Research Article

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On some new Hermite-Hadamard and Ostrowski type inequalities for s-convex functions in \((p, q)\)-calculus with applications

https://doi.org/10.1515/math-2022-0037
received September 23, 2021; accepted April 8, 2022

Abstract: In this study, we establish some new Hermite-Hadamard type inequalities for s-convex functions in the second sense using the post-quantum calculus. Moreover, we prove a new \((p, q)\)-integral identity to prove some new Ostrowski type inequalities for \((p, q)\)-differentiable functions. We also show that the newly discovered results are generalizations of comparable results in the literature. Finally, we give application to special means of real numbers using the newly proved inequalities.

Keywords: Hermite-Hadamard inequality, Ostrowski inequality, \((p, q)\)-integral, post-quantum calculus, s-convex functions

MSC 2020: 26D10, 26D15, 26A51

1 Introduction

The Hermite-Hadamard (HH) inequality, which was independently found by Hermite and Hadamard (see, also [1], and [2, p. 137]), is particularly important in convex functions theory:

\[
f\left(\frac{\pi_1 + \pi_2}{2}\right) \leq \frac{1}{\pi_2 - \pi_1} \int_{\pi_1}^{\pi_2} f(x) dx \leq \frac{f(\pi_1) + f(\pi_2)}{2}, \tag{1}\]

where \(f\) is a convex function on \([\pi_1, \pi_2]\) in this case. The aforementioned inequality is true in reverse order for concave mappings.

In [3], Hudzik and Maligranda defined s-convex functions in the second sense as follows: a mapping \(f : \mathbb{R}^+ \rightarrow \mathbb{R}\), where \(\mathbb{R}^+ = [0, \infty)\) is called s-convex in the second sense if

\[f(tx + (1 - t)y) \leq t^s f(x) + (1 - t)^s f(y)\]
for all \(x, y \in \mathbb{R}^n\) and \(t \in [0, 1]\) and \(s \in (0, 1]\). Dragomir and Fitzpatrick [4] then used this newly discovered class of functions to prove the HH inequality on \([n_1, n_2]\) as follows:

\[
2^{s-1} \left( \frac{n_1 + n_2}{2} \right) \leq \frac{1}{n_2 - n_1} \int_{n_1}^{n_2} f(x) \, dx \leq \frac{f(n_1) + f(n_2)}{s + 1}. \tag{2}
\]

On the other hand, several works in the field of \(q\)-analysis are being carried out, beginning with Euler, to achieve mastery in the mathematics that underpins quantum computing. The link between physics and mathematics is referred to as \(q\)-calculus. It has a wide range of applications in different areas of pure and applied mathematics [5,6]. Euler is thought to be the inventor of this significant branch of mathematics. In Newton’s work on infinite series, he used the \(q\) parameter. Later, Jackson [7,8] presented the \(q\)-calculus that knew without limits calculus in a logical approach. Al-Salam [9] presented the \(q\)-analogue of the \(q\)-fractional integral and the \(q\)-Riemann-Liouville fractional in 1966. Since then, the amount of study in this area has steadily expanded. In particular, in 2013, Tariboon and Ntouyas introduced \(D_q\)-difference operator and \(q_n\)-integral in [10]. In 2020, Bermudo et al. introduced the notion of \(n\)-\(D_q\)-derivative and \(q^{n^r}\)-integral in [11]. Sadjiang generalized to quantum calculus and introduced the notions of post-\(q\)-calculus or shortly \((p, q)\)-calculus in [12]. Soontharapun and Sithithiwathath [13] introduced the notions of fractional \((p, q)\)-calculus later on. In [14], Tunç and Göv gave the post-\(q\)-quantum variant of \(n\)-\(D_q\)-difference operator and \(q_n\)-integral. Recently, in 2021, Vivas-Cortez et al. introduced the notions of \(n\)-\(D_{p,q}\)-derivative and \((p, q)^{n^r}\)-integral in [15].

Many integral inequalities have been studied using quantum integrals for various types of functions. For example, in [16–19,11,20–23], the authors used \(n\)-\(D_q\)-\(n\)-\(D_q\)-derivatives and \(q_{\pi}\)-\(q_{\pi}\)-integrals to prove HH integral inequalities and their left-right estimates for convex and co-ordinated convex functions. In [24], Noor et al. presented a generalized version of quantum HH integral inequalities. For generalized quasi-convex functions, Nwaeeze and Tameru proved certain parameterized quantum integral inequalities in [25]. Khan et al. proved quantum HH inequality using the Green function in [26]. Budak et al. [27], Ali et al. [28,29] and Vivas-Cortez et al. [30] developed new quantum Simpson’s and quantum Newton’s type inequalities for convex and co-ordinated convex functions. For quantum Ostrowski’s inequalities for convex and co-ordinated convex functions, readers refer to [31–33]. Kunt et al. [34] generalized the results of [18] and proved Hermite-Hadamard type inequalities and their left estimates using \(n\)-\(D_{p,q}\)-difference operator and \((p, q)^{n^r}\)-integral. Recently, Latif et al. [35] found the right estimates of Hermite-Hadamard type inequalities proved by Kunt et al. [34].

Inspired by these ongoing studies, in the context of \((p, q)\)-calculus, we prove several new Hermite-Hadamard and Ostrowski type inequalities for \(s\)-convex functions in the second sense.

The following is the structure of this article: Section 2 provides a brief overview of the fundamentals of \(q\)-calculus as well as other related studies in this field. In Section 3, we go over some basic \((p, q)\)-calculus notions and inequalities. In Section 4, we show the relationship between the results presented here and related results in the literature by proving post-quantum HH inequalities for \(s\)-convex functions in the second sense. Post-quantum Ostrowski type inequalities for \(s\)-convex functions in the second are presented in Section 5. In Section 6, we present some applications to special means of real numbers for newly established inequalities. Section 7 concludes with some recommendations for future research.

## 2 Preliminaries of \(q\)-calculus and some inequalities

In this section, we revisit several previously regarded ideas. In addition, throughout the paper, \(s \in (0, 1]\), and we use the following notations (see, [6]):

\[
[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \cdots + q^{n-1}, \quad q \in (0, 1).\]
In [8], Jackson gave the \( q \)-Jackson integral from 0 to \( \pi_2 \) for \( 0 < q < 1 \) as follows:

\[
\int_0^{\pi_2} f(x) dq(x) = (1 - q)\pi_2 \sum_{n=0}^{\infty} q^n f(q^n \pi_2) \tag{3}
\]

provided the sum converge absolutely.

**Definition 1.** [10] For a function \( f : [\pi_1, \pi_2] \to \mathbb{R} \), the \( q \)-\( \pi \)-derivative of \( f \) at \( x \in [\pi_1, \pi_2] \) is characterized by the expression:

\[
x_q D_qf(x) = \frac{f(x) - f(qx + (1 - q)\pi_1)}{(1 - q)(x - \pi_1)}, \quad x \neq \pi_1.
\]

If \( x = \pi_1 \), we define \( n_q D_qf(\pi_1) = \lim_{n \to \pi_1} n_q D_qf(x) \) if it exists, and it is finite.

**Definition 2.** [11] For a function \( f : [\pi_1, \pi_2] \to \mathbb{R} \), the \( q^n \)-\( \pi \)-derivative of \( f \) at \( x \in [\pi_1, \pi_2] \) is characterized by the expression:

\[
x_q D_q^n f(x) = \frac{f(q^n (x - \pi_2) + \pi_2)(x - \pi_1) + \pi_2)}{(1 - q)(x - \pi_1)} - f(x), \quad x \neq \pi_2.
\]

If \( x = \pi_2 \), we define \( n_q D_q^n f(\pi_2) = \lim_{n \to \pi_2} n_q D_q^n f(x) \) if it exists and it is finite.

**Definition 3.** [10] Let \( f : [\pi_1, \pi_2] \to \mathbb{R} \) be a function. Then, the \( q \)-\( \pi \)-definite integral on \([\pi_1, \pi_2]\) is defined as follows:

\[
\int_{\pi_1}^{\pi_2} f(x) n_q d_q x = (1 - q)(\pi_2 - \pi_1) \sum_{n=0}^{\infty} q^n f(q^n \pi_2 + (1 - q^n)\pi_1) = (\pi_2 - \pi_1) \int_0^{\pi_2} f((1 - t)\pi_1 + t\pi_2) dt_q. \tag{6}
\]

**Definition 4.** [11] Let \( f : [\pi_1, \pi_2] \to \mathbb{R} \) be a function. Then, the \( q^n \)-\( \pi \)-definite integral on \([\pi_1, \pi_2]\) is defined as follows:

\[
\int_{\pi_1}^{\pi_2} f(x) n_q^n d_q x = (1 - q)(\pi_2 - \pi_1) \sum_{n=0}^{\infty} q^n f(q^n \pi_1 + (1 - q^n)\pi_2) = (\pi_2 - \pi_1) \int_0^{\pi_2} f(t\pi_1 + (1 - t)\pi_2) dt_q. \tag{7}
\]

In [11], Bermudo et al. established the following quantum HH type inequality.

**Theorem 1.** For the convex mapping \( f : [\pi_1, \pi_2] \to \mathbb{R} \), the following inequality holds

\[
f\left(\frac{\pi_1 + \pi_2}{2}\right) \leq \frac{1}{2(\pi_2 - \pi_1)} \left[ \int_{\pi_1}^{\pi_2} f(x) n_q d_q x + \int_{\pi_1}^{\pi_2} f(x) n_q^n d_q x \right] \leq \frac{f(\pi_1) + f(\pi_2)}{2}. \tag{8}
\]

In [33], Budak et al. proved the following Ostrowski inequality by using the concepts of quantum derivatives and integrals.

**Theorem 2.** Let \( f : [\pi_1, \pi_2] \subset \mathbb{R} \to \mathbb{R} \) be a function and \( q^n D_q f \) and \( n_q d_q x \) be two continuous and integrable functions on \([\pi_1, \pi_2]\). If \( |q^n D_q f(t)|, |n_q d_q x| \leq M \) for all \( t \in [\pi_1, \pi_2] \), then we have the following quantum quantum Ostrowski inequality:

\[
\left| f(x) - \frac{1}{\pi_2 - \pi_1} \left[ \int_{\pi_1}^{x} f(t) n_q d_q x + \int_{x}^{\pi_2} f(t) n_q^n d_q x \right] \right| \leq \frac{qM}{(\pi_2 - \pi_1)} \left[ \frac{(x - \pi_2)^2 + (\pi_1 - x)^2}{2}\right]. \tag{9}
\]
Recently, Asawasamrit et al. [36] gave the following generalizations of inequalities (8) and (9) using the s-convexity.

**Theorem 3.** Assume that the mapping \( f : [0, \infty) \to \mathbb{R} \) is s-convex in the second sense and \( \pi_1, \pi_2 \in [0, \infty) \) with \( \pi_1 < \pi_2 \), then the following inequality holds for \( s \in (0, 1) \):

\[
2^{s-1} \left( \frac{\pi_1 + \pi_2}{2} \right) \leq \frac{1}{2(\pi_2 - \pi_1)} \left[ \int_{\pi_1}^{\pi_2} f(x) d_s x + \int_{\pi_1}^{\pi_2} f(x) d_{\pi_2} x \right] \leq \frac{f(\pi_1) + f(\pi_2)}{[s + 1]_q}.
\] (10)

**Theorem 4.** Let \( f : [\pi_1, \pi_2] \subset \mathbb{R} \to \mathbb{R} \) be function and \( \pi_1 D_{\pi_2} f \) and \( \pi_2 D_{\pi_1} f \) be two continuous and integrable functions on \([\pi_1, \pi_2]\). If \( |\pi_1 D_{\pi_2} f(t)|, |\pi_2 D_{\pi_1} f(t)| \leq M \) for all \( t \in [\pi_1, \pi_2] \), then we have the following quantum Ostrowski inequality for s-convex functions in the second sense:

\[
\left| f(x) - \frac{1}{\pi_2 - \pi_1} \left[ \int_{\pi_1}^{\pi_2} f(x) d_{\pi_1} x + \int_{\pi_1}^{\pi_2} f(x) d_{\pi_2} x \right] \right| \leq \frac{Mq}{\pi_2 - \pi_1} \left( \frac{1}{[s + 2]_q} + \Theta_{11} \right) [(x - \pi_1)^2 + (\pi_2 - x)^2],
\] (11)

where

\[
\Theta_{11} = \int_0^1 t(1 - t)^2 d_t t.
\]

### 3 Post-quantum calculus and some inequalities

In this section, we review some fundamental notions and notations of \((p, q)\)-calculus.

The \([n]_{p,q}\) is said to be \((p, q)\)-integers and expressed as follows:

\[
[n]_{p,q} = \frac{p^n - q^n}{p - q}
\]

with \(0 < q < p \leq 1\). The \([n]_{p,q}\) and \(\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\) are called \((p, q)\)-factorial and \((p, q)\)-binomial, respectively, and expressed as follows:

\[
[n]_{p,q} = \prod_{k=1}^{n} [k]_{p,q}, \quad n \geq 1, \quad [0]_{p,q} = 1,
\]

\[
\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}}{[n-k]_{p,q} [k]_{p,q}}.
\]

**Definition 5.** [12] The \((p, q)\)-derivative of mapping \( f : [\pi_1, \pi_2] \to \mathbb{R} \) is given as follows:

\[
D_{p,q} f(x) = \frac{f(px) - f(qx)}{(p - q)x}, \quad x \neq 0
\]

with \(0 < q < p \leq 1\).

**Definition 6.** [14] The \((p, q)_{\pi_2}\)-derivative of mapping \( f : [\pi_1, \pi_2] \to \mathbb{R} \) is given as follows:

\[
\pi_2 D_{p,q} f(x) = \frac{f(px + (1 - p)\pi_1) - f(qx + (1 - q)\pi_1)}{(p - q)(x - \pi_1)}, \quad x \neq \pi_1
\] (12)

with \(0 < q < p \leq 1\). For \( x = \pi_1 \), we state \( \pi_1 D_{p,q} f(\pi_1) = \lim_{x \to \pi_1} D_{p,q} f(x) \) if it exists and it is finite.
Definition 7. [15] The \((p,q)^{\pi_2}\)-derivative of mapping \(f : [\pi_1,\pi_2] \to \mathbb{R}\) is given as follows:
\[
\pi_2^{D_{p,q}}f(x) = \frac{f(qx + (1 - q)\pi_2) - f(px + (1 - p)\pi_2)}{(p - q)(\pi_2 - x)}, \quad x \neq \pi_2.
\]
with \(0 < q < p \leq 1\). For \(x = \pi_2\), we state \(\pi_2^{D_{p,q}}f(\pi_2) = \lim_{h \to 0}\pi_2^{D_{p,q}}f(x)\) if it exists and it is finite.

Remark 1. It is clear that if we use \(p = 1\) in (12) and (13), then the equalities (12) and (13) reduce to (4) and (5), respectively.

Definition 8. [14] The definite \((p,q)^{\pi_1}\)-integral of mapping \(f : [\pi_1,\pi_2] \to \mathbb{R}\) on \([\pi_1,\pi_2]\) is stated as follows:
\[
\int_{\pi_1}^{\pi_2} f(\tau) d_{p,q}\tau = (p - q)(\pi_2 - \pi_1) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left( \frac{q^n}{p^{n+1}} \right) \left( 1 - \frac{q^n}{p^{n+1}} \right) \pi_1
\]
with \(0 < q < p \leq 1\).

Definition 9. [15] The definite \((p,q)^{\pi_2}\)-integral of mapping \(f : [\pi_1,\pi_2] \to \mathbb{R}\) on \([\pi_1,\pi_2]\) is stated as follows:
\[
\int_{\pi_1}^{\pi_2} f(\tau) d_{p,q}\tau = (p - q)(\pi_2 - \pi_1) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left( \frac{q^n}{p^{n+1}} \right) \left( 1 - \frac{q^n}{p^{n+1}} \right) \pi_1
\]
with \(0 < q < p \leq 1\).

Remark 2. It is evident that if we pick \(p = 1\) in (14) and (15), then the equalities (14) and (15) change into (6) and (7), respectively.

Remark 3. If we take \(\pi_1 = 0\) and \(x = \pi_2 = 1\) in (14), then we have
\[
\int_{0}^{1} f(\tau) d_{p,q}\tau = (p - q) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left( \frac{q^n}{p^{n+1}} \right).
\]
Similarly, by taking \(x = \pi_1 = 0\) and \(\pi_2 = 1\) in (15), then we obtain that
\[
\int_{0}^{1} f(\tau) d_{p,q}\tau = (p - q) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left( 1 - \frac{q^n}{p^{n+1}} \right).
\]

Lemma 1. [15] We have the following equalities:
\[
\int_{\pi_1}^{\pi_2} (\pi_2 - x)^{a_{\pi_2}} d_{p,q}\tau = \frac{(\pi_2 - \pi_1)^{a_1}}{[a + 1]_{p,q}}
\]
\[
\int_{\pi_1}^{\pi_2} (x - \pi_1)^{a_{\pi_1}} d_{p,q}\tau = \frac{(\pi_2 - \pi_1)^{a_1}}{[a + 1]_{p,q}},
\]
where \(a \in \mathbb{R} - \{-1\}\).

Recently, Vivas-Cortez et al. [15] proved the following HH type inequalities for convex functions using the \((p,q)^{\pi_2}\)-integral:

Theorem 5. [15] For a convex mapping \(f : [\pi_1,\pi_2] \to \mathbb{R}\), which is differentiable on \([\pi_1,\pi_2]\), the following inequalities hold for \((p,q)^{\pi_2}\)-integral:
\[
F\left(\frac{p\eta_1 + q\eta_2}{2}\right) \leq \frac{1}{p(\eta_2 - \eta_1)} \int_{\eta_1}^{\eta_2} f(x)^{\eta_1}d_{p,q}x \leq \frac{pF(\eta_1) + qF(\eta_2)}{2},
\]
where \(0 < q < p \leq 1\).

**Theorem 6.** [15] For a convex function \(f : [\eta_1, \eta_2] \rightarrow \mathbb{R}\), the following inequality holds:

\[
F\left(\frac{\eta_1 + \eta_2}{2}\right) \leq \frac{1}{2p(\eta_2 - \eta_1)} \int_{\eta_1}^{\eta_2} f(x)^{\eta_1}d_{p,q}x + \int_{\eta_1}^{\eta_2} f(x)^{\eta_2}d_{p,q}x \leq \frac{f(\eta_1) + f(\eta_2)}{2},
\]
where \(0 < q < p \leq 1\).

### 4 Hermite-Hadamard inequalities

In this section, we prove HH inequalities for s-convex functions in the second kind using the post-quantum integrals.

**Theorem 7.** Assume that the mapping \(f : [0, \infty) \rightarrow \mathbb{R}\) is s-convex in the second sense and \(\eta_1, \eta_2 \in [0, \infty)\) with \(\eta_1 < \eta_2\), then the following inequality holds for \(s \in (0, 1]\):

\[
2^{s-1}F\left(\frac{\eta_1 + \eta_2}{2}\right) \leq \frac{1}{2p(\eta_2 - \eta_1)} \int_{\eta_1}^{\eta_2} f(x)^{\eta_1}d_{p,q}x + \int_{\eta_1}^{\eta_2} f(x)^{\eta_2}d_{p,q}x \leq \frac{f(\eta_1) + f(\eta_2)}{s + 1}.\]

**Proof.** We have s-convexity, as we know from s-convexity

\[
2^{s}f\left(\frac{x + y}{2}\right) \leq f(x) + f(y).
\]

We obtain the following by putting \(x = t\eta_2 + (1 - t)\eta_1\) and \(y = t\eta_1 + (1 - t)\eta_2\) in (19)

\[
2^{s}f\left(\frac{\eta_1 + \eta_2}{2}\right) \leq f(t\eta_2 + (1 - t)\eta_1) + f(t\eta_1 + (1 - t)\eta_2).
\]

From Definitions 8 and 9, we have

\[
2^{s-1}F\left(\frac{\eta_1 + \eta_2}{2}\right) \leq \frac{1}{2p(\eta_2 - \eta_1)} \int_{\eta_1}^{\eta_2} f(x)^{\eta_1}d_{p,q}x + \int_{\eta_1}^{\eta_2} f(x)^{\eta_2}d_{p,q}x,
\]
and the first inequality in (18) is proved.

To prove the second inequality, we use the s-convexity, and we have

\[
f(t\eta_2 + (1 - t)\eta_1) \leq t^s f(\eta_1) + (1 - t)^s f(\eta_2)
\]
and

\[
f(t\eta_1 + (1 - t)\eta_2) \leq t^s f(\eta_1) + (1 - t)^s f(\eta_2).
\]
By adding (20) and (21), from Definitions 8 and 9, we have

\[
\frac{1}{2p(\eta_2 - \eta_1)} \int_{\eta_1}^{\eta_2} f(x)^{\eta_1}d_{p,q}x + \int_{\eta_1}^{\eta_2} f(x)^{\eta_2}d_{p,q}x \leq \frac{f(\eta_1) + f(\eta_2)}{s + 1},
\]
and the proof is completed. \(\square\)
Example 1. For s-convex function \( f(x) = x^s \), from inequality (18) with \( a = s = 1, b = 2, p = \frac{1}{2}, \) and \( q = \frac{1}{4}, \) we have

\[
2^{s-1} \left( \frac{\pi_1 + \pi_2}{2} \right) = \frac{3}{2},
\]

\[
\frac{1}{2p(\pi_2 - \pi_1)} \left[ \frac{\pi_2}{n_1} \int_{n_1}^{\pi_2} f(x) \, d_{p,q}x + \frac{\pi_1}{n_1} \int_{n_1}^{\pi_1} f(x)^n \, d_{p,q}x \right]
= \left( \frac{1}{2} \right)^{-1} - \left( \frac{1}{4} \right) \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n \left( \frac{1}{2} \right)^{n+1} + \left( \frac{1}{2} \right)^n + \left( \frac{1}{2} \right)^{n+1} \right)
\]

and

\[
\frac{f(\pi_1) + f(\pi_2)}{1 + s}_{p,q} = \frac{1 + 2 + \frac{1}{2}}{4} = 4.
\]

Thus,

\[
\frac{3}{2} < 3 < 4,
\]

which shows that the inequality proved in Theorem 7 is true.

Remark 4. If we set \( s = 1 \) in Theorem 7, then we recapture the inequality (17).

Remark 5. In Theorem 7, if we take the limit as \( p = 1 \), then inequality (18) becomes the inequality (10).

Remark 6. In Theorem 7, if we take \( p = 1 \) and later take the limit as \( q \to \Gamma \), then inequality (18) becomes the inequality (2).

5 Ostrowski’s inequalities

In this section, we prove post-quantum Ostrowski type inequalities for s-convex functions in the second sense.

We begin with the following identity.

Lemma 2. Let \( f : [\pi_1, \pi_2] \subset \mathbb{R} \to \mathbb{R} \) be a function. If \( n_1D_{p,q}f \) and \( n_2D_{p,q}f \) are two continuous and integrable functions on \([\pi_1, \pi_2] \), then for all \( x \in [\pi_1, \pi_2] \), we have

\[
f(x) = \frac{1}{p(\pi_2 - \pi_1)} \left[ \int_{\pi_1}^{\pi_2} f(t) \, d_{p,q}t + \int_{px+(1-p)\pi_1}^{\pi_2} f(t) \, d_{p,q}t \right]
\]

\[
eq \frac{(x - \pi_1)^2}{\pi_2 - \pi_1} \int_{0}^{\pi_1} t \, D_{p,q}f(tx + (1-t)\pi_1) \, dt - \frac{q(\pi_2 - x)^2}{\pi_2 - \pi_1} \int_{0}^{1} D_{p,q}f(tx + (1-t)\pi_2) \, dt.
\]

Proof. From Definitions 6 and 7, we have

\[
n_1D_{p,q}f(tx + (1-t)\pi_1) = \frac{f(ptx + (1-p)t\pi_1) - f(qtx + (1-qt)\pi_1)}{t(x-\pi_1)(p-q)}
\]
and
\[ n^0 \mathcal{D}_{p,q}f(tx + (1 - t)\pi_2) = \frac{f(qtx + (1 - qt)\pi_2) - f(ptx + (1 - pt)\pi_2)}{t(\pi_2 - x)(p - q)}. \]

By using Definition 9, we have
\[
I_1 = \int_0^1 n^0 \mathcal{D}_{p,q}f(tx + (1 - t)\pi_2) d_{p,q} t \\
= \frac{1}{(\pi_2 - x)(p - q)} \int_0^1 [f(qtx + (1 - qt)\pi_2) - f(ptx + (1 - pt)\pi_2)] d_{p,q} t \\
= \frac{1}{\pi_2 - x} \sum_{n=0}^\infty \frac{q^n}{p^n+1} \left( \frac{q^n x + \left(1 - \frac{q^n}{p^n}\right) \pi_2}{p^{n+1}} \right) - \sum_{n=0}^\infty \frac{q^n}{p^n+1} \left( \frac{q^n x + \left(1 - \frac{q^n}{p^n}\right) \pi_2}{p^{n+1}} \right) \\
= \frac{1}{\pi_2 - x} \left( \frac{1 - \frac{1}{q}}{p} \sum_{n=0}^\infty \frac{q^n}{p^n+1} \left( \frac{q^n x + \left(1 - \frac{q^n}{p^n}\right) \pi_2}{p^{n+1}} \right) - \frac{1}{q} f(x) \right) \\
= \frac{1}{\pi_2 - x} \left( \frac{p - q}{pq} \sum_{n=0}^\infty \frac{q^n}{p^n+1} \left( \frac{q^n x + \left(1 - \frac{q^n}{p^n}\right) \pi_2}{p^{n+1}} \right) - \frac{1}{q} f(x) \right) \\
= \frac{1}{\pi_2 - x} \int_{p^{-1}(1-p)\pi_2}^\pi f(x)^0 d_{p,q} x - \frac{1}{q} f(x).
\]

Similarly, from Definition 8, we have
\[
I_2 = \int_0^1 t^0 \mathcal{D}_{p,q}f(tx + (1 - t)\pi_2) d_{p,q} t = \int_0^1 \frac{1}{x - \pi_1} \left[ f(x) - \frac{1}{pq(x - \pi_2)} \int_{\pi_1}^{pq(x - \pi_2)} f(x)^0 d_{p,q} x \right].
\]

Thus, we obtain the resultant equality (22) by subtracting (23) from (24).

**Remark 7.** In Lemma 2, if we set \( p = 1 \), then we obtain the equality:
\[
f(x) = \frac{1}{(\pi_2 - \pi_1)} \int_{\pi_1}^x f(t)^0 d_{p,q} t + \frac{\pi_2}{n^0} \int_0^1 f(t)^0 d_{p,q} t \\
- \frac{q(\pi_2 - x)^2}{p\pi_2 - \pi_1} \int_0^1 t^0 \mathcal{D}_{p,q}f(tx + (1 - t)\pi_2) d_{p,q} t,
\]
which is proved by Budak et al. in [33].

**Remark 8.** In Lemma 2, if we set \( p = 1 \) and later taking the limit as \( q \to 1 \), then we obtain [37, Lemma 1].

**Theorem 8.** Assume that the mapping \( f : I \subset [0, \infty) \to \mathbb{R} \) is differentiable and \( n_1, n_2 \in I \) with \( n_1 < n_2 \). If \( |n^0 \mathcal{D}_{p,q}f| \) and \( |p^0 \mathcal{D}_{p,q}f| \) are s-convex mappings in the second sense, then the following inequality holds:
\[
\left| f(x) - \frac{1}{p(n_2 - n_1)} \int_{n_1}^{n_2} f(t)^0 d_{p,q} t + \int_{p^{-1}(1-p)n_2}^{pq(n_2 - x)} f(t)^0 d_{p,q} t \right| \\
\leq \left| f(x) - \frac{1}{p(n_2 - n_1)} \int_{n_1}^{n_2} f(t)^0 d_{p,q} t + \int_{p^{-1}(1-p)n_2}^{pq(n_2 - x)} f(t)^0 d_{p,q} t \right|.
\]
\[
\leq \frac{q(x - \pi)}{\pi - \pi} \left[ \frac{1}{s + 2|p|} |p| D_{p,q}f(x) + \Theta \right] + \frac{q(x - \pi)}{\pi} \left[ \frac{1}{s + 2|p|} |p| D_{p,q}f(x) + \Theta \right] \],
\]

where
\[
\Theta_1 = \int_0^1 t(1 - t) d_{p,q} t
\]
and
\[
\Theta_2 = \int_0^1 t(1 - t) s d_{p,q} t.
\]

**Proof.** From Lemma 2 and properties of the modulus, we have
\[
\left| f(x) - \frac{1}{p(x - \pi)} \int_{\pi}^{p(x - \pi)} f(t) n d_{p,q} t + \int_{p(x - \pi)}^{x} f(t)^n d_{p,q} t \right| \leq \frac{q(x - \pi)}{\pi} \int_0^1 t^s D_{p,q} f(tx + (1 - t)\pi) n d_{p,q} t + \frac{q(x - \pi)}{\pi} \int_0^1 t^s D_{p,q} f(tx + (1 - t)\pi) n d_{p,q} t \]
\[
\leq \frac{q(x - \pi)}{\pi} \int_0^1 t^s D_{p,q} f(tx + (1 - t)\pi) n d_{p,q} t + \frac{q(x - \pi)}{\pi} \int_0^1 t^s D_{p,q} f(tx + (1 - t)\pi) n d_{p,q} t \]
\[
\leq \frac{1}{s + 2|p|} |p| D_{p,q} f(x) + \Theta |p| D_{p,q} f(\pi)\]
and
\[
\leq \frac{1}{s + 2|p|} |p| D_{p,q} f(x) + \Theta |p| D_{p,q} f(\pi)\]

We obtain the resultant inequality (25) by putting (27) and (28) in (26).

**Corollary 1.** If we set \( s = 1 \) in Theorem 8, then we obtain the following new Ostrowski type inequality for convex functions:
\[
\left| f(x) - \frac{1}{p(x - \pi)} \int_{\pi}^{p(x - \pi)} f(t) n d_{p,q} t + \int_{p(x - \pi)}^{x} f(t)^n d_{p,q} t \right| \leq \frac{q(x - \pi)}{\pi} \left[ \frac{1}{3|p|} |p| D_{p,q} f(x) + \frac{3|p| - 2|p|}{3|p|} |p| D_{p,q} f(\pi) \right] + \frac{q(x - \pi)}{\pi} \left[ \frac{1}{3|p|} |p| D_{p,q} f(x) + \frac{3|p| - 2|p|}{3|p|} |p| D_{p,q} f(\pi) \right].
\]

We obtain the resultant inequality (25) by putting (27) and (28) in (26).
Remark 9. In Theorem 8, if we set \( p = 1 \), then Theorem 8 reduces to [36, Theorem 4.1].

Remark 10. In Corollary 1, if we set \( p = 1 \), then we obtain the following inequality:

\[
\left| f(x) - \frac{1}{p_2 - p_1} \left[ \int \frac{x}{n_1} f(t)^{x}d_{p,q}t + \int \frac{n_1}{x} f(t)^{n_1}d_{p,q}t \right] \right| \leq \frac{q}{(p_2 - p_1)[2]=[q][3]} [ (x - p_2)^2 [2]_{p,q} D_{p,q}f(x) + q^2]_{p_2} D_{p,q}f(p_2)] + (p_2 - x)^2 [2]_{p,q} D_{p,q}f(x) + q^2]_{p_2} D_{p,q}f(p_2)],
\]

which is given by Budak et al. in [33].

Corollary 2. If we assume \( |_{p,q} D_{p,q}f(x)|, |_{p_2} D_{p,q}f(x)| \leq M \) in Theorem 8, then we have following post-quantum Ostrowski type inequality for s-convex functions in the second sense:

\[
\left| f(x) - \frac{1}{p_2 - p_1} \left[ \int \frac{x}{n_1} f(t)^{x}d_{p,q}t + \int \frac{n_1}{x} f(t)^{n_1}d_{p,q}t \right] \right| \leq M q (x - p_2)^2 \left[ \int \frac{1}{s + 2} D_{p,q}f(x) + \Theta_1 \right] + M q (p_2 - x)^2 \left[ \int \frac{1}{s + 2} D_{p,q}f(x) + \Theta_2 \right].
\]

Remark 11. In Corollary 2, if we set \( p = 1 \) in Corollary 2, then Corollary 2 reduces to [36, Corollary 4.1].

Remark 12. If we set \( s = p = 1 \) in Corollary 2, then we recapture inequality (9).

Remark 13. In Corollary 2, if we set \( p = 1 \) and later take the limit as \( q \to 1 \), then Corollary 2 reduces to [38, Theorem 2].

Theorem 9. Assume that the mapping \( f : I \subset [0, \infty) \to \mathbb{R} \) is differentiable and \( p_1, p_2 \in I \) with \( p_1 < p_2 \). If \( |_{n} D_{p,q}f(x)|, |_{n} D_{p,q}f(x)| \leq M \) in Theorem 8, then the following inequality holds:

\[
\left| f(x) - \frac{1}{p_2 - p_1} \left[ \int \frac{x}{n_1} f(t)^{x}d_{p,q}t + \int \frac{n_1}{x} f(t)^{n_1}d_{p,q}t \right] \right| \leq \frac{q}{(p_2 - p_1)[2]=[q][3]} [ (x - p_2)^2 [2]_{p,q} D_{p,q}f(x) + q^2]_{p_2} D_{p,q}f(p_2)] + (p_2 - x)^2 [2]_{p,q} D_{p,q}f(x) + q^2]_{p_2} D_{p,q}f(p_2)],
\]

Proof. From Lemma 2, by using properties of the modulus and power mean inequality, we have

\[
\left| f(x) - \frac{1}{p_2 - p_1} \left[ \int \frac{x}{n_1} f(t)^{x}d_{p,q}t + \int \frac{n_1}{x} f(t)^{n_1}d_{p,q}t \right] \right| \leq q (x - p_2)^2 \int \frac{1}{s + 2} D_{p,q}f(x) \left[ (1 - t) f(x) + q (p_2 - x)^2 \right]_{p_2} D_{p,q}f(p_2)t.
\]
\[
\frac{q(x - \pi_1)^2}{\pi_2 - \pi_1} \left( \int_0^1 t_0 d_{p,q} t \right)^{1 - \frac{1}{p^*}} \leq \frac{1}{(2p,q)} \left( \int t^n D_{p,q} f(tx + (1 - t)\pi_1) d_{p,q} t \right)^{\frac{1}{p^*}} \\
+ \frac{q(\pi_2 - x)^2}{\pi_2 - \pi_1} \left( \int_0^1 t^n d_{p,q} t \right)^{1 - \frac{1}{p^*}} \leq \frac{1}{(2p,q)} \left( \int t^n D_{p,q} f(tx + (1 - t)\pi_2) d_{p,q} t \right)^{\frac{1}{p^*}}.
\]

(31)

Since the mapping \(|n D_{p,q} f|^p|\) and \(|^n D_{p,q} f|^p|\) are \(s\)-convexities in the second sense, therefore

\[
\left( \int_0^1 t^n d_{p,q} t \right)^{1 - \frac{1}{p^*}} \leq \frac{1}{(2p,q)} \left( \int t^n D_{p,q} f(tx + (1 - t)\pi_1) d_{p,q} t \right)^{\frac{1}{p^*}}
\]

(32)

and

\[
\left( \int_0^1 t^n d_{p,q} t \right)^{1 - \frac{1}{p^*}} \leq \frac{1}{(2p,q)} \left( \int t^n D_{p,q} f(tx + (1 - t)\pi_2) d_{p,q} t \right)^{\frac{1}{p^*}}.
\]

(33)

We obtain the resultant inequality (30) by putting (32) and (33) in (31).

\[\square\]

**Corollary 3.** If we set \(s = 1\) in Theorem 9, then we obtain the following new Ostrowski type inequality for convex functions:

\[
\left| f(x) - \frac{1}{p(\pi_2 - \pi_1)} \int_{\pi_1}^{px + (1 - p)\pi_1} f(t) d_{p,q} t + \int_{px + (1 - p)\pi_2}^{\pi_2} f(t) d_{p,q} t \right| \\
\leq \frac{q}{\pi_2 - \pi_1} \left( \frac{1}{(2p,q)} \right)^{1 - \frac{1}{p}} \left( x - \pi_1 \right)^{\frac{1}{p}} \left[ \frac{1}{|3|_{p,q}} \left| D_{p,q} f(x) \right|^p\pi_1 + \frac{[3]_{p,q} - [2]_{p,q}}{|3|_{p,q} [2]_{p,q}} \left| D_{p,q} f(\pi_1) \right|^p \right]^{\frac{1}{p^*}} \\
+ \left( \pi_2 - x \right)^{\frac{1}{p}} \left[ \frac{1}{|3|_{p,q}} \left| D_{p,q} f(x) \right|^p\pi_2 + \frac{[3]_{p,q} - [2]_{p,q}}{|3|_{p,q} [2]_{p,q}} \left| D_{p,q} f(\pi_2) \right|^p \right]^{\frac{1}{p^*}}.
\]

**Remark 14.** In Theorem 9, if we set \(p = 1\), then Theorem 9 reduces to [36, Theorem 4.2].

**Remark 15.** In Corollary 3, if we set \(p = 1\), then we obtain the following inequality:

\[
\left| f(x) - \frac{1}{\pi_2 - \pi_1} \int_x^{\pi_1} f(t) d_{q} t + \int_x^{\pi_2} f(t) d_{q} t \right| \\
\leq \frac{q}{(\pi_2 - \pi_1) |2|_q} \left( x - \pi_1 \right)^{\frac{1}{p}} \left[ \frac{[2]_q |D_q f(x)|^p + q^2 |D_q f(\pi_1)|^p}{|3|_q} \right]^{\frac{1}{p^*}} \\
+ \left( \pi_2 - x \right)^{\frac{1}{p}} \left[ \frac{[2]_q |D_q f(x)|^p + q^2 |D_q f(\pi_2)|^p}{|3|_q} \right]^{\frac{1}{p^*}},
\]

which is proved by Budak et al. in [33].
Corollary 4. If we assume $|n_i D_{p,q} f(x)|$, $|n_i D_{p,q} f(x)| \leq M$ in Theorem 9, then we have following post-quantum Ostrowski type inequality for $s$-convex functions in the second sense:

$$
\left| f(x) - \frac{1}{p(n_2 - n_1)} \left[ \int_{n_1}^{\text{px}(1-p)n_1} f(t) \, dn_{p,q} + \int_{\text{px}(1-p)n_2}^{n_2} f(t) \, dn_{p,q} \right] \right| \\
\leq \frac{Mq}{n_2 - n_1} \left[ \left( \frac{1}{[2n_1 + 1]p,q} \right) \left( x - n_1 \right)^2 \left( \frac{1}{[s + 1]p,q} + n_1 \right) \left( n_2 - x \right)^2 \left( \frac{1}{[s + 1]p,q} + n_2 \right) \right].
$$

Remark 16. In Corollary 4, if we set $p = 1$, then Corollary 4 reduces to [36, Corollary 4.2].

Remark 17. In Corollary 4, if we set $p = 1$ and later take the limit as $q \to 1^+$, then Corollary 4 reduces to [38, Theorem 4].

Theorem 10. Assume that the mapping $f : I \subset [0, \infty) \to \mathbb{R}$ is differentiable and $n_1, n_2 \in I$ with $n_1 < n_2$. If $|n_i D_{p,q} f|^p_i$ and $|n_i D_{p,q} f|^q_i$, $p_1 > 1$ are s-convex mappings in the second sense, then the following inequality holds:

$$
\left| f(x) - \frac{1}{p(n_2 - n_1)} \left[ \int_{n_1}^{\text{px}(1-p)n_1} f(t) \, dn_{p,q} + \int_{\text{px}(1-p)n_2}^{n_2} f(t) \, dn_{p,q} \right] \right| \\
\leq \frac{q}{n_2 - n_1} \left( \frac{1}{[n_1 + 1]p,q} \right) \left( x - n_1 \right)^2 \left( \frac{1}{[s + 1]p,q} \right) \left( n_2 - x \right)^2 \left( \frac{1}{[s + 1]p,q} \right)
$$

where $r_i^{-1} + p_i^{-1} = 1$.

Proof. From Lemma 2, by using properties of the modulus and Hölder’s inequality, we have

$$
\left| f(x) - \frac{1}{p(n_2 - n_1)} \left[ \int_{n_1}^{\text{px}(1-p)n_1} f(t) \, dn_{p,q} + \int_{\text{px}(1-p)n_2}^{n_2} f(t) \, dn_{p,q} \right] \right| \\
\leq \frac{q(x - n_1)^2}{n_2 - n_1} \left( \int_{n_1}^{\frac{1}{[n_1]p,q}} f(t) \, dt \right)^\frac{1}{n_1} \left( \int_{\frac{1}{[n_2]p,q}}^{n_2} f(t) \, dt \right)^\frac{1}{n_2} \\
\leq \frac{q(x - n_1)^2}{n_2 - n_1} \left( \int_{n_1}^{\frac{1}{[n_1]p,q}} f(t) \, dt \right)^\frac{1}{n_1} \left( \int_{\frac{1}{[n_2]p,q}}^{n_2} f(t) \, dt \right)^\frac{1}{n_2} \\
\leq \frac{q(x - n_1)^2}{n_2 - n_1} \left( \int_{n_1}^{\frac{1}{[n_1]p,q}} f(t) \, dt \right)^\frac{1}{n_1} \left( \int_{\frac{1}{[n_2]p,q}}^{n_2} f(t) \, dt \right)^\frac{1}{n_2} \\
\leq \frac{q(x - n_1)^2}{n_2 - n_1} \left( \int_{n_1}^{\frac{1}{[n_1]p,q}} f(t) \, dt \right)^\frac{1}{n_1} \left( \int_{\frac{1}{[n_2]p,q}}^{n_2} f(t) \, dt \right)^\frac{1}{n_2}.
$$

Since the mapping $|n_i D_{p,q} f|^p_i$ and $|n_i D_{p,q} f|^q_i$ are s-convexities in the second sense, therefore

$$
\left( \int_{n_1}^{\frac{1}{[n_1]p,q}} f(t) \, dt \right)^\frac{1}{n_1} \left( \int_{\frac{1}{[n_2]p,q}}^{n_2} f(t) \, dt \right)^\frac{1}{n_2} \\
\leq \left( \frac{1}{[n_1 + 1]p,q} \right)^\frac{1}{n_1} \left( \frac{1}{[s + 1]p,q} \left( |n_i D_{p,q} f|^p_i + |n_i D_{p,q} f|^q_i \right) \right).
$$
and
\[
\left( \int_0^1 t^{n_1} d_{p,q} t \right)^2 \int_0^1 t^{n_2} D_{p,q} f(t x + (1 - t) \pi_2) d_{p,q} t \right)^\frac{1}{2} 
\leq \left( \frac{1}{[s + 1]_{p,q}} \right)^\frac{1}{2} \left( \frac{1}{[1 + 1]_{p,q}} \right)^\frac{1}{2} \left( |\pi_2 D_{p,q} f(x)| \pi_1 + |\pi_2 D_{p,q} f(\pi_2)| \pi_1 \right) \right). 
\]
(37)

We obtain the resultant inequality (34) by putting (36) and (37) in (35).

\[\Box\]

**Corollary 5.** If we set \( s = 1 \) in Theorem 10, then we obtain the following new Ostrowski type inequality for convex functions:

\[
\left| f(x) - \frac{1}{p(\pi_2 - \pi_1)} \left[ \int_0^x f(t) \pi_1 d_{p,q} t + \int_x^{\pi_2} f(t) \pi_2 d_{p,q} t \right] \right|
\]

\[
\leq \frac{q}{\pi_2 - \pi_1} \left( \frac{1}{[1 + 1]_{q} \pi_2} \right)^\frac{1}{2} \left( x - \pi_1 \right)^\frac{1}{2} \left( \left| \pi_2 D_{q} f(x) \right| \pi_1 + \left| \pi_2 D_{q} f(\pi_2) \right| \pi_1 \right) \right)^\frac{1}{2} + \left( \pi_2 - x \right)^\frac{1}{2} \left( \frac{\left| \pi_2 D_{q} f(x) \right| \pi_1 + \left| \pi_2 D_{q} f(\pi_2) \right| \pi_1 }{[2]_{q} \pi_1} \right)^\frac{1}{2},
\]

which is proved by Budak et al. in [33].

**Remark 18.** In Corollary 5, if we set \( p = 1 \), then Corollary 5 reduces to [36, Theorem 4.3].

**Remark 19.** In Corollary 5, if we set \( p = 1 \), then we obtain the following inequality:

\[
\left| f(x) - \frac{1}{\pi_2} \int_0^x f(t) \pi_1 d_{q} t + \int_x^{\pi_2} f(t) \pi_2 d_{q} t \right|
\]

\[
\leq \frac{q}{\pi_2} \left( \frac{1}{[1 + 1]_{q} \pi_2} \right)^\frac{1}{2} \left( x - \pi_1 \right)^\frac{1}{2} \left( \left| \pi_2 D_{q} f(x) \right| \pi_1 + \left| \pi_2 D_{q} f(\pi_2) \right| \pi_1 \right) \right)^\frac{1}{2} + \left( \pi_2 - x \right)^\frac{1}{2} \left( \frac{\left| \pi_2 D_{q} f(x) \right| \pi_1 + \left| \pi_2 D_{q} f(\pi_2) \right| \pi_1 }{[2]_{q} \pi_1} \right)^\frac{1}{2},
\]

which is proved by Budak et al. in [33].

\[\Box\]

**Corollary 6.** If we assume \( |\pi_2 D_{p,q} f(x)|, |\pi_1 D_{p,q} f(x)| \leq M \) in Theorem 10, then we have following post-quantum Ostrowski type inequality for \( s \)-convex functions in the second sense:

\[
\left| f(x) - \frac{1}{p(\pi_2 - \pi_1)} \left[ \int_0^x f(t) \pi_1 d_{p,q} t + \int_x^{\pi_2} f(t) \pi_2 d_{p,q} t \right] \right|
\]

\[
\leq \frac{M q}{\pi_2 - \pi_1} \left( \frac{1}{[1 + 1]_{p,q} \pi_2} \right)^\frac{1}{2} \left( \frac{2}{[s + 1]_{p,q} \pi_2} \right)^\frac{1}{2} \left( x - \pi_1 \right)^2 + \left( \pi_2 - x \right)^2 \right].
\]
(38)

**Remark 20.** In Corollary 6, if we set \( p = 1 \), then Corollary 6 reduces to [36, Corollary 4.3].

**Remark 21.** In Corollary 6, if we set \( p = 1 \) and later take the limit as \( q \to 1 \), then Corollary 6 reduces to [38, Theorem 3].
6 Applications to special means

For arbitrary positive numbers \( \pi_1, \pi_2 (\pi_1 \neq \pi_2) \), we consider the means as follows:
1. The arithmetic mean

\[
\mathcal{A} = \mathcal{A}(\pi_1, \pi_2) = \frac{\pi_1 + \pi_2}{2}.
\]

2. The logarithmic mean

\[
\mathcal{L}_0^\sigma = \mathcal{L}_0^\sigma(\pi_1, \pi_2) = \frac{\pi_1^{\sigma + 1} - \pi_2^{\sigma + 1}}{(\sigma + 1)(\pi_2 - \pi_1)}.
\]

**Proposition 1.** For \( 0 < \pi_1 < \pi_2 \) and \( 0 < q < p \leq 1 \), the following inequality is true:

\[
\frac{1}{s + 1}[\mathcal{A}^{s+1}(\pi_1, \pi_2) - \mathcal{A}(\kappa_1, \kappa_2)] \leq \frac{q(\pi_2 - \pi_1)}{2} \left[ \frac{1}{[s + 2]_{p,q}} \left( \mathcal{L}_0^\sigma \left( \frac{\pi_2 - \pi_1}{2} + \pi_1, p \left( \frac{\pi_2 - \pi_1}{2} + \pi_1 \right) \right) + \pi_1 \right) + \mathcal{L}_0^\sigma \left( \pi_2 - q \left( \frac{\pi_2 - \pi_1}{2} \right), \pi_2 - p \left( \frac{\pi_2 - \pi_1}{2} \right) \right) \right] + \Theta_1 \mathcal{A}(\pi_1^q, \pi_2^q) + \Theta_2 \mathcal{A}(\pi_1^p, \pi_2^p),
\]

where

\[
\kappa_1 = (p - q) \sum_{n=0}^{\infty} \frac{q^n}{p^n} \left( \frac{\pi_2 - \pi_1}{2} + \pi_1 \right)^{n+1},
\]

\[
\kappa_2 = (p - q) \sum_{n=0}^{\infty} \frac{q^n}{p^n} \left( \frac{\pi_2 - \pi_1}{2} \right)^{n+1}.
\]

**Proof.** The inequality (25) in Theorem 8 with \( x = \frac{\pi_1 + \pi_2}{2} \) for \( f(x) = \frac{x^{s+1}}{s+1} \), where \( x > 0 \) leads to this conclusion.

**Proposition 2.** For \( 0 < \pi_1 < \pi_2 \) and \( 0 < q < p \leq 1 \), the following inequality is true:

\[
\frac{1}{s + 1}[\mathcal{A}^{s+1}(\pi_1, \pi_2) - \mathcal{A}(\kappa_1, \kappa_2)] \leq \frac{Mq(\pi_2 - \pi_1)}{4} \left[ \frac{2}{[s + 2]_{p,q}} + \Theta_1 + \Theta_2 \right].
\]

**Proof.** The inequality (29) in Corollary 2 with \( x = \frac{\pi_1 + \pi_2}{2} \) for \( f(x) = \frac{x^{s+1}}{s+1} \), where \( x > 0 \) leads to this conclusion.

**Proposition 3.** For \( 0 < \pi_1 < \pi_2 \) and \( 0 < q < p \leq 1 \), the following inequality is true:

\[
\frac{1}{s + 1}[\mathcal{A}^{s+1}(\pi_1, \pi_2) - \mathcal{A}(\kappa_1, \kappa_2)] \leq \frac{q(\pi_2 - \pi_1)}{2} \left[ \frac{1}{[s + 2]_{p,q}} \left( \mathcal{L}_0^\sigma \left( \frac{\pi_2 - \pi_1}{2} + \pi_1, p \left( \frac{\pi_2 - \pi_1}{2} + \pi_1 \right) \right) + \pi_1 \right) + \Theta_1 |\pi_1|^p \right]^{\frac{1}{p}}
\]

\[
+ \left( \frac{1}{[s + 2]_{p,q}} \left| \mathcal{L}_0^\sigma (\pi_2 - q \left( \frac{\pi_2 - \pi_1}{2} \right), \pi_2 - p \left( \frac{\pi_2 - \pi_1}{2} \right)) \right|^{\frac{1}{p}} \right]^{\frac{1}{p}} + \Theta_2 |\pi_2|^p \right].
\]

**Proof.** The inequality (30) in Theorem 9 with \( x = \frac{\pi_1 + \pi_2}{2} \) for \( f(x) = \frac{x^{s+1}}{s+1} \), where \( x > 0 \) leads to this conclusion.
Proposition 4. For $0 < \pi_1 < \pi_2$ and $0 < q < p \leq 1$, the following inequality is true:

$$\frac{1}{s+1} \left[ A^{s+1}(\pi_1, \pi_2) - A(k_1, k_2) \right] \leq \frac{q(\pi_2 - \pi_1)}{2} \left( \frac{1}{|n_1 + 1|_{p,q}} \right)^{1/s} \left( \frac{1}{|s + 1|_{p,q}} \right)^{1/s} \left[ \sum_{i=1}^{p} \left( p_i \left( \frac{\pi_2 - \pi_1}{2} \right) + \pi_i \right) + \left( \pi_i^{1/p_i} \right)^{1/s} \right].$$

Proof. The inequality (34) in Theorem 10 with $x = \frac{n_1 + n_2}{2}$ for $f(x) = \frac{x^{p+1}}{s+1}$, where $x > 0$ leads to this conclusion. \(\square\)

Proposition 5. For $0 < \pi_1 < \pi_2$ and $0 < q < p \leq 1$, the following inequality is true:

$$\frac{1}{s+1} \left[ A^{s+1}(\pi_1, \pi_2) - A(k_1, k_2) \right] \leq \frac{Mq(\pi_2 - \pi_1)}{2} \left( \frac{1}{|n_1 + 1|_{p,q}} \right)^{1/s} \left( \frac{2}{|s + 1|_{p,q}} \right)^{1/s}.$$

Proof. The inequality (38) in Corollary 6 with $x = \frac{n_1 + n_2}{2}$ for $f(x) = \frac{x^{p+1}}{s+1}$, where $x > 0$ leads to this conclusion. \(\square\)

7 Conclusion

In this work, we proved some new variants of post-quantum Hermite-Hadamard and Ostrowski type inequalities using the $(p, q)$-differentiable $s$-convex functions in the second sense. We also proved that the newly established results are strong generalizations of the related existing results. Finally, we presented various applications based on the newly established inequalities to demonstrate the utility of our findings. It is a new and interesting problem that upcoming researchers can obtain similar inequalities for different kinds of convexity in their future work.

Acknowledgements: The authors are thankful to the Nanjing Normal University for wonderful research environment provided to the researchers.

Funding information: The work was supported by Philosophy and Social Sciences of Educational Commission of Hubei Province of China (20Y109), and Foundation of Hubei Normal University (2021JSKCSZY06, 2021056). This work was also supported by King Mongkut’s University of Technology North Bangkok (Contract no. KMUTNB-63-KNOW-021).

Author contributions: All authors contributed equally to the writing of this article. All authors read and approved the final manuscript.

Conflict of interest: The authors declare that they do not have any conflict of interests.

Data availability statement: Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.
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