

We thus conclude
\[ \frac{D}{2} \int_0^t \|\tilde{\vartheta}_t\|^2 \, dt \leq C t \| (\mu_1, v_1, \omega_1, \vartheta_1) - (\mu_2, v_2, \omega_2, \vartheta_2) \|_X. \] (2.44)

The inequality (2.26) is proved.
3 \textit{A priori} estimates

In this section, we establish the \textit{a priori} estimates for solutions of \((u, v, w, \theta)\) with the initial boundary value problems (1.7)–(1.11).

Lemma 3.1. There holds for any \(t \in [0, T_c]\)

\[
\begin{aligned}
&\int_0^t \|u\|^2 \, dr + \int_0^t \|v\|^2 \, dr + \int_0^t \|u_{xx}\|^2 \, dr + \int_0^t \|v_{xx}\|^2 \, dr \\
&+ \int_0^t \|u\|_{L^2(\Omega)}^2 \, dr + \|v\|_{L^2(\Omega)}^2 \, dr + D \int_0^t \|\theta\|_{H^1(\Omega)}^2 \, dr + \|\theta\|^2 \leq C.
\end{aligned}
\]

(3.1)

Proof. Differentiating the free energy \(F\) formally with respect to \(t\) and integrating with respect to \(t\) yield

\[
F[u(t, x), v(t, x), w(t, x), \theta(t, x)] \leq F[u(0, x), v(0, x), w(0, x), \theta(0, x)],
\]

(3.2)

we have

\[
\begin{aligned}
&|uv - vu|_{L^\infty(0, t; L^2(\Omega))} \leq C, \quad \|(1 - v)u_x - uv_x\|_{L^\infty(0, t; L^2(\Omega))} \leq C, \\
&\|\theta\|_{L^\infty(0, t; L^2(\Omega))} \leq C, \quad \|uv\|_{L^\infty(0, t; L^2(\Omega))} \leq C.
\end{aligned}
\]

(3.3)

Formally, multiplying (1.7) and (1.8) by \(-u_x\) and \(-v_{xx}\), respectively, and integrating and adding together, we get

\[
\frac{1}{2} \frac{d}{dt}(\|u\|^2 + \|v\|^2) + \frac{1}{2}(ay - \lambda^2 - ye)d\|u_{xx}\|^2 + \frac{1}{2}(ya - \beta^2 - ae)d\|v_{xx}\|^2 \\
+ \frac{1}{2}(6a_1 - |3a_1 - a_2| - |5a_1 - 3a_3 - a_2|^2)d\|u_x\|^2 \\
+ \frac{1}{2}(2a_1 + 2a_3 - |3a_2 - a_3| - |5a_1 - 3a_3 - a_2|^2)d\|v_x\|^2 \\
+ \frac{1}{2}(6a_2 - |3a_2 - a_3| - |5a_2 - 3a_3 - a_1|^2)d\|v\|^2 \\
+ \frac{1}{2}(2a_2 + 2a_3 - |3a_2 - a_3| - |5a_2 - 3a_3 - a_1|^2)d\|u\|^2 \\
\leq C(\|u\|^2 + \|v\|^2 + \|\theta\|^2 + 1),
\]

(3.5)

where \(ay - \lambda^2 - ye > 0, ya - \beta^2 - ae > 0,\)

\[
\begin{aligned}
6a_1 - |3a_1 - a_2| - |5a_1 - 3a_3 - a_2|^2 > 0, & \quad 2a_1 + 2a_3 - |3a_2 - a_3| - |5a_1 - 3a_3 - a_2|^2 > 0, \\
2a_2 - |3a_2 - a_3| - |5a_2 - 3a_3 - a_1|^2 > 0, & \quad 2a_2 + 2a_3 - |3a_2 - a_3| - |5a_2 - 3a_3 - a_1|^2 > 0,
\end{aligned}
\]

and

\(\|u\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)}, \|w\|_{L^\infty(\Omega)}\) are suitably small.

By using the Gronwall inequality (3.5), one can easily obtain

\[
\|u\|^2 + \|v\|^2 + \frac{1}{2}(ay - \lambda^2 - ye)\int_0^t \|u_{xx}\|^2 \, dr + \frac{1}{2}(ya - \beta^2 - ae)\int_0^t \|v_{xx}\|^2 \, dr \leq C_t.
\]

(3.6)

By using the Poincaré inequality and the result of regularity theory of elliptic equations, we have

\[
\begin{aligned}
\|u\|_{L^\infty(0, t; H^1(\Omega))} & \leq C_t, \quad \int_0^t \|u\|_{H^1(\Omega)}^2 \, dr \leq C_t, \\
\|v\|_{L^\infty(0, t; H^1(\Omega))} & \leq C_t, \quad \int_0^t \|v\|_{H^1(\Omega)}^2 \, dr \leq C_t.
\end{aligned}
\]

(3.7)
By using the Gagliardo-Nirenberg inequality, we get
\[
\int_0^t \int_\Omega u^4 dx dt \leq \int_0^t \|u\|_{L^6(\Omega)}^2 \|u\|_2^2 dt \leq \|u\|_{L^\infty(\Omega)}^2 \int_0^t \|u\|_6^2 dt \leq C \int_0^t \|u\|_6^2 dt \leq C. \tag{3.8}
\]
We get in the same way
\[
\int_0^t \int_\Omega v^4 dx dt \leq C. \tag{3.9}
\]
Multiplying (1.7) and (1.8) by \(u, v\), respectively, adding and integrating with respect to \((t, x) \in (0, T) \times \Omega\), using Hölder’s inequality and Young’s inequality, we find
\[
\int_0^t \|u\|^2 dt \leq C, \quad \int_0^t \|v\|^2 dt \leq C. \tag{3.10}
\]
Multiplying (1.9) by \(\theta\) and integrating and using Hölder’s inequality and Young’s inequality, we find
\[
\frac{1}{2} \frac{d}{dt} \|\theta\|^2 + D\|\theta\|^2 \leq C(\|u\| + \|v\|)\|\theta\|. \tag{3.11}
\]
From this, we get
\[
\|\theta\| \leq C. \tag{3.12}
\]
We thus conclude from (3.11) that
\[
\|\theta\|_{L^2(0, t; L^1(\Omega))} \leq C. \tag{3.13}
\]
By using the Poincaré inequality, we have
\[
\|\theta\|_{L^\infty(0, t; L^2(\Omega))} \leq C. \tag{3.14}
\]
We thus infer from (1.9) by the inequalities (3.12) and (3.14)
\[
\|\theta\|_{L^2(0, t; H^{-1}(\Omega))} \leq C. \tag{3.15}
\]
This means that \((u, v, \theta)\) is a weak solution of (1.7)–(1.9). This allows us to extend the solution step by step to \(T_e\).

**Theorem 3.1. (Uniqueness)** Assume the \(u, v, \) and \(\theta\) are the weak solution of (1.7)–(1.9) for \((t, x) \in (0, T_e) \times \Omega\). Then, the weak solution is unique.

**Proof.** If \(u_1, v_1, \theta_1\) and \(u_2, v_2, \theta_2\) are two solutions, write \(\bar{u} = u_1 - u_2, \bar{v} = v_1 - v_2, \bar{\theta} = \theta_1 - \theta_2\). We replace (1.12)–(1.16) by \(\varphi_1 = u_1 - u_2, \varphi_2 = v_1 - v_2, \varphi_3 = \theta_1 - \theta_2\), and integrating. By using the Young inequality, one can obtain
\[
\frac{1}{2} \frac{d}{dt} (\|ar{u}\|^2 + \|ar{v}\|^2 + \frac{\alpha e}{D} \|ar{\theta}\|^2) + \frac{1}{2} \frac{d}{dt} \|ar{u}\|^2 + \frac{\alpha e}{D} \|ar{\theta}\|^2 \leq C(\|ar{u}\|^2 + \|ar{v}\|^2 + \frac{\alpha e}{D} \|ar{\theta}\|^2)(1 + \|u_1\|^2 + \|v_1\|^2 + \|\theta_1\|^2).
\]
By using Gronwall’s inequality, we thus conclude \(u_1 = u_2, v_1 = v_2, \theta_1 = \theta_2\) for almost everywhere \(Q_{T_e}\). □
4 Global attractor

In this section, we will prove the asymptotic of solutions as \( t \to \infty \) to problems (1.7)–(1.11).

4.1 Global attractor

**Lemma 4.1.** (*The uniform Gronwall lemma*). Let \( g, h \), and \( y \) be three positive locally integral functions on \((t_0, \infty)\) such that \( y' \) is locally integrable on \((t_0, \infty)\), which satisfies

\[
\frac{dy}{dt} \leq gy + h \quad \text{for } t \geq t_0,
\]

\[
\int_t^{t+r} g(s)ds \leq a_1, \quad \int_t^{t+r} h(s)ds \leq a_2, \quad \int_t^{t+r} y(s)ds \leq a_3 \quad \text{for } t \geq t_0,
\]

where \( r, a_1, a_2, \) and \( a_3 \) are positive constants. Then,

\[
y(t + r) \leq \left( \frac{a_3}{r} + a_2 \right)e^{ar}, \quad \forall t \geq t_0.
\]

This version, which we need here, is proved in [15].

**Proof of Theorem 1.2.**

(a) Absorbing set in \( L^2(\Omega) \) of \( u, v \). Multiplying (1.7) and (1.8) by \( u \) and \( v \), respectively, integrating and adding together, and using the Young inequality, we get

\[
\frac{1}{2} \frac{d}{dt}(\|u\|^2 + \|v\|^2) + \frac{1}{2y}(ay - \lambda^2)\|u\|^2 + \frac{1}{2a}(y\alpha - \beta^2)\|v\|^2
\]

\[+ \frac{1}{64}(128\alpha_1 - 33|3\alpha_1 - \alpha| - |3\alpha_2 - \alpha| - 64\epsilon)\|u\|^4_{L^4(\Omega)} + a_2\|u\|^2
\]

\[+ \frac{1}{64}(128\alpha_2 - 33|3\alpha_2 - \alpha| - |3\alpha_1 - \alpha| - 64\epsilon)\|v\|^4_{L^4(\Omega)} + a_3\|v\|^2
\]

\[+ \frac{1}{32}(32\alpha_1 + 64\alpha_3 + 32\alpha_2 - 15|3\alpha_1 - \alpha| - 15|3\alpha_2 - \alpha|)\|uv\|^2
\]

\[\leq C_0\|\theta\|^2 + \epsilon(\|u\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}) + C_1|\Omega|,
\]

\(|\Omega| = \text{the measure (volume) of } \Omega, \text{ where } ay - \beta^2 > 0, \lambda^2 > 0.\]

By using the Poincaré inequality, we have

\[\|u\| \leq c_0\|u\|, \quad \forall u \in H^1_0(\Omega), \quad \|v\| \leq c_0\|v\|, \quad \forall v \in H^1_0(\Omega).
\]

By using the Gronwall lemma, we have

\[\|u\|^2 + \|v\|^2 \leq (\|u_0\|^2 + \|v_0\|^2)e^{-2\epsilon t} + e^{-2\epsilon t}\int_0^t e^{2\epsilon \tau}(2C(\Omega) + C_2\|u\|^2)d\tau
\]

\[\leq (\|u_0\|^2 + \|v_0\|^2)e^{-2\epsilon t} + e^{-2\epsilon t}\int_0^t e^{2\epsilon \tau}(2C(\Omega) + C_2)d\tau
\]

\[\leq (\|u_0\|^2 + \|v_0\|^2)e^{-2\epsilon t} + \frac{1}{\epsilon}(C(\Omega) + C_2)(1 - e^{-\epsilon t}).
\]

Thus,

\[\limsup_{t \to \infty}(\|u\|^2 + \|v\|^2) \leq \rho_0, \quad \rho_0^2 = \frac{1}{\epsilon}(C(\Omega) + C_2).
\]
The inequality (4.3) implies the existence of an absorbing set \( \mathcal{B}(0, \rho'_0) \) in \( L^2(\Omega) \) (see [6]). If \( \mathcal{B} \subset B(0, R) \) is a bounded set of \( L^2(\Omega) \), then \( S(t) \mathcal{B} \subset B(0, \rho'_0) \) for \( t \geq t(\mathcal{B}; \rho'_0) \)

\[
 t_0 = \frac{1}{2 \varepsilon} \log \left( \frac{R^2}{(\rho'_0)^2 - \rho_0^2} \right). \tag{4.4}
\]

We also infer from (4.1) that

\[
 \frac{1}{32} (128a_1 - 33|3a_1 - a_3| - |3a_2 - a_3| - 64\varepsilon) \int_t^{t+\tau} \|u\|_{L^4(\Omega)}^4 \, dt + 2\varepsilon \int_t^{t+\tau} \|u\|^2 \, dt
\]

\[
 + \frac{1}{32} (128a_2 - 33|3a_2 - a_3| - |3a_2 - a_3| - 64\varepsilon) \int_t^{t+\tau} \|v\|_{L^4(\Omega)}^4 \, dt + 2\varepsilon \int_t^{t+\tau} \|v\|^2 \, dt
\]

\[
 + \frac{1}{16} (32a_1 + 64a_3 + 32a_2 - 15|3a_1 - a_3| - 15|3a_2 - a_3|) \int_t^{t+\tau} \|uv\|^2 \, dt
\]

\[
 \leq r(C_e + C(\Omega)) + 2(\|u\|^2 + \|v\|^2), \quad \forall t > 0.
\]

By using the relation (4.3), we have

\[
 \limsup_{t \to \infty} \left\{ 2\varepsilon \int_t^{t+\tau} \left( \|u\|^2 + \|v\|^2 \right) \, dt + \frac{1}{32} (128a_1 - 33|3a_1 - a_3| - |3a_2 - a_3| - 64\varepsilon) \int_t^{t+\tau} \|u\|_{L^4(\Omega)}^4 \, dt \right.
\]

\[
 + \frac{1}{32} (128a_2 - 33|3a_2 - a_3| - |3a_2 - a_3| - 64\varepsilon) \int_t^{t+\tau} \|v\|_{L^4(\Omega)}^4 \, dt
\]

\[
 + \left. \frac{1}{16} (32a_1 + 64a_3 + 32a_2 - 15|3a_1 - a_3| - 15|3a_2 - a_3|) \int_t^{t+\tau} \|uv\|^2 \, dt \right\}
\]

\[
 \leq r(C_e + C(\Omega)) + 2\rho_0^2, \quad \forall t > 0,
\]

and if \( u_0, v_0 \in \mathcal{B} \) and \( t \geq t_0(\mathcal{B}, \rho'_0) \), we have

\[
 2\varepsilon \int_t^{t+\tau} \left( \|u\|^2 + \|v\|^2 \right) \, dt + \frac{1}{32} (128a_1 - 33|3a_1 - a_3| - |3a_2 - a_3| - 64\varepsilon) \int_t^{t+\tau} \|u\|_{L^4(\Omega)}^4 \, dt
\]

\[
 + \frac{1}{32} (128a_2 - 33|3a_2 - a_3| - |3a_2 - a_3| - 64\varepsilon) \int_t^{t+\tau} \|v\|_{L^4(\Omega)}^4 \, dt
\]

\[
 + \frac{1}{16} (32a_1 + 64a_3 + 32a_2 - 15|3a_1 - a_3| - 15|3a_2 - a_3|) \int_t^{t+\tau} \|uv\|^2 \, dt
\]

\[
 \leq r(C_e + C(\Omega)) + 2\rho_0^2, \quad \forall t > 0.
\]

(b) Absorbing set in \( H_0^1(\Omega) \) of \( u \) and \( v \). Multiplying (1.7) and (1.8) by \(-u_{xx}\) and \(-v_{xx}\), respectively, and integrating and adding together, we get

\[
 \frac{1}{2} \frac{d}{dt} \|u\|^2 + \|v\|^2 + \frac{1}{2c_1^2} (acy^2 - \lambda^2 - \gamma e - c_1^2 a_2 y) \|u\|^2 + \frac{1}{2c_1^2} (\gamma a - \beta^2 - \alpha e - c_1^2 a_2 \alpha) \|v\|^2
\]

\[
 + \frac{1}{2} \left( 6a_1 - 3a_1 - a_3 - (5a_1 - 3a_3 - a_2) \right) \int_\Omega w^2 u^2 \, dx
\]

\[
 \leq r(C_e + C(\Omega)) + 2\rho_0^2, \quad \forall t > 0.
\]
+ (2a_1 + 2a_3 - |3a_1 - a_3| - (5a_1 - 3a_3 - a_3)^2) \int_\Omega v^2 u_x^2 \, dx 
+ \frac{1}{2} \left( 6a_2 - |3a_2 - a_3| - (5a_2 - 3a_3 - a_3)^2 \right) \int_\Omega v^2 v_x^2 \, dx 
+ (2a_2 + 2a_3 - |3a_1 - a_3| - (5a_2 - 3a_3 - a_3)^2) \int_\Omega u^2 v_x^2 \, dx 
\leq C(\|\theta\|^2 + 1).

There exists a constant \( c_\Omega = c(\Omega) \) [15, Chapter II] such that
\[ \|\nabla u\| \leq c_\Omega \|\nabla u\|, \quad \|\nabla v\| \leq c_\Omega \|\nabla v\|, \quad \forall u, v \in H^1_\Omega. \]  
\tag{4.9}

We find
\[ \frac{d}{dt}(\|u_\xi\|^2 + \|v_\xi\|^2) \leq C(\|\theta\|^2 + 1) + \kappa(\|u_\xi\|^2 + \|v_\xi\|^2), \]  
\tag{4.10}

where \( \kappa = \max(c_\Omega^2 a_1, c_\Omega^2 a_2) \).

For arbitrary fixed \( r > 0 \), by using the uniform Gronwall Lemma 4.1, we obtain
\[ \|u_\xi(t + r)\|^2 + \|v_\xi(t + r)\|^2 \leq \left( \frac{a_4}{r} + a_5 \right) e^{kr}, \quad t \geq t_0, \]  
\tag{4.11}

where
\[ \int_t^{t+\tau} \left( \|u_\xi\|^2 + \|v_\xi\|^2 \right) \, dt \leq a_4 + \frac{1}{2\kappa}(C_\varepsilon + C(\Omega) + 2\rho_0^2), \quad \forall t \geq t_0. \]  
\tag{4.12}

\[ \int_t^{t+\tau} C(\|\theta\|^2 + 1) \, dt = Cr = a_5, \quad \forall t \geq t_0. \]  
\tag{4.13}

Therefore, the ball \((0, \rho_t)\) of \( H^1_\Omega \) is absorbing in \( H^1_\Omega \), when
\[ \rho_t^2 = \left( \frac{a_4}{r} + a_5 \right) e^{kr}, \]  

and if \( \|u_0\|^2 + \|v_0\|^2 \subset B(0, R) \), then \( u \) and \( v \) enter this absorbing set denoted \( B \). The uniform compactness of \( S(t) \) is proved.

(c) Absorbing set in \( L^2(\Omega) \) of \( \theta \). By using the relation (4.11), we find
\[ \|u_\xi(t + r)\|^2 + \|v_\xi(t + r)\|^2 + \frac{1}{2} (ay - \lambda^2 - yz) \int_t^{t+\tau} \|u_\xi\|^2 \, dt + \frac{1}{2\lambda}(ya - \beta^2 - az) \int_t^{t+\tau} \|v_\xi\|^2 \, dt 
+ \int_\Omega \int_0^{\tau} \left( \frac{1}{2} (6a_1 - |3a_1 - a_3| - (5a_1 - 3a_3 - a_3)^2) u^2 u_x^2 
+ (2a_1 + 2a_3 - |3a_2 - a_3| - (5a_1 - 3a_3 - a_3)^2) v^2 v_x^2 
+ \frac{1}{2} (6a_2 - |3a_2 - a_3| - (5a_2 - 3a_3 - a_3)^2) v^2 v_x^2 + (2a_2 + 2a_3 - |3a_1 - a_3| - (5a_2 - 3a_3 - a_3)^2) u^2 u_x^2 \right) \, dx \, dr
\]  
\[ \leq Cr + a_5. \]  
\tag{4.14}
Multiplying (1.7) and (1.8) by $u_t$ and $v_t$, respectively, and integrating over $(t, x) \in (t, t + r) \times \Omega$ and adding together, we have

$$
(1 - \varepsilon) \int_t^{t+r} (\|u_t\|^2 + \|v_t\|^2) \, dt \leq C_r + C_d a_s.
$$

(4.15)

Multiplying (1.9) by $\theta$ and integrating, we get

$$
\frac{1}{2} \frac{d}{dt} \|\theta\|^2 + D\|\theta_t\|^2 \leq \varepsilon \|\theta\|^2 + c_r (\|u_t\|^2 + \|v_t\|^2) \|u_t\|^2 + c_r (\|u_t\|^2 + \|v_t\|^2) \|v_t\|^2.
$$

(4.16)

By using the Poincaré inequality, we infer from (4.16) that

$$
\frac{1}{2} \frac{d}{dt} \|\theta\|^2 + \left(\frac{D}{c_0} - \varepsilon\right) \|\theta_t\|^2 \leq \frac{c_e}{1 - \varepsilon} (C_r + C_d a_s).
$$

By using the uniform Gronwall Lemma 4.1, we have

$$
\|\theta\|^2 \leq e^{\frac{-\varepsilon}{c_0}} \|\theta_0\|^2 + c_r \frac{C_r + C_d a_s}{1 - \varepsilon} \left(1 - e^{\frac{-\varepsilon}{c_0}}\right)
$$

(4.17)

Thus,

$$
\limsup_{t \to \infty} \|\theta\| \leq \rho_2, \quad \rho_2 = c_e \frac{C_r + C_d a_s}{1 - \varepsilon}.
$$

(4.18)

The inequality (4.18) implies the existence of an absorbing set $\mathcal{B}_2$ in $L^2(\Omega)$. If $\mathcal{B}_3 \subset B(0, R_2)$ is a bounded set of $L^2(\Omega)$, then $S(t)\mathcal{B}_3 \subset B(0, \rho_2^2)$ for $t \geq t_0(\mathcal{B}, \rho_2^2)$

$$
t_0 = \frac{c_0^2}{D - c_0^2 \varepsilon} \log \frac{R_2^2}{(\rho_2^2)^2 - \rho_2^2}.
$$

We also infer from (4.16) that

$$
\limsup_{t \to \infty} 2D \int_t^{t+r} \|\theta_t\|^2 \, dt \leq (1 + 2r) \rho_2^2 + \frac{2c_e}{1 - \varepsilon} (C_r + C_d a_s), \quad \forall r > 0,
$$

and if $\theta_0 \in \mathcal{B} \subset B(0, R_2)$ and $t \geq t_0(\mathcal{B}, \rho_2^2)$, then

$$
2D \int_t^{t+r} \|\theta_t\|^2 \, dt \leq (1 + 2r) \rho_2^2 + \frac{2c_e}{1 - \varepsilon} (C_r + C_d a_s), \quad \forall r > 0.
$$

\[ \square \]

4.2 Large-time behavior of the solutions

Lemma 4.2. Let $f = f(t)$ be nonnegative, which satisfies $f \in L^1(\mathbb{R}^+)$, $\frac{d f}{d t} \in L^1(\mathbb{R}^+)$ and

$$
\int_0^\infty |f(t)| \, dt \leq C, \quad \int_0^\infty \left| \frac{d f}{d t} \right| \, dt \leq C,
$$

Then,

$$
\lim_{t \to \infty} f(t) = 0.
$$

(4.19)
Theorem 4.1. Let \((u(t, x), v(t, x), \theta(t, x))\) satisfies \(\|u\|^2 \in L^1(\mathbb{R}^+), \quad \frac{d}{dt}\|u\|^2 \in L^1(\mathbb{R}^+), \quad \|\theta\|^2 \in L^1(\mathbb{R}^+), \quad \frac{d}{dt}\|\theta\|^2 \in L^1(\mathbb{R}^+),\) and

\[
\int_0^\infty \|u_t\|^2 \, dt \leq C, \quad \int_0^\infty \frac{d}{dt}\|u_t\|^2 \, dt \leq C,
\]

\[
\int_0^\infty \|v_t\|^2 \, dt \leq C, \quad \int_0^\infty \frac{d}{dt}\|v_t\|^2 \, dt \leq C,
\]

\[
\int_0^\infty \|\theta_t\|^2 \, dt \leq C, \quad \int_0^\infty \frac{d}{dt}\|\theta_t\|^2 \, dt \leq C.
\]

Then,

\[
\lim_{t \to \infty} \|u\|_{L^\infty(\Omega)} = 0, \quad \lim_{t \to \infty} \|v\|_{L^\infty(\Omega)} = 0, \quad \lim_{t \to \infty} \|\theta\| = 0. \tag{4.20}
\]

Proof. Multiplying (1.7) and (1.8) by \(u\) and \(v\), respectively, integrating and adding together, and using Hölder’s inequality and Young’s inequality, we have

\[
\frac{1}{2} \frac{d}{dr}(\|u\|^2 + \|v\|^2) + \frac{1}{2}\gamma(\alpha - \beta^2 - \lambda^2)\|u_t\|^2 + \frac{1}{2\alpha}(\alpha - \beta^2 - \lambda^2)\|v_t\|^2
\]

\[
+ \frac{1}{64}(28a_1 - 33|3a_1 - a_3| - |3a_2 - a_3|)\|u_t\|^2_{L^2(\Omega)}
\]

\[
+ \frac{1}{64}(28a_2 - 33|3a_2 - a_3| - |3a_1 - a_3|)\|v_t\|^2_{L^2(\Omega)}
\]

\[
+ \frac{1}{32}(2a_1 + 64a_1 + 2a_2 - 15|3a_1 - a_3| - 15|3a_2 - a_3|)\|uv\|^2
\]

\[
\leq C_\gamma\|\theta_t\|^2 + \frac{1}{32}(a_1 + a_2 + \varepsilon)(18\|u\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)})
\]

\[
\leq C_\gamma\|\theta_t\|^2 + \frac{9}{16}c_0^2(a_1 + a_2 + \varepsilon)(\|u_t\|^2_{L^2(\Omega)} + \|v_t\|^2_{L^2(\Omega)}),
\]

where

\[
\sigma = \min\left(\frac{1}{2\gamma}(\alpha - \beta^2 - \lambda^2) - \frac{18}{32}\gamma(\alpha + a_2 + \varepsilon), \frac{1}{2\alpha}(\alpha - \beta^2 - \lambda^2) - \frac{18}{32}\gamma^2(\alpha + a_2 + \varepsilon)\right) > 0,
\]

\[
28a_1 - 33|3a_1 - a_3| - |3a_2 - a_3| > 0, \quad 28a_2 - 33|3a_2 - a_3| - |3a_1 - a_3| > 0,
\]

\[
2a_1 + 64a_1 + 2a_2 - 15|3a_1 - a_3| - 15|3a_2 - a_3| > 0.
\]

Integrating (4.21) in \(\tau \in (0, t)\), we find

\[
(\|u\|^2 + \|v\|^2) + \sigma \int_0^t (\|u_t\|^2 + \|v_t\|^2) \, dt
\]

\[
+ \frac{1}{64}(28a_1 - 33|3a_1 - a_3| - |3a_2 - a_3|) \int_0^t \|u_t\|^2_{L^2(\Omega)} \, dt
\]

\[
+ \frac{1}{64}(28a_2 - 33|3a_2 - a_3| - |3a_1 - a_3|) \int_0^t \|v_t\|^2_{L^2(\Omega)} \, dt
\]

\[
+ \frac{1}{32}(2a_1 + 64a_1 + 2a_2 - 15|3a_1 - a_3| - 15|3a_2 - a_3|) \int_0^t |uv|^2 \, dt
\]

\[
\leq C_\gamma \int_0^t \|\theta_t\|^2 \, dt + \|u_t\|^2 + \|v_t\|^2 \leq C.
\]
We thus conclude
\[
\int_0^t (\|u\|_t^2 + \|v\|_t^2) \, dt \leq C, \quad t \to \infty, \tag{4.23}
\]
\[
\int_0^t \int_\Omega u^2 v^2 \, dx \, dt \leq C, \quad t \to \infty, \tag{4.24}
\]
\[
\int_0^t \left( \|u\|_{L^4(\Omega)}^4 + \|v\|_{L^4(\Omega)}^4 \right) \, dt \leq C, \quad t \to \infty. \tag{4.25}
\]

Multiplying (1.7) and (1.8) by \( -u_\alpha \) and \( -v_\alpha \), respectively, integrating and adding together, using Hölder’s inequality and Young’s inequality, we get
\[
\frac{1}{2} \left( \frac{d}{dt} \|u\|_2^2 + \|v\|_2^2 \right) + \frac{1}{2y} (\alpha y - \lambda^2 - y \varepsilon) \|u_\alpha\|_2^2 + \frac{1}{2a} (\gamma \alpha - \beta^2 - \alpha \varepsilon) \|v_\alpha\|_2^2
\]
\[
+ \frac{1}{2} \left( 6a_1 - |3a_1 - a_3| - |5a_1 - 3a_3 - a_3^2| \right) \|u_\alpha\|_2^2 + \frac{1}{2} \left( 2a_1 + 2a_3 - |3a_1 - a_3| - |5a_1 - 3a_3 - a_3^2| \right) \|v_\alpha\|_2^2
\]
\[
+ \frac{1}{2} \left( 6a_2 - |3a_2 - a_3| - |5a_2 - 3a_3 - a_3^2| \right) \|v_\alpha\|_2^2 + \frac{1}{2} \left( 2a_2 + 2a_3 - |3a_1 - a_3| - |5a_2 - 3a_3 - a_3^2| \right) \|u_\alpha\|_2^2
\]
\[
\leq C \|\theta\|_2^2 + \frac{a_1}{2} \|u_\alpha\|_2^2 + \frac{a_2}{2} \|v_\alpha\|_2^2 + 2(a_1 + a_2) \|uv\|_2^2.
\]

Integrating (4.26) into \( \tau \in (0, t) \), using relations (4.23)–(4.25), we find
\[
\left( \|u\|_t^2 + \|v\|_t^2 \right) + \frac{1}{2y} \alpha y - \lambda^2 - y \varepsilon \int_0^t \|u_\alpha\|_2^2 \, d\tau + \frac{1}{2a} \gamma \alpha - \beta^2 - \alpha \varepsilon \int_0^t \|v_\alpha\|_2^2 \, d\tau
\]
\[
+ \frac{1}{2} \left( 6a_1 - |3a_1 - a_3| - |5a_1 - 3a_3 - a_3^2| \right) \int_0^t \int_\Omega u^2 v^2 \, dx \, d\tau
\]
\[
+ \frac{1}{2} \left( 2a_1 + 2a_3 - |3a_1 - a_3| - |5a_1 - 3a_3 - a_3^2| \right) \int_0^t \int_\Omega v^2 u^2 \, dx \, d\tau
\]
\[
+ \frac{1}{2} \left( 6a_2 - |3a_2 - a_3| - |5a_2 - 3a_3 - a_3^2| \right) \int_0^t \int_\Omega v^2 v^2 \, dx \, d\tau
\]
\[
+ \frac{1}{2} \left( 2a_2 + 2a_3 - |3a_2 - a_3| - |5a_2 - 3a_3 - a_3^2| \right) \int_0^t \int_\Omega u^2 v^2 \, dx \, d\tau
\]
\[
\leq C \int_0^t \|\theta\|_2^2 \, d\tau + \frac{a_1}{2} \int_0^t \|u_\alpha\|_2^2 \, d\tau + \frac{a_2}{2} \int_0^t \|v_\alpha\|_2^2 \, d\tau + 2(a_1 + a_2) \int_0^t \|uv\|_2^2 \, d\tau + \|u_\alpha(0)\|_2^2 + \|v_\alpha(0)\|_2^2
\]
\[
\leq C.
\]

We thus conclude
\[
\int_0^t \left( \|u_\alpha\|_t^2 + \|v_\alpha\|_t^2 \right) \, dt \leq C, \quad t \to \infty, \tag{4.28}
\]
\[
\int_0^t \int_\Omega (u^2 u_\alpha^2 + v^2 v_\alpha^2 + u^2 v_\alpha^2) \, dx \, d\tau \leq C, \quad t \to \infty. \tag{4.29}
\]
Multiplying (1.7) and (1.8) by \(-u_x\) and \(-v_x\), respectively, integrating and adding together, we get

\[
\frac{1}{2} \left| \frac{d}{dt} \|u_x\|^2 \right| + \frac{1}{2} \left| \frac{d}{dt} \|v_x\|^2 \right| \\
\leq a \|u_x\|^2 + y \|v_x\|^2 + \beta \int_\Omega |uv_xu_{xx}| \, dx + a_1 \int_\Omega |u(1-u-v)(1-2u-v)u_{xx}| \, dx \\
+ a_2 \int_\Omega |v(1-u-v)(1-u-2v)v_{xx}| \, dx + a_3 \int_\Omega |uv(v-u)u_{xx}| + uv(u-v)v_{xx}| \, dx \\
+ \lambda \int_\Omega \left| |v_x| v_x|x \right| \, dx + \int_\Omega \left| \{\psi(h_u - h_v(u, v, 0) - h_w(u, 0, w))u_{xx} \right| \\
+ \left| \{\psi(h_v - h_u(u, v, 0) - h_w(v, 0, w))v_{xx} \right| \, dx.
\]  

(4.30)

Integrating it in \(\tau \in (0, t)\), using the relations (4.23)–(4.25) and (4.28) and (4.29), we have

\[
\int_0^t \left( \left| \frac{d}{dt} \|u_x\|^2 \right| + \left| \frac{d}{dt} \|v_x\|^2 \right| \right) \, d\tau \leq C \int_0^t \left( \|u_x\|^2 + \|v_x\|^2 + \|\theta_x\|^2 + \|u\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2 + \|uv\|^2 \right) \, d\tau \leq C.
\]  

(4.31)

Thus, we obtain

\[
\int_0^t \left| \frac{d}{dt} \|u_x\|^2 \right| + \left| \frac{d}{dt} \|v_x\|^2 \right| \, d\tau \leq C, \quad t \to \infty,
\]  

(4.32)

By using Lemma 4.2 and the relations (4.23) and (4.32), we obtain

\[
\lim_{t \to \infty} (\|u_x\|^2 + \|v_x\|^2) = 0.
\]  

(4.33)

By the Poincaré inequality, we have

\[
\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} \leq C(\|u_x\| + \|v_x\|) \to 0, \quad t \to \infty.
\]  

(4.34)

Multiplying (1.7) and (1.8) by \(u_t\) and \(v_t\), respectively, and integrating in \(\tau \in (0, t)\) and adding together, we get

\[
(1 - \varepsilon) \int_0^t (\|u_t\|^2 + \|v_t\|^2) \, d\tau \leq C \int_0^t \left( \|u_x\|^2 + \|v_x\|^2 \right) \, d\tau + C \int_0^t \left( \|u\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2 + \|uv\|^2 + \|\theta_t\|^2 \right) \, d\tau \\
+ \int_0^t (\|u_t(0)\|^2 + \|v_t(0)\|^2) \, d\tau \leq C.
\]  

(4.35)

Multiplying (1.9) by \(\theta\) and integrating, we have

\[
\frac{1}{2} \frac{d}{dt} \|\theta\|^2 + D \|\theta_t\|^2 \leq \frac{1}{2} (h_u - h_w) u_t, \theta + \frac{1}{2} (h_v - h_w) v_t, \theta \\
\leq \left( (\|h_u\|_{L^\infty(\Omega)} + \|h_w\|_{L^\infty(\Omega)}) \|u_t\| + (\|h_v\|_{L^\infty(\Omega)} + \|h_w\|_{L^\infty(\Omega)}) \|v_t\| \right) \|\theta\| \\
\leq C(\|u_t\| + \|v_t\|) \|\theta\| \\
\leq C(\|u_x\| + \|v_x\|) \|\theta\|.
\]  

(4.36)

Thus,

\[
\frac{1}{2} \frac{d}{dt} \|\theta\|^2 + (D - \varepsilon) \|\theta_t\|^2 \leq c_\varepsilon(\|u_x\|^2 + \|v_x\|^2),
\]  

(4.37)
where \( D - \varepsilon > 0 \). Integrating (4.37) into \( \tau \in (0, t) \),

\[
|\theta|^2 + (D - \varepsilon) \int_0^t \|\theta\|^2 \, dr \leq C. \tag{4.38}
\]

Here, we employ the results \( \|\theta\|^2 \leq C \) and \( \int_0^t \|\theta\|^2 \, dr \leq C \) in (4.40) and (4.41), respectively.

By using the Poincaré inequality, we infer from (4.36) that

\[
\frac{1}{2} \frac{d}{dt} \|\theta\|^2 + \frac{1}{c_0^2} (D - c_0^2 \varepsilon) \|\theta\|^2 \leq c_0 (\|\nu\|^2 + \|\nu\|^2), \tag{4.39}
\]

where \( D - c_0^2 \varepsilon > 0 \). Integrating (4.39) into \( \tau \in (0, t) \), we have

\[
\|\theta\|^2 + \frac{1}{c_0^2} (D - c_0^2 \varepsilon) \int_0^t \|\theta\|^2 \, dr \leq C. \tag{4.40}
\]

Thus, by using the relation (4.38), we follow from (4.36) that

\[
\int_0^t \left( \frac{1}{2} \frac{d}{dt} \|\theta\|^2 + \int_0^t \|\theta\|^2 \, dr \right) \, dt \leq C \int_0^t (\|\nu\|^2 + \|\nu\|^2) \, dr \leq C. \tag{4.41}
\]

By Poincaré’s inequality, we have

\[
\lim_{t \to \infty} \|\theta\| = 0. \tag{4.42}
\]

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