Research Article

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Generalized Munn rings

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Abstract: Generalized Munn rings exist extensively in the theory of rings. The aim of this note is to answer when a generalized Munn ring is primitive (semiprimitive, semiprime and prime, respectively). Sufficient and necessary conditions are obtained for a generalized Munn ring with a regular sandwich matrix to be primitive (semiprimitive, semiprime and prime, respectively). Also, we obtain sufficient and necessary conditions for a Munn ring over principal ideal domains to be prime (semiprime, respectively). Our results can be regarded as the generalizations of the famous result in the theory of rings that for a ring $R$, $R$ is primitive (semiprimitive and semiprime, respectively) if and only if so is $M_n(R)$. As applications of our results, we consider the primeness and the primitivity of generalized matrix rings and generalized path algebras. In particular, it is proved that a path algebra is a semiprime if and only if it is semiprimitive.

Keywords: generalized Munn ring, generalized path algebra, semiprimitive ring, semiprime ring, generalized matrix ring

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1 Introduction

Throughout this note, we shall use the standard notions and notations, and each of the considered rings is associative but has possibly no identity.

The class of generalized matrix rings has been extensively studied. Examples of generalized matrix rings include piecewise domains (see [1]), incidence algebras of directed graphs (see, [2,3]), structural matrix rings (see [4] and subsequent papers), endomorphism rings, and Morita context rings. Sands [5] observed that if $[S, V, W, T]$ is a Morita context, then

$$
\begin{pmatrix}
    S & V \\
    W & T
\end{pmatrix}
$$

is a ring. These Morita context rings are precisely generalized matrix rings with idempotent sets $E$ such that $|E| = 2$, and they have been widely studied. In particular, we note Amitsur’s paper [6], the survey paper [7], McConnell and Robson’s treatment [8] and Müller’s computation of the maximal quotient ring [9]. Indeed, cellular algebras, affine cellular algebras and standardly based algebras there exist some “local” structures of generalized Munn algebras (for example, see [10–12]).

Brown [13] considered generalized matrix algebras of finite dimension over a field of characteristic 0. He proved that such a generalized matrix algebra is either simple or nonsemisimple and simple modulo its radical, and it is simple if and only if it possesses an identity. Sands [5] gave the prime radical of generalized matrix rings with a finite idempotent set. Zhang [14] considered the prime radical of the general case.
In 1989, Wauters and Jespers [15] determined when a generalized matrix ring with a finite idempotent set is semiprime. Classical quotient rings of generalized matrix rings with finite idempotent sets had been attracting due attention. There are a series of papers on this field (see [16–18]). Indeed, any generalized matrix ring can be viewed as a subring of some generalized Munn ring. It is natural to research generalized Munn rings. Li [19] considered the regularity of Munn rings. The main aim of this study is to answer when a generalized Munn ring is semiprimitive (semiprime and prime, respectively). A sufficient and necessary condition is established for a generalized Munn ring with a regular sandwich matrix to be primitive (semiprimitive, semiprime and prime, respectively) (Theorems 2.5 and 2.7). Moreover, we answer when a Munn ring over a principal ideal domain is semiprime (prime, respectively) (Proposition 2.11). In Section 3, we determine when a generalized matrix ring is primitive (semiprimitive, semiprime and prime, respectively) (Theorems 3.2 and 3.3). Finally, we consider the primeness and the primitivity of generalized Munn rings. It is proved that for a quiver $Q$ and a field $K$, the path algebra $K(Q)$ is semiprime if and only if the path-connected quiver of $Q^{PC}$ is the disjoint union of complete quivers; if and only if $K(Q)$ is semiprimitive (Theorem 4.9). And, $K(Q)$ is prime if and only if $Q^{PC}$ is a complete quiver (Theorem 4.10).

2 Generalized Munn rings

The aim of this section is to consider the primeness and the primitivity of generalized Munn rings.

2.1 Regular sandwich matrices

To begin with, we give the notion of regular matrices.

**Definition 2.1.** Let $A$ be a ring and $X = (x_{im})$ be a generalized $I \times M$ matrix over $A$. A nonzero entry $x_{im}$ of $X$ is called a unit entry of $X$ if there exists a nonzero idempotent $e \in A$ such that $x_{im}$ is a unit in $eAe$.

Notice that a group has exactly one idempotent, which is just the identity of the group. This means that in Definition 2.1, the idempotent $e$ is indeed unique. So, the unique idempotent $e$ in Definition 2.1 is denoted by $x_{im}^{\circ}$.

**Definition 2.2.** Let $A$ be a ring. An $I \times M$ matrix $X = (x_{im})$ over $A$ is said regular in $A$ if the following conditions hold:

(RM1) For any $i \in I$, there exists $m \in M$ such that $x_{im}$ is a unit entry of $X$.

(RM2) For any $n \in M$, there exists $j \in I$ such that $x_{jn}$ is a unit entry of $X$.

(RM3) If $x_{im_{0}}$ is a unit entry of $X$, then

(i) $x_{im_{0}}x_{jm} = x_{jm}$ for any $m \in M$;

(ii) $x_{im}x_{im_{0}} = x_{im}$ for any $i \in I$.

By definition, any $m \times m$ matrix without zero rows and zero columns are regular in the field $C$ of complex numbers. Also, for a ring $A$ with unity, any $I \times M$ matrix over $A$, in which each row and each column contains at least one unit of $A$, must be regular in $A$.

Let $M, I$ be nonempty sets, and $A$ an associative ring and $Q = (q_{im})$ a generalized $M \times I$ matrix over $A$. Consider the set $\mathfrak{N}(A, I, M)$ consisting of all generalized $I \times M$ matrices over $A$ with only finite nonzero entries, such an $I \times M$ matrix is usually said to be bounded. For $C = (a_{im}), D = (b_{im}) \in \mathfrak{N}(A, I, M)$, define

$C + D = (c_{im} + d_{im})$, where $c_{im} = a_{im} + d_{im}$ for $i \in I, m \in M$;

$C \cdot D = CQD$, where the product on the right side is the product of matrices;

$\lambda C = (\lambda a_{im})$ for $\lambda \in R$.

By definition, a routine calculation shows that with these operations, $\mathfrak{N}(A, I, M)$ is an associative ring.
**Definition 2.3.** The above ring \( \mathfrak{M}(\mathcal{A}, I, M) \) is called a generalized Munn ring \( \mathcal{A} \) with the sandwich matrix \( Q \), in notation, \( \mathfrak{M}(\mathcal{A}, I, M; Q) \).

If \( I \) is finite, then we identity it with the set \( \{1, 2, \ldots, i\} \), where \( i \) is the cardinality of \( I \), and we write \( \mathfrak{M}(\mathcal{A}, I, M; Q) \) as \( \mathfrak{M}(\mathcal{A}, i, M; Q) \). Similarly, the notation \( \mathfrak{M}(\mathcal{A}, I, m; Q) \) is used if \( |M| = m < \infty \). Denote by \( M_{m,d}(\mathcal{A}) \) the set of all \( m \times n \) matrices over \( \mathcal{A} \).

Recall from [20] that the generalized Munn ring \( \mathfrak{M}(\mathcal{A}, m, n; Q) \) is called the Munn matrix ring over \( \mathcal{A} \) with sandwich matrix \( Q \). It is obvious that \( M_{n}(\mathcal{A}) \) is the Munn \( n \times n \) matrix ring over \( \mathcal{A} \) with sandwich matrix \( \Delta \), where \( \Delta \) is the unit matrix; that is, the diagonal matrix each of whose entries in the diagonal positions is the unity of \( \mathcal{A} \).

**Definition 2.4.** Let \( T \) be a nonempty subset of \( \mathfrak{M}(\mathcal{A}, I, M) \). An \( M \times I \) matrix \( X = (x_{mi}) \) over \( \mathcal{A} \) is said to be cancellable in \( T \) if for any nonzero element \( Y \in T \), \( YX \) and \( XY \) are neither zero, where \( YX \) and \( XY \) are usual matrix products.

Evidently, for a ring \( \mathcal{A} \) with identity 1, the \( I \times I \) unit matrix \( \Delta \) is cancellable in any subset of \( \mathfrak{M}(\mathcal{A}, I, I) \). And, any invertible \( n \times n \) matrix must be cancellable in \( M_{n}(\mathcal{A}) \), but not all of cancellable matrices in \( M_{n}(\mathcal{A}) \) are invertible in the matrix algebra.

**Example.** Let \( \mathcal{Z} \) be the ring of integers. It is easy to check that the matrix

\[
A = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}
\]

is cancellable in the matrix ring \( M_{2}(\mathcal{Z}) \). But \( A \) is not invertible in \( M_{2}(\mathcal{Z}) \).

For convenience, we denote

\[
(a)_{im}: \text{the generalized } I \times M \text{ matrix with } a \text{ in the } (i, m) \text{ position and } 0 \text{ elsewhere};
\]

\[
(B)_{im}: \text{the set } \{(b)_{im}: b \in B\} \text{ for } B \subseteq \mathcal{A};
\]

\[
(a_{im})_{i \in I, m \in N}: \text{the generalized } I \times M \text{ matrix with } a_{im} \text{ in the } (i, m) \text{ position for } i \in I,
\]

\[
m \in N \text{ and } 0 \text{ elsewhere. Especially, if } N = \{n\}, \text{ we simply write } m \in N
\]

as \( m = n \), and the similar sign for the case: \( |I| = 1 \).

**Theorem 2.5.** Let \( \mathfrak{m} = \mathfrak{M}(\mathcal{A}, I, M; Q) \) be a generalized Munn ring. If \( Q \) is regular in \( \mathcal{A} \), then \( \mathfrak{M}(\mathcal{A}, I, M; Q) \) is semiprime (semiprimitive, respectively) if and only if the following conditions are satisfied:

(i) \( Q \) is cancellable in \( \mathfrak{M}(\mathcal{A}, I, M) \);

(ii) for any unit entries \( q_{mi}, q_{nj} \) of \( Q \), if \( x \) is a nonzero element of \( q_{mi}^{-1} \mathcal{A} q_{nj}^{-1} \), then \( q_{mi}^{-1} \mathcal{A} q_{nj}^{-1} x \neq 0 \) and \( xq_{nj}^{-1} \mathcal{A} q_{mi}^{-1} \neq 0 \);

(iii) for any unit entry \( q_{mi} \) of \( Q \), \( q_{mi}^{-1} \mathcal{A} q_{mi}^{-1} \) is semiprime (semiprimitive, respectively).

**Proof.** Let \( q_{mi} \) be a unit entry of \( Q \) and denote by \( q_{mi}^{-1} \) the inverse of \( q_{mi} \) in \( q_{mi}^{-1} \mathcal{A} q_{mi}^{-1} \). Obviously \( q_{mi}^{-1} = q_{mi} q_{mi}^{-1} = q_{mi}^{-1} q_{mi} \). Then \( (q_{mi}^{-1})_{im} \) is an idempotent of \( \mathfrak{m} \), and

\[
(q_{mi}^{-1})_{im} \mathfrak{m} \ast (q_{mi}^{-1})_{im} = (q_{mi}^{-1} \mathcal{A} q_{mi}^{-1})_{im}
\]

(2.1)

since \( q_{mi} \) is a unit in \( q_{mi}^{-1} \mathcal{A} q_{mi}^{-1} ). A routine calculation shows that the mapping

\[
\phi: \mathfrak{m} \to (q_{mi}^{-1} \mathcal{A} q_{mi}^{-1})_{im}
\]

\[\phi: (x)_{mi} \mapsto xq_{mi}^{-1} \text{ is an isomorphism from } (q_{mi}^{-1} \mathcal{A} q_{mi}^{-1})_{im} \text{ onto } q_{mi}^{-1} \mathcal{A} q_{mi}^{-1}. \]

\[(2.2.1) \text{ The proof for the semiprime case. If } \mathfrak{m} \text{ is semiprime, then by (2.1), } (q_{mi}^{-1} \mathcal{A} q_{mi}^{-1})_{im} \text{ is semiprime, so that by } \phi \text{ is an isomorphism, } q_{mi}^{-1} \mathcal{A} q_{mi}^{-1} \text{ is semiprime. It results (iii). To see (i), assume on the contrary that } Q \text{ is not cancellable in } \mathfrak{M}(\mathcal{A}, I, M), \text{ then there is a nonzero element } X \in \mathfrak{m} \text{ such that } XQ = 0 \text{ or } QX = 0. \] Without
loss of generality, let \( XQ = 0 \), so that \( X \circ (U \circ X) = XQUQX = 0 \) for any \( U \in \mathfrak{M} \), whence \( X \circ \mathfrak{M} \circ X = 0 \), contrary to that \( \mathfrak{M} \) is semiprime. Thus, \( Q \) is cancellable in \( \mathfrak{M}(\mathfrak{A}, I, M) \).

We next verify (ii). To the end, we assume contrariwise that there exists a nonzero element \( x \in q_{m_i}^\mathfrak{A} q_{m_j}^\mathfrak{A} \) such that \( xq_{m_i}^\mathfrak{A} \mathfrak{A} q_{m_j}^\mathfrak{A} = 0 \). Obviously, \( xq_{m_j}^\mathfrak{A} = x = q_{m_j}^m x \), so that

\[
(x)_{im} \circ \mathfrak{M} \circ (x)_{im} \subseteq (x \mathfrak{A} x)_{im} = (xq_{m_i}^\mathfrak{A} \mathfrak{A} q_{m_j}^\mathfrak{A} x)_{im} = 0,
\]

contrary to the hypothesis that \( \mathfrak{M} \) is semiprime. Therefore, \( xq_{m_j}^\mathfrak{A} \mathfrak{A} q_{m_i}^\mathfrak{A} \neq 0 \). Similarly, \( q_{m_j}^\mathfrak{A} \mathfrak{A} q_{m_i}^\mathfrak{A} x \neq 0 \). We have now proved that (ii) is valid.

For the converse, we contrariwise let \( w = (w_{im}) \) be a nonzero generalized \( I \times M \) matrix in \( \mathfrak{M} \) such that \( w \circ \mathfrak{M} \circ w = 0 \). Because \( Q \) is cancellable, \( QwQ = (u_{im})_{\text{mct}} \neq 0 \), and we assume that \( u_{m_0} \neq 0 \).

If \( wQ = (v_{ij})_{i,j \in I} \), then

\[
u_{m_0} = \sum_{j \in I} q_{m_j}^\mathfrak{A} v_{ij} j_{i_0}.
\]

When \( Q \) is regular, there is \( j_0 \in I \) such that \( q_{m_0}^\mathfrak{A} j_{j_0} \) is a unit entry of \( Q \). It follows that \( q_{m_0}^\mathfrak{A} q_{m_0} = q_{m_0} \). Now by (2.3),

\[
u_{m_0} = \sum_{j \in I} q_{m_0}^\mathfrak{A} q_{m_0} v_{ij} j_{i_0} = q_{m_0}^\mathfrak{A} \left( \sum_{j \in I} q_{m_j} v_{ij} j_{i_0} \right) = q_{m_0}^\mathfrak{A} u_{m_0} j_{i_0},
\]

and similarly, there exists a unit entry \( q_{m_0}^\mathfrak{A} \) of \( Q \) such that \( u_{m_0} q_{m_0}^\mathfrak{A} = u_{m_0} \). Therefore, \( u_{m_0} \in q_{m_0}^\mathfrak{A} \mathfrak{A} q_{m_0}^\mathfrak{A} \). Furthermore, by (ii), there is \( x \in q_{m_0}^\mathfrak{A} \mathfrak{A} q_{m_0}^\mathfrak{A} \) such that \( 0 \neq u_{m_0} x \in q_{m_0}^\mathfrak{A} \mathfrak{A} q_{m_0}^\mathfrak{A} \). Clearly,

\[
q_{m_0}^\mathfrak{A} u_{m_0} x = u_{m_0} x = (u_{m_0} x) q_{m_0}^\mathfrak{A}.
\]

It follows that

\[
u = (q_{m_0}^\mathfrak{A})_{j \in m_0} QwQ(x)_{j \in m_0} = (q_{m_0}^\mathfrak{A})_{j \in m_0} u_{m_0} j_{i_0} (x)_{j \in m_0} = (q_{m_0}^\mathfrak{A} u_{m_0} x)_{j \in m_0} = (q_{m_0}^\mathfrak{A} j_{j_0})_{j \in m_0} \circ w \circ (x)_{j \in m_0} \neq 0.
\]

Now

\[
(u_{m_0} x) q_{m_0}^\mathfrak{A} \mathfrak{A} q_{m_0}^\mathfrak{A} \cdot u_{m_0} x)_{j \in m_0} = (u_{m_0} x q_{m_0}^\mathfrak{A} j_{j_0} \cdot \mathfrak{A} \cdot q_{m_0}^\mathfrak{A} u_{m_0} x)_{j \in m_0} = (u_{m_0} x)_{j \in m_0} Q q_{m_0}^\mathfrak{A} \mathfrak{A} q_{m_0}^\mathfrak{A} Q^{-1} u_{m_0} x)_{j \in m_0} \subseteq (u_{m_0} x)_{j \in m_0} Q \mathfrak{M} Q (u_{m_0} x)_{j \in m_0} = (u_{m_0} x)_{j \in m_0} \circ \mathfrak{M} \circ (u_{m_0} x)_{j \in m_0} = (q_{m_0}^\mathfrak{A} j_{j_0} \circ w \circ (x)_{j \in m_0}) \circ \mathfrak{M} \circ (q_{m_0}^\mathfrak{A} j_{j_0} \circ w \circ (x)_{j \in m_0}) \subseteq (q_{m_0}^\mathfrak{A} j_{j_0} \circ (w \circ \mathfrak{M} \circ w) \circ (x)_{j \in m_0}) = 0,
\]

so that \( u_{m_0} x \cdot q_{m_0}^\mathfrak{A} \mathfrak{A} q_{m_0}^\mathfrak{A} \cdot u_{m_0} x = 0 \). This is contrary to that \( q_{m_0}^\mathfrak{A} \mathfrak{A} q_{m_0}^\mathfrak{A} \) is semiprime. Consequently, \( \mathfrak{M} \) is semiprime.

(2.5.2) The proof for the semiprimitive case. It is well known that for a semiprimitive algebra \( \mathfrak{A} \) and an idempotent \( e \in \mathfrak{A} \), \( e \mathfrak{A} e \) is still semiprimitive. So, the same reason as in (2.5.1) shows that the “if” part is valid. With notations in (2.5.1), if \( w \in \text{rad}(\mathfrak{M}) \setminus \{0\} \), then by (2.5), \( 0 \neq (u_{m_0} x)_{j \in m_0} \in \text{rad}(\mathfrak{M}) \). Notice that \( (q_{m_0}^\mathfrak{A} j_{j_0})_{j \in m_0} \) is an idempotent of \( \mathfrak{M} \), we can obtain that
\[(u_{m_{0},k_{0}}x)_{m_{0},m_{0}} = (q_{m_{0},k_{0}}^{\omega}u_{m_{0},k_{0}}xq_{m_{0},k_{0}}^{\omega})_{m_{0},m_{0}} \quad \text{(by (2.3))} \]
\[= (q_{m_{0},k_{0}}^{-1})_{m_{0},m_{0}} \circ (u_{m_{0},k_{0}}x)_{m_{0},m_{0}} \circ (q_{m_{0},k_{0}}^{-1})_{m_{0},m_{0}} \]
\[\in (q_{m_{0},k_{0}}^{-1})_{m_{0},m_{0}} \circ \text{rad}(\mathfrak{m}) \circ (q_{m_{0},k_{0}}^{-1})_{m_{0},m_{0}} \]
\[= \text{rad}((q_{m_{0},k_{0}}^{-1})_{m_{0},m_{0}} \circ \mathfrak{m} \circ (q_{m_{0},k_{0}}^{-1})_{m_{0},m_{0}}) \]
\[= \text{rad}((q_{m_{0},k_{0}}^{\omega}q_{m_{0},k_{0}}^{\omega})_{m_{0},m_{0}}). \]

This means that \((q_{m_{0},k_{0}}^{\omega}q_{m_{0},k_{0}}^{\omega})_{m_{0},m_{0}}\) is not semiprimitive. Now by (2.2), we observe that \(q_{m_{0},k_{0}}^{\omega}q_{m_{0},k_{0}}^{\omega}\) is not semiprimitive. This is contrary to the hypothesis. It results the “only if” part. \(\square\)

**Remark 2.6.** Let us turn back to the proof of Theorem 2.5. In (2.5.1), the proof of Condition (i) in the direct part has indeed proved that if \(\mathfrak{m}(\mathcal{A}, I, M; Q)\) is semiprime, then \(Q\) is cancellable in \(\mathfrak{m}(\mathcal{A}, I, M)\).

**Theorem 2.7.** Let \(\mathfrak{m} = \mathfrak{m}(\mathcal{A}, I, M; Q)\) be a generalized Munn ring. If \(Q\) is regular in \(\mathcal{A}\), then \(\mathfrak{m}(\mathcal{A}, I, M; Q)\) is prime (primitive, respectively) if and only if the following conditions are satisfied:

(i) \(Q\) is cancellable in \(\mathfrak{m}(\mathcal{A}, I, M)\);

(ii) for any unit entries \(q_{m_{i}}\), \(q_{n_{j}}\), \(q_{k_{q}}\), \(q_{l_{t}}\) of \(Q\), if \(x\) and \(y\) are nonzero elements of \(q_{m_{i}}^{\omega}\mathcal{A}q_{n_{j}}^{\omega}\) and \(q_{k_{q}}^{\omega}\mathcal{A}q_{l_{t}}^{\omega}\), respectively, then \(q_{m_{i}}^{\omega}\mathcal{A}q_{n_{j}}^{\omega}y \neq 0\); and

(iii) for any unit entry \(q_{m_{i}}\) of \(Q\), \(q_{m_{i}}^{\omega}\mathcal{A}q_{m_{i}}^{\omega}\) is prime (primitive, respectively).

**Proof.** (2.7.1) The proof for the prime case. If \(\mathfrak{m}\) is prime, then by (2.1), \((q_{m_{i}}^{\omega}\mathcal{A}q_{m_{i}}^{\omega})_{\mathfrak{m}}\) is prime, so that by \(\phi\) is an isomorphism, \(q_{m_{i}}^{\omega}\mathcal{A}q_{m_{i}}^{\omega}\) is prime. By Remark 2.6, \(Q\) is cancellable in \(\mathfrak{m}(\mathcal{A}, I, M)\).

We next verify (ii). We contrariwise let \(x \in q_{m_{i}}^{\omega}\mathcal{A}q_{n_{j}}^{\omega}\setminus\{0\}, y \in q_{k_{q}}^{\omega}\mathcal{A}q_{l_{t}}^{\omega}\setminus\{0\}\) such that \(q_{m_{i}}^{\omega}\mathcal{A}q_{n_{j}}^{\omega}y = 0\). Then, \(q_{m_{i}}^{\omega} = x\) and \(q_{n_{j}}^{\omega}y = y\). Moreover,
\[(x)_{m_{i}} \circ \mathfrak{m} \circ (y)_{m_{i}} = (x \cdot \mathcal{A}y)_{m_{i}} = (x \cdot q_{m_{i}}^{\omega}\mathcal{A}q_{n_{j}}^{\omega} \cdot y)_{m_{i}} = (0)_{m_{i}} = 0.\]

It is contrary to the hypothesis that \(\mathfrak{m}\) is prime. We have now proved the necessity.

To see the converse part, we assume conversely that there exist nonzero elements \(A, B \in \mathfrak{m}\) such that \(A \circ \mathfrak{m} \circ B = 0\). It is not difficult to see that Condition (ii) in Theorem 2.7 implies Condition (ii) in Theorem 2.5. Indeed, by (ii), for any \(x \in q_{m_{i}}^{\omega}\mathcal{A}q_{n_{j}}^{\omega}\), \(q_{m_{i}}^{\omega}\mathcal{A}q_{n_{j}}^{\omega}x \neq 0\), so that \(q_{m_{i}}^{\omega}\mathcal{A}q_{n_{j}}^{\omega} \neq 0\); similarly, \(q_{k_{q}}^{\omega}\mathcal{A}q_{l_{t}}^{\omega} \neq 0\), and it results immediately in Condition (ii) in Theorem 2.5. Now by (2.5.1) (precisely, see (2.4) and (2.5)), there are \(m, n \in M, j, k \in I\) and \(C_{1}, C_{2}, D_{1}, D_{2} \in \mathfrak{m}\) such that

(a) \(q_{m_{i}}, q_{n_{j}}\) are unit entries of \(Q\); and

(b) \(C_{1} \circ A \circ D_{1} = (a)_{m_{i}}\) and \(C_{2} \circ B \circ D_{2} = (b)_{k_{q}}\), where \(a\) and \(b\) are nonzero elements in \(q_{m_{i}}^{\omega}\mathcal{A}q_{m_{i}}^{\omega}\) and \(q_{k_{q}}^{\omega}\mathcal{A}q_{l_{t}}^{\omega}\), respectively.

Furthermore by (ii), we have \(u \in q_{m_{i}}^{\omega}\mathcal{A}q_{m_{i}}^{\omega}\) such that \(aub \neq 0\). Obviously, \(q_{m_{i}}^{\omega}u = u = auq_{m_{i}}^{\omega}\).

Compute
\[(aub)_{m_{i}} \circ \mathfrak{m} \circ (aub)_{m_{i}} = (q_{m_{i}}^{\omega}q_{m_{i}}^{\omega}u)_{m_{i}} \circ \mathfrak{m} \circ (auq_{m_{i}}^{\omega}b)_{m_{i}} \]
\[= (u_{m_{i},k_{i}}q_{m_{i},k_{i}}^{\omega}ub)_{m_{i}} \circ \mathfrak{m} \circ (auq_{m_{i}}^{\omega}b)_{m_{i}} \]
\[= (a)_{m_{i}} \circ (q_{m_{i}}^{\omega}ub)_{m_{i}} \circ \mathfrak{m} \circ (auq_{m_{i}}^{\omega}b)_{m_{i}} \]
\[\subseteq (a)_{m_{i}} \circ \mathfrak{m} \circ (b)_{k_{q}} \]
\[= C_{1} \circ A \circ D_{1} \circ \mathfrak{m} \circ C_{2} \circ B \circ D_{2} \]
\[\subseteq C_{1} \circ A \circ \mathfrak{m} \circ B \circ D_{2} \]
\[= 0. \quad (2.6)\]
This shows that \( \mathcal{M} \) is not a semiprime ring. But by Theorem 2.5, \( \mathcal{M} \) is semiprime. It is a contradiction. Therefore, \( \mathcal{M} \) is prime.

(2.7.2) The proof for the primitive case. By the well-known result (for example, see [21, Ex. 10, p. 339]): for any primitive algebra \( \mathcal{A} \) and any idempotent \( e \) in \( \mathcal{A} \), \( e\mathcal{A}e \) is still primitive, and since any primitive algebra is prime, a similar argument as in (2.5.1) can verify the “if” part. For the converse, if given conditions hold, then by (2.7.1), \( \mathcal{M} \) is prime. The rest follows from a famous result of Lanahi et al. [22] showed that for a prime ring \( R \), if \( e \) is a nonzero idempotent in \( R \), then \( R \) is primitive if and only if \( eRe \) is primitive.

Based on Theorems 2.5 and 2.7, we may prove the following proposition.

**Proposition 2.8.** Let \( \mathcal{M}(\mathcal{A}, I, M; Q) \) be a generalized Munn ring. Assume that

1. \( \mathcal{A} \) has a unity;
2. each row and each column of \( Q \) contains at least one unit of \( \mathcal{A} \).

Then \( \mathcal{M}(\mathcal{A}, I, M; Q) \) is prime (semiprime, primitive and semiprimitive, respectively) if and only if the following conditions are satisfied:

(i) \( Q \) is cancellable in \( \mathcal{M}(\mathcal{A}, I, I) \);
(ii) \( \mathcal{A} \) is prime (semiprime, primitive and semiprimitive, respectively).

**Proof.** By definition, \( q_{mi}^\diamond \) is the unity of \( \mathcal{A} \) for any unit entry \( q_{mi} \) of \( Q \) satisfying Condition (2); in this case, \( q_{mi}^\diamond \mathcal{A}q_{mi}^\diamond = \mathcal{A} \). Obviously, \( Q \) is regular in \( \mathcal{A} \).

Let \( q_{mi} \) be an arbitrary unit entry of \( Q \). By Condition (2), there is an entry \( q_{mi0} \) of \( Q \) such that \( q_{mi0}^\diamond \) is a unit in \( \mathcal{A} \). But \( Q \) is regular in \( \mathcal{A} \), so \( q_{mi}^\diamond q_{mi0} = q_{mi0}^\diamond \), and it follows that \( q_{mi}^\diamond \) must be the unity of \( \mathcal{A} \). We have now proved that any unit entry of \( Q \) is a unit in \( \mathcal{A} \). This shows that Condition (ii) in Theorem 2.5 is satisfied and that Condition (ii) in Theorem 2.7 is satisfied whenever \( \mathcal{A} \) is prime.

The rest follows immediately from Theorems 2.5 and 2.7.

For a ring \( \mathcal{A} \) with unity, denote by \( \Delta \) the generalized \( I \times I \) matrix over \( \mathcal{A} \) each of whose entries in the diagonal positions is the unity of \( \mathcal{A} \) and 0 elsewhere. Obviously, \( \Delta \) is cancellable in \( \mathcal{M}(\mathcal{A}, I, I) \). It is easy to see that the following corollary is an easy consequence of Proposition 2.8.

**Corollary 2.9.** Let \( \mathcal{A} \) be a ring with unity. Then \( \mathcal{A} \) is prime (semiprime, primitive and semiprimitive, respectively) if and only if for any [for some] nonempty set \( I \), \( \mathcal{M}(\mathcal{A}, I, I; \Delta) \) is prime (semiprime, primitive and semiprimitive, respectively).

Let us turn back to the proof of Theorem 2.7. Assume now that \( \mathcal{M} \) is semiprime and the condition:

**PM** If \( q_{mj}, q_{ak} \) are unit entries of \( Q \), then \( aq_{mj}^\diamond \mathcal{A}q_{ak}^\diamond b \neq 0 \) for any nonzero elements \( a \in q_{mj}^\diamond \mathcal{A}q_{mj}^\diamond, b \in q_{ak}^\diamond \mathcal{A}q_{ak}^\diamond \).

In this case, \( \mu \) in (2.5) exists in \( \mathcal{M} \). Moreover, we can derive Conditions (a) and (b) in the proof of Theorem 2.7, and whence (2.6). So, we have indeed proved the following theorem.

**Theorem 2.10.** Let \( \mathcal{M}(\mathcal{A}, I, M; Q) \) be a generalized Munn ring. If \( Q \) is regular in \( \mathcal{A} \), then \( \mathcal{M} \) is prime if and only if \( \mathcal{M} \) is semiprime and (PM) is satisfied.

Comparing with Theorem 2.10, it raises a natural conjecture as follows:

**Conjecture 2.11.** Let \( \mathcal{M}(\mathcal{A}, I, M; Q) \) be a generalized Munn ring, and assume that \( Q \) is regular in \( \mathcal{A} \). Then the following conditions are equivalent:

(i) \( \mathcal{M} \) is primitive;
(ii) \( \mathcal{M} \) is semiprimitive and (PM) is satisfied;
(iii) \( \mathcal{M} \) is both semiprimitive and prime.
2.2 Principal ideal domains

In this subsection, we study the primeness of Munn rings over a principal ideal domain. We first provide one property of cancellable matrices over a principal ideal domain.

Lemma 2.12. Let \( \mathcal{A} \) be a principal ideal domain, and \( Q \) a \( m \times n \) matrix over \( \mathcal{A} \). Then \( Q \) is cancellable in \( \mathcal{M}(\mathcal{A}, n, m) \) if and only if \( m = n = r_Q \), where \( r_Q \) is the rank of \( Q \).

Proof. (Necessity). Assume that \( Q \) is cancellable in \( \mathcal{M}(\mathcal{A}, I, M) \). We suppose contrariwise that \( m = n = r_Q \) is not valid. By [23, Proposition III.2.11], there exist invertible matrices \( U, V \) such that

\[
UQV = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix},
\]

where \( D \) is a diagonal \( r_Q \times r_Q \) matrix with nonzero diagonal entries. So, there exists a nonzero matrix \( A_{22} \) over \( \mathcal{A} \) such that

\[
\begin{pmatrix} 0 & 0 \\ 0 & A_{22} \end{pmatrix}UQV = 0.
\]

Moreover,

\[
\begin{pmatrix} 0 & 0 \\ 0 & A_{22} \end{pmatrix}UQ = 0,
\]

so that

\[
\begin{pmatrix} 0 & 0 \\ 0 & A_{22} \end{pmatrix} = 0
\]

since, by hypothesis, \( Q \) is cancellable in \( \mathcal{M}(\mathcal{A}, I, M) \). It follows that \( A_{22} = 0 \). It is a contradiction. Thus, \( m = r_Q = n \).

(Sufficiency). If \( m = r_Q = n \), then by [23, Proposition III.2.11], there exist invertible matrices \( U, V \) such that

\[
UQV = \text{diag}(d_1, d_2, \ldots, d_m),
\]

where \( d_i \neq 0 \) for \( i = 1, 2, \ldots, m \). For any \( X = (x_{ij}) \in M_n(\mathcal{A}) \), we have

\[
XQ = 0 \Leftrightarrow (XU^{-1})UQV = (y_{ij}d_i) = 0, \quad \text{where } XU^{-1} = (y_{ij});
\]

\[
\Rightarrow y_{ij}d_i = 0 \quad \text{for } i, j = 1, 2, \ldots, n;
\]

\[
\Rightarrow y_{ij} = 0 \quad \text{for } i, j = 1, 2, \ldots, n;
\]

\[
\Rightarrow XU^{-1} = 0;
\]

\[
\Rightarrow X = 0,
\]

and similarly, \( QX = 0 \) if and only if \( X = 0 \). Therefore, \( Q \) is cancellable in \( \mathcal{M}(\mathcal{A}, m, n) \).

Proposition 2.13. Let \( \mathcal{M}(\mathcal{A}, m, n; Q) \) be a Munn ring. If \( \mathcal{A} \) is a principal ideal domain with unity 1, then the following conditions are equivalent:

(i) \( \mathcal{M} \) is semiprime;

(ii) \( m = r_Q = n \);

(iii) \( \mathcal{M} \) is prime.

Proof. (i) \( \Rightarrow \) (ii). By Remark 2.6, \( Q \) is cancellable in \( \mathcal{M}(\mathcal{A}, m, n) \). Now Lemma 2.12 results (ii).

(ii) \( \Rightarrow \) (iii). Let \( A_1 \) and \( A_2 \) be an arbitrary nonzero \( n \times n \) matrices over \( \mathcal{A} \). By [23, Proposition III.2.11], there exist invertible matrices \( U_i, V_i \), \( i = 1, 2 \) such that

\[
U_iA_iV_i = \text{diag}(d^{(i)}_1, d^{(i)}_2, \ldots, d^{(i)}_m, 0, \ldots, 0),
\]

where \( d^{(i)}_i \neq 0 \) for \( i = 1, 2, \ldots, m \).
where $d_k^{(i)} \neq 0$ for any $1 \leq k \leq m$. Let $U$ and $V$ be invertible matrices such that $Q = U \text{diag}(c_1, c_2, \ldots, c_n)V$, where $c_j \neq 0$ for $j = 1, 2, \ldots, n$. Because

$$V^{-1}_1(U(x_{ij}), BU^{-1}_1 = (y_{ij})$$

are both invertible, there exist $1 \leq j_0, i_0 \leq n$ such that $x_{ij_0} \neq 0, y_{i_0j_0} \neq 0$. Compute

$$(1)_{11}(U_{A_1}QV^{-1}(1)_{i_0j_0}U^{-1}QA_2V_2(1))_{11} = (1)_{11}(U_{A_1}V(XV^{-1}_1U)\text{diag}(c_1, \ldots, c_n)(1)_{i_0j_0}(U^{-1}QV^{-1})(VU^{-1}_2(U_{A_2}V_2(1)))_{11}$$

$$= (1)_{11}\text{diag}(d_1^{(1)}, \ldots, d_m^{(1)}, 0 \cdots, 0)(x_{ij_0})\text{diag}(c_1, \ldots, c_n)(y_{ij_0})\text{diag}(d_1^{(2)}, \ldots, d_m^{(2)}, 0 \cdots, 0)(1)_{11}$$

$$= (d_1^{(1)}x_{ij_0}c_jc_{i_0}y_{i_0j_0}d_2^{(2)})_{11} \neq 0,$$

so that the entry in the $(1, 1)$ position of

$$U_{A_1}QV^{-1}(1)_{i_0j_0}U^{-1}QA_2V_2 = U_{A_1} \circ V^{-1}(1)_{i_0j_0}U^{-1} \circ A_2V_2$$

is equal to the nonzero element $d_1^{(1)}x_{ij_0}c_jc_{i_0}y_{i_0j_0}d_2^{(2)}$. It follows that $A_1 \circ V^{-1}(1)_{i_0j_0}U^{-1} \circ A_2 \neq 0$ since $\mathcal{A}$ is a principal ideal domain, giving $d_1^{(1)}x_{ij_0}c_jc_{i_0}y_{i_0j_0}d_2^{(2)} \neq 0$. Thus, $A_1 \circ \mathfrak{M} \circ A_2 \neq 0$ and whence $\mathfrak{M}$ is prime.

(iii) $\Rightarrow$ (i). It is obvious.

Notice that for a principal ideal domain, the unity is the only nonzero idempotent. We observe that a generalized matrix over a principal ideal domain is regular if and only if each of its rows and each of its columns contain at least one unit. By Propositions 2.8 and 2.13, the following corollary is immediate from that any domain is prime.

Corollary 2.14. Let $\mathfrak{M}(\mathcal{A}, m, n; Q)$ be a Munn ring. If

1. $\mathcal{A}$ is a principal ideal domain;
2. $Q$ is regular in $\mathcal{A}$,

then $\mathfrak{M}$ is prime if and only if $m = r_0 = n$.

### 3 Generalized matrix rings

In this section, we shall consider the primeness and the primitivity of generalized matrix rings. We first recall the definition of generalized matrix rings.

Let $I$ be a nonempty set. For any $i, j, l \in I$, let $A_{ij}$ be a ring with identity $1_i$, and $A_{ij}$ a unitary $(A_{ii}, A_{ij})$-bimodule. Assume that there is a module homomorphism $\mu_{ij} : A_{ij} \times A_{ij}$ into $A_{ii}$, written $\mu_{ij}(x, y) = xy$, for any $i, j, l \in I$. If the following conditions hold:

(G1) $(x + y)z = xz + yz, \quad w(x + y) = wx + wy$;
(G2) $w(xz) = (wx)z$,

for any $x, y, z \in A_{ij}, w \in A_{ii}$, then the triple $(A_{ij}, I, \mu_{ij})$ is called a $\Gamma$-system with index $I$.

Given a $\Gamma$-system $(A_{ij}, I, \mu_{ij})$, let $\mathcal{G}\mathfrak{M} = \mathcal{G}\mathfrak{M}(A_{ij}, I, \mu_{ij})$ be the external direct sum of $\{A_{ij} : i, j \in I\}$. We shall use $[x_{ij}]$ to denote the external direct sum of $x_{ij}$ with $i, j \in I$. Now we define the multiplication in $\mathcal{G}\mathfrak{M}$ as

$$xy = \left\{ \sum_k X_{ik}Y_{kj} \right\},$$
where \( x = \{x_0\} \) and \( y = \{y_0\} \). It is easy to check that \( G\mathcal{M} \) is a ring (possibly without unity). We call \( G\mathcal{M} \) a generalized matrix ring, or a gm ring for short, written \( G\mathcal{M}(A_{ij}, I, \mu_{ij}) \) or \( G\mathcal{M} \) for short. If \( e_{ii} \) is a nonzero element of \( A_{ii} \) satisfying that \( e_{ii}x = x = xe_{ii} \) for all \( x \in A_{ii} \), then the set \( \{e_{ii} : i \in I\} \) is called a generalized matrix unit of the \( \Gamma \)-system \( (A_{ij}, I, \mu_{ij}) \) (for example, see [14]).

In what follows, we still write the element \( x = \{x_0\} \) satisfying that \( x_0 = 0 \) if \( i \neq i_0 \), \( j \neq j_0 \) and \( x_{i_0j_0} = u \) as \( [u]_{i_0j_0} \), especially, write \( \{1\}_{i_0j_0} = 1_{i_0} \). Also, we use \( A_{ij} \) to stand for the set \( \{[a]_{ij} : a \in A \} \). And, we write \( x = \{x_{ij} \}_{i \in A, j \in B} \) if \( x_{ij} = 0 \) whenever \( i \in I \setminus A \) or \( j \in I \setminus B \). It is easy to check that the set \( \{1_i : i \in I\} \) is a generalized matrix unit of the \( \Gamma \)-system \( (A_{ij}, I, \mu_{ij}) \).

**Proposition 3.1.** The generalized matrix ring \( G\mathcal{M}(A_{ij}, I, \mu_{ij}) \) is a subring of the generalized Munn ring \( \mathcal{M}(G\mathcal{M}, I, I; \Xi) \), where \( \Xi \) is the generalized \( I \times I \) matrix in which any entry in the \((i, i)\) position is \( 1 \), for any \( i \in I \), and \( 0 \) elsewhere.

**Proof.** Consider the mapping

\[ \phi : G\mathcal{M} \to \mathcal{M}(G\mathcal{M}, I, I; \Xi); \{x_0\} \mapsto \sum(x_{ij}i_j). \]

A routine calculation shows that \( \phi \) is an injective homomorphism, and here, we omit the detail. \( \square \)

We can now describe the main results of this section.

**Theorem 3.2.** Let \( G\mathcal{M} = G\mathcal{M}(A_{ij}, I, \mu_{ij}) \) be a generalized matrix ring. Then \( G\mathcal{M} \) is semiprime (semiprimitive, respectively) if and only if the following conditions are satisfied:

(i) for any \( i, j \in I \), if \( x \) is a nonzero element in \( A_{ij} \), then \( xA_{ij} \neq 0 \) and \( A_{ij}x \neq 0 \);

(ii) for any \( i \in I \), \( A_{ii} \) is semiprime (semiprimitive, respectively).

**Proof.** A routine calculation shows that \( \{1\}_{ii} \) is an idempotent, for all \( i \in I \), and the mapping \( \varphi : \{x\}_{ii} \mapsto x \) is an isomorphism from \( A_{ii} \) onto \( A_{ii} \). Compute

\[ \{1\}_{ii}G\mathcal{M}\{1\}_{ii} = A_{ii}. \]

So,

\[ \{1\}_{ii}G\mathcal{M}\{1\}_{ii} \cong A_{ii}. \quad (3.1) \]

(3.2.1) *The proof for the semiprime case.* If \( G\mathcal{M} \) is semiprime, then as for all \( i \in I \), \( \{1\}_{ii} \) is an idempotent in \( G\mathcal{M} \), we obtain that \( \{1\}_{ii}G\mathcal{M}\{1\}_{ii} \neq 0 \) and \( \{1\}_{ii}G\mathcal{M}\{1\}_{ii} = A_{ii} \).

We contrariwise let \( i_0, j_0 \in I \) such that \( A_{i_0j_0} \neq 0 \) but \( A_{j_0i_0} = 0 \). Pick a nonzero element \( a \) in \( A_{i_0j_0} \). Of course, \( \{a\}_{i_0j_0} \) is a nonzero element in \( G\mathcal{M} \). Compute

\[ \{a\}_{i_0j_0}G\mathcal{M}\{a\}_{i_0j_0} = \{a\}_{i_0j_0}\{A_{i_0j_0}\}_{i_0j_0}\{a\}_{i_0j_0} = \{AA_{i_0j_0}a\}_{i_0j_0} \subseteq \{A_{i_0j_0}\}_{i_0j_0} = 0, \quad (3.2) \]

contrary to the hypothesis that \( G\mathcal{M} \) is semiprime. Therefore we have now proved that for any \( i, j \in I \),

\[ A_{ij} \neq 0 \Rightarrow A_{ii} \neq 0. \quad (3.3) \]

To see (i), we assume contrarily that \( x \) is a nonzero element in \( A_{i_0j_0} \) such that \( xA_{i_0j_0} = 0 \). By (3.2), we have \( \{x\}_{i_0j_0}G\mathcal{M}\{x\}_{i_0j_0} = \{xA_{i_0j_0}x\}_{i_0j_0} = 0 \), contrary to the hypothesis that \( G\mathcal{M} \) is semiprime. So, \( xA_{ii} \neq 0 \). Dually, we may prove that \( A_{ji}x \neq 0 \).

For the converse, assume that given Conditions (i) and (ii) hold. We oppositely let \( u = \{u_{ij}\} \) be a nonzero element in \( G\mathcal{M} \) such that \( uG\mathcal{M}u = 0 \). Notice that

- \( \Xi \) is regular in \( G\mathcal{M} \) and cancellable in \( \mathcal{M}(G\mathcal{M}, I, I) \);
- \( \{1\}_{ii} \), \( i \in I \) are all unit entries of \( \Xi \). Obviously, \( \{1\}_{ii} = \{1\}_{ii} \) and furthermore, \( \{1\}_{ii}G\mathcal{M}\{1\}_{ii} = \{A_{ii}\}_{ii} \). Together with Condition (i), it is easy to see that Condition (ii) in Theorem 2.5.
By Theorem 2.5, the generalized Munn ring \( G M(I, I; \Xi) \) is semiprime. Denote
\[
J = \{k \in I : \text{for some } j \in I \text{ or } u_{ik} \neq 0 \text{ for some } i \in I \}.
\]
It is not difficult to check that
(a) \( \varepsilon = \sum_{i \in I} (1)_{ii} \) is an idempotent. Moreover, \( e M(I, I; \Xi) \) is semiprime.
(b) \( e \phi(u) = \phi(u) \varepsilon \), where \( \phi \) has the same meanings as in the proof of Proposition 3.1.

Moreover,
\[
\phi(u)(e M(I, I; \Xi) \varepsilon) \phi(u) \subseteq \phi(u) \phi(G M) \phi(u) = \phi(u G M u) = 0,
\]
contrary to the foregoing proof that \( e M(I, I; \Xi) \varepsilon \) is semiprime. Therefore \( u = 0 \) and whence \( G M \)

is semiprime.

(3.2.2) The proof for the semiprimitive case. Similar as (3.2.1), we may prove the necessity.

For the converse, we contrariwise assume that \( u \) is a nonzero element in \( r M(G M) \). With notations in
(3.2.1), we denote \( X = \{x_{i j} \mid i, j \} \subseteq G M \). It is easy to see that
(a) \( X \) is a subalgebra of \( G M \);
(b) \( \tau = \sum_{i \in I} (1)_{ii} \) is an idempotent in \( G M \). Moreover, \( \phi(\tau) = \varepsilon, \tau u = u \tau \) and \( r M G M \tau = X \);
(c) \( \phi(X) = e M(G M, I, I; \Xi) \varepsilon \).

Therefore \( u \in r M(r G M \tau) = r M(X) \). Notice that \( \phi \) is an injective homomorphism. We observe that
\( e M(G M, I, I; \Xi) \varepsilon \) is isomorphic to \( X \). It follows that
\[
\phi(u) \in r M(e M(G M, I, I; \Xi) \varepsilon),
\]
so that \( e M(G M, I, I; \Xi) \varepsilon \) is not semiprimitive. Indeed, by the proof of the converse part in (3.2.1), we can
gain that \( M(G M, I, I; \Xi) \) is semiprimitive if and only if the following conditions are satisfied:
(i) \( \text{for any } i, j, k, l \in I, \text{ if } x \text{ and } y \text{ are respectively nonzero elements in } A_{ij} \text{ and in } A_{kl}, \text{ then } x A_{ij} y \neq 0; \)
(ii) \( \text{for any } i \in I, \text{ } A_i \text{ is primitive, respectively}. \)

**Theorem 3.3.** Let \( G M = G M(A_i, I, \mu_g) \) be a generalized matrix ring. Then \( G M \) is prime (primitive, respectively) if and only if the following conditions are satisfied:
(i) \( \text{for any } i, j, k, l \in I, \text{ if } x \text{ and } y \text{ are respectively nonzero elements in } A_{ij} \text{ and in } A_{kl}, \text{ then } x A_{ij} y \neq 0; \)
(ii) \( \text{for any } i \in I, \text{ } A_i \text{ is prime (primitive, respectively)}. \)

**Proof.** (3.3.1) The proof for the prime case. Similar as in (3.2.1), we may prove the necessity.

For the sufficiency, we contrarily assume that \( u = \{u_{ij}\} \) and \( v = \{v_{ki}\} \) are nonzero elements in \( G M \) such that \( u G M v = 0 \). Obviously, there are \( l_0, j_0, k_0, \) and \( l_0 \in I \) such that \( u_{l_0 j_0} \) and \( v_{k_0 l_0} \) are neither equal to 0. Further, by Condition (i), there is \( x \in A_{k_0 l_0} \) such that \( u_{l_0 j_0} x v_{k_0 l_0} \neq 0 \). By the same reason, we have \( y \in A_{l_0 k_0} \)
such that \( w_{l_0 k_0} x v_{k_0 l_0} y u_{i j} x v_{k_0 l_0} \neq 0 \). So that \( 0 \neq w_{l_0 k_0} x v_{k_0 l_0} y u_{i j} x v_{k_0 l_0} \). Compute
\[
\{u_{l_0 j_0} x v_{k_0 l_0} y \cdot A_{i j} \cdot u_{l_0 k_0} x v_{k_0 l_0} y\}_{l_0 k_0} = \{u_{l_0 j_0} x v_{k_0 l_0} y\}_{l_0 k_0} G M \{u_{l_0 k_0} x v_{k_0 l_0} y\}_{l_0 k_0} = \{u_{l_0 j_0} x v_{k_0 l_0} y\}_{l_0 k_0} G M \{u_{l_0 k_0} x v_{k_0 l_0} y\}_{l_0 k_0} = 0.
\]
It follows that \( u_{l_0 j_0} x v_{k_0 l_0} y \cdot A_{i j} \cdot u_{l_0 k_0} x v_{k_0 l_0} y = 0 \). This means that \( A_{i j} \) is not semiprime, contrary to Condition
(ii). Therefore, \( G M \) is prime.

(3.3.2) The proof for the primitive case. It follows from the proof in (2.5.2).
4 Generalized path algebras

In this section, we consider the primeness and the primitivity of generalized path algebras. We first provide some results on quivers.

4.1 Quivers

We start with the basic definitions. A quiver \( Q = (V, E) \) is an oriented graph, where \( V \) is the vertex set and \( E \) is the arrow set. We denote by \( \mathcal{E} : E \to V \) and \( \mathcal{S} : E \to V \) the mappings, where \( \mathcal{E}(a) = i \) and \( \mathcal{S}(a) = j \) when \( a : i \to j \) is an arrow from \( i \) to \( j \). A path in the quiver \( Q \) is an ordered sequence of arrows \( p = a_n \cdots a_1 \) with \( \mathcal{E}(a_l) = \mathcal{S}(a_{l+1}) \) for \( 1 < l < n \), or the symbol \( e_i \) for \( i \in V \). We call the path \( e_i \) trivial path and define \( \mathcal{E}(e_i) = i = \mathcal{S}(e_i) \). For a nontrivial path \( p = a_n \cdots a_1 \), we define \( \mathcal{E}(p) = \mathcal{E}(a_n) \) and \( \mathcal{S}(p) = \mathcal{S}(a_0) \). A nontrivial path \( p = a_n \cdots a_1 \) is said to be
(i) an oriented cycle if \( \mathcal{E}(p) = \mathcal{S}(p) \);
(ii) a loop from \( i \) to \( i \) if \( n = 1 \) and \( \mathcal{E}(p) = i = \mathcal{S}(p) \).

**Definition 4.1.**

(i) A quiver \( G \) with vertex set \( V \) is said to be a complete quiver if for any \( a, b \in V \) with \( a \neq b \), there are one arrow from \( a \) to \( b \) and one arrow from \( b \) to \( a \).

(ii) Let \( G_1 \) and \( G_2 \) be quivers with vertex set \( V_1 \) and arrow set \( E_1 \), and with vertex set \( V_2 \) and arrow set \( E_2 \), respectively. A quiver \( G \) is said to be a union of \( G_1 \) and \( G_2 \) if the vertex set of \( G \) is \( V_1 \sqcup V_2 \) and the arrow set of \( G \) is \( E_1 \sqcup E_2 \). If, in addition, both \( V_1 \sqcup V_2 \) and \( E_1 \sqcup E_2 \) are disjoint unions, then we shall call \( G \) to be a disjoint union of \( G_1 \) and \( G_2 \).

By an empty graph, we mean a graph without arrows. Obviously, we have the following observations:

(OB1) The empty graph is a complete quiver if and only if it has exactly one vertex.

Also,

(OB2) Let \( Q = (V, E) \) be a quiver without loops. \( Q \) is a disjoint union of complete quivers if and only if for any \( a, b \in V \) with \( a \neq b \), if there is a path from \( a \) to \( b \), then there is one arrow from \( a \) to \( b \).

Indeed, by definition, the necessity is evident. Conversely, we define a relation on the vertex set \( V \) as follows:

\[ a \mathcal{D} b \quad \text{if} \quad a = b; \quad \text{or there is a path from} \quad a \quad \text{to} \quad b. \]

It is not difficult to see that \( \mathcal{D} \) is an equivalence relation on \( V \). Consider the quotient \( V / \mathcal{D} = \{V_a : a \in A\} \) and for \( V_a \), construct a subquiver \( Q_a = (V_a, E_a) \) of \( Q \) as follows: for \( a, b \in V_a \),

- there is an arrow from \( a \) to \( b \) in \( Q_a \) if and only if there is an arrow from \( a \) to \( b \) in \( Q \).

It follows that \( Q \) is a disjoint union of the quivers \( Q_a \) with \( a \in A \). We next prove that each \( Q_a \) is a complete quiver. We consider the following two cases:

- If \( Q_a \) has exactly one vertex, then \( Q_a \) is an empty graph because it has no loops; thus, \( Q_a \) is a complete quiver.

- If \( Q_a \) has more than two vertices, then there are any two vertices \( u, v \) of \( Q_a \), there is a path from \( u \) to \( v \), and furthermore by hypothesis, there is an arrow from \( v \) to \( u \) in \( Q \). Therefore, there is an arrow from \( v \) to \( u \) in \( Q_a \), and by definition, \( Q_a \) is a complete quiver.

However, \( Q_a \) is a complete quiver. Consequently, \( Q \) is a disjoint union of complete quivers.

**Definition 4.2.** Let \( Q = (V, E) \) be a quiver with vertex set \( V \) and arrow set \( E \). Construct a quiver \( \overline{Q}^{PC} \) with vertex set \( V \) and in which for \( u, v \in V \), there is an arrow from \( u \) to \( v \) in \( \overline{Q}^{PC} \) if \( u \neq v \) and there is a path from \( u \) to \( v \) in \( Q \). The quiver \( \overline{Q}^{PC} \) is called the path-connected quiver of \( Q \), written \( \overline{Q}^{PC} \).
4.2 Generalized path algebras

We recall the definition of generalized path algebras.

Let $I$ be a nonempty set and $K$ a field. For any $i, j, u, v \in I$, $A_{ij}$ is a vector space over the field $K$, and there exists $K$-linear mapping $\mu_{ij}$ from $A_{ij}$ to $A_{uv}$, written $\mu_{ij}(x \otimes y) = xy$, such that $x(yz) = (xy)z$ for any $x \in A_{ij}$, $y \in A_{uj}$, $z \in A_{uv}$, then the set $\{A_{ij}, I, \mu_{ij}\}$ is a $\Gamma$-system with index $I$ over the field $K$. Similar to the generalized matrix ring, we obtain a $K$-algebra, called a generalized matrix algebra, or a gm algebra in short, and written as $GM(\mathcal{A}(A_{ij}, I, \mu_{ij}))$, or $GM(\mathcal{A})$ in short.

Assume that $D = (V, E)$ is a quiver (possibly an infinite quiver and also not a simple graph) with vertex set $V$ and arrow set $E$. Let $\Omega = GM(\mathcal{A}(\Omega_i, V, \mu_{ij}))$ be a generalized matrix algebra over the field $K$ satisfying the following conditions:

1. $\Omega$ has a generalized matrix unit $\{e_i : i \in V\}$.
2. $\Omega_{ij} = 0$ for any $i, j \in V$ with $i \neq j$.

The sequence $x = a_{ij}x_{ij}a_{ij}x_{ij} \cdots x_{ij}a_{ij}$ is called a generalized path, or an $\Omega$-path, from $i_0$ to $i_n$ via arrows $x_{ij}, x_{ij}, \cdots, x_{ij}, x_{ij}$, where $0 \neq a_{ij} \in \Omega_{ij}$ for $p = 0, 1, 2, \ldots, n$. In this case, $n$ is called the length of $x$, written $l(x)$.

For two $\Omega$-paths $x = a_{ij}x_{ij}a_{ij}x_{ij} \cdots x_{ij}a_{ij}$ and $y = b_{ij}y_{ij}b_{ij}y_{ij} \cdots y_{ij}b_{ij}b_{ij}$ with $i_m = j_0$, we define the multiplication of $x$ and $y$ as follows:

$$xy = a_{ij}x_{ij}a_{ij}x_{ij} \cdots x_{ij}a_{ij}(a_{ij}b_{ij})y_{ij}b_{ij}y_{ij} \cdots y_{ij}b_{ij}b_{ij}.$$  (4.1)

Denote by $A_{ij}$ the vector space over the field $K$ with basis consisting of all $\Omega$-paths from $i$ to $j$ with length $\geq 1$. Let $B_{ij}$ be the subspace spanned by all elements:

$$a_{ij}x_{ij}a_{ij}x_{ij} \cdots x_{ij}a_{ij}\left(\sum_{r=1}^{n} a_{ij}^{(r)}\right)x_{ij} \cdots x_{ij}a_{ij},$$  (4.2)

where $i_0 = i, i_m = j, a_{ij}^{(r)} \in \Omega_{ij}$, and $x_{ij}$, is an arrow, $p = 0, 1, \ldots, m - 1$. Let $A_{ij} = B_{ij}/B_{ij}$ when $i \neq j$ and $A_{ii} = (A_{ii} + \Omega_{ii})/B_{ii}$, written $[a] = a + B_{ii}$ for any generalized path $a$ from $i$ to $j$. We can obtain a $K$-linear mapping $x_{ij}$ from $A_{ij}$ to $A_{ij}$ induced by (4.1). We write $a$ instead of $[a]$ when $a \in \Omega$. So, $(A_{ij}, V, \kappa_{ij})$ is a $\Gamma$-system. It is not difficult to know that $[e_{ij}x_{ij}] = x_{ij}$ for any $x_{ij}$ from $i$ to $j$. Moreover, $\{e_i : i \in V\}$ is a generalized matrix unit of the $\Gamma$-system $(A_{ij}, V, \kappa_{ij})$.

The notion of generalized path algebras is originally defined in [24]. For generalized path algebras, also see [25].

Definition 4.3. The aforementioned generalized matrix algebra $GM(A_{ij}, V, \kappa_{ij})$ is called the generalized path algebra of the quiver $D$ over the generalized matrix algebra $\Omega$, or the $\Omega$-path algebra, written $K(D, \Omega)$. If, in addition, $\Omega_{ij} = K\epsilon_{ij}$ for any $i \in V$, then $K(D, \Omega)$ is called a path algebra of the quiver $D$ over the field $K$, written $K(D)$.

It is worthy to record here that for a generalized path algebra $K(D, \Omega)$, by (4.2), it follows that for any nonzero elements,

$$x = a_{ij}x_{ij}a_{ij}x_{ij} \cdots x_{ij}a_{ij}, y \in K(D, \Omega),$$

we have

(E1) $x = 0$ if and only if $a_{ij} = 0$ for some $0 \leq r \leq m$;

(E2) $|x| = |y|$ in $K(D, \Omega)$ if and only if $x = y$ regarded as sequences.
Let $G M = G M(A_{ij}, I, \mu_{ij})$ be a generalized matrix ring, and construct a quiver $Q(G M)$ with vertex set $I$ and in which there is an arrow from $i$ to $j$ if and only if $i \neq j$ and $A_{ij} \neq 0$. We call the quiver $Q(G M)$ the $\Gamma$-quiver of the generalized matrix algebra $G M(A_{ij}, I, \mu_{ij})$. Obviously, $Q(G M)$ is a quiver without loops.

We next establish the relationship between a quiver and the $\Gamma$-quiver of its generalized path algebra.

**Lemma 4.4.** Let $D = (V, E)$ be a quiver. If $K(D, \Omega)$ is a generalized path algebra of $D$ over the generalized matrix ring $\Omega$, then

(i) $Q(K(D, \Omega))$ is just the path connected quiver $D^{PC}$ of $D$.

(ii) For any $i, j, u \in V$, if $x$ is a nonzero element of $A_{ij}$, then $xA_{ju} \neq 0$ whenever $A_{ju} \neq 0$.

**Proof.** (i) Notice that $Q(K(D, \Omega))$ and $D^{PC}$ have the same vertex set. So, we need only to see whether $Q(K(D, \Omega))$ and $D^{PC}$ have the same arrow set. It follows from the following implications: There is an arrow from $u$ to $v$ in $Q(K(D, \Omega))$ if and only if $u \neq v$ and $A_{uv} \neq \emptyset$; if and only if $u \neq v$ and $A_{uv} \neq \emptyset$; if and only if there is a $\Omega$-path $a_{ij}a_{ii}a_{ij} \cdots a_{ik}a_{ik}a_{ii}$, where $i_0 = u$, $i_n = v$; if and only if there is a $\Omega$-path $e_{ij}e_{ii}e_{ij} \cdots e_{ik}e_{ik}e_{ii}$, where $i_0 = u$, $i_n = v$; if and only if there is a path $x_{ij}x_{ii} \cdots x_{ik}x_{ik}$, where $i_0 = u$, $i_n = v$; if and only if there is an edge from $u$ to $v$ in $D^{PC}$.

(ii) Let $x = a_{ij}a_{ii}a_{ij} \cdots a_{ik}a_{ik}a_{ii}$ where $i_0 = i$, $i_n = j$ and $0 \neq a_p$ for $p = 0, 1, 2, \ldots, n - 1$. Obviously, $y = e_{ij}e_{ii}e_{ij} \cdots e_{ik}e_{ik}e_{ii}$ where $j_0 = j$, $j_n = u$ and $y_{ijp+1} \neq 0$ for $p = 0, 1, \ldots, n - 1$, is a nonzero element of $A_{uj}$. Then,

$$xy = a_{ij}a_{ii}a_{ij} \cdots a_{ik}a_{ik}a_{ii}(a_{ij}a_{ii}a_{ij} \cdots a_{ik}a_{ik}a_{ii})y_{ij}y_{ij} \cdots y_{ik}y_{ik} \neq 0,$$

which results (ii). $\square$

By Theorem 3.2 and Lemma 4.4, we may prove the following theorem.

**Theorem 4.5.** Let $K(D, \Omega)$ be a generalized path algebra. Then $K(D, \Omega)$ is semiprime (semiprimitive, respectively) if and only if

(i) $D^{PC}$ of $D$ is a disjoint union of complete quivers;

(ii) for any $i \in V$, $A_{ij}$ is semiprime (semiprimitive, respectively).

**Proof.** Suppose that $K(D, \Omega)$ is semiprime (semiprimitive, respectively). Theorem 3.2 immediately results (ii). Notice that $A_{ij}$ is nonzero if and only if there is a $\Omega$-path from $i$ to $j$; if and only if there is a path from $i$ to $j$ in the quiver $D$; if and only if there is an arrow from $i$ to $j$ in $D^{PC}$. We can observe that if in $D^{PC}$, there is a path $i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_n$, then $A_{ii_1i_2} \neq 0$ for $i = 1, 2, \ldots, n - 1$, so that by Lemma 4.4, $A_{ii_1i_2} \cdots A_{ii_n} \neq 0$. This means that there is a $\Omega$-path from $i_1$ to $i_n$. It follows that $A_{ii_n} \neq 0$. By (3.3) in the proof of Theorem 3.2, this implies that $A_{ii_n} \neq 0$, thereby there is an arrow from $i_n$ to $i$ in $Q(K(D, \Omega))$. It follows from Lemma 4.4 (i) that there is an arrow in $D^{PC}$. Now by (OB2), $D^{PC}$ is a disjoint union of complete quivers.

Conversely, assume that given conditions hold. For a nonzero element $x \in A_{ij}$, we have $A_{ij} \neq 0$, so that there is a path from $i$ to $j$ in $D^{PC}$, it follows from (OB2) that there is a path from $j$ to $i$ in $D^{PC}$, thus $A_{ji} \neq 0$. Again by Lemma 4.4, $A_{ij}x \neq 0$ and similarly, $A_{ji}x \neq 0$. Now by Theorem 3.2, $K(D, \Omega)$ is semiprime (semiprimitive, respectively). $\square$

Also, by Theorem 3.3 and Lemma 4.4, we have

**Theorem 4.6.** Let $K(D, \Omega)$ be a generalized path algebra. Then $K(D, \Omega)$ is prime (primitive, respectively) if and only if

(i) $D^{PC}$ is a complete quiver;

(ii) for any $i \in V$, $A_{ij}$ is prime (primitive, respectively).
Proof. For the necessity, it suffices to verify that $\mathcal{D}^{\text{PC}}$ is a complete quiver.

- If $|I| = 1$, then $\mathcal{Q}(K(D, \Omega))$ is an empty graph since it has no loops, and by Lemma 4.4, $\mathcal{D}^{\text{PC}}$ is a complete quiver.

- Assume that $|I| \geq 2$. For any $i, j \in I$ with $i \neq j$, by definition, $A_{ij} \neq 0$ and $A_{ji} \neq 0$, and by Theorem 3.3 (i), $A_{ii}A_{ij}A_{ji} \neq 0$. It follows that $A_{ij} \neq 0$, so that there is an arrow form $i$ to $j$ in $\mathcal{Q}(K(D, \Omega))$. Similarly, we may prove that there is an arrow form $j$ to $i$ in $\mathcal{Q}(K(D, \Omega))$. Therefore, $\mathcal{Q}(K(D, \Omega))$ is a complete quiver, and by Lemma 4.4 (i), $\mathcal{D}^{\text{PC}}$ is a complete quiver.

To verify the sufficiency, we consider the following two cases:

- If $|I| = 1$, then $\mathcal{Q}(K(D, \Omega))$ is an empty graph, and $K(D, \Omega) \cong A_{ii}$, and this means that Condition (i) in Theorem 3.3 holds since each $A_{ii}$ has a unity. It follows that $K(D, \Omega)$ is prime.

- Assume that $|I| \geq 2$. In this case, by $\mathcal{Q}(K(D, \Omega))$ is a complete quiver, there is an arrow from $j$ to $k$ in $\mathcal{Q}(K(D, \Omega))$ for any $j, k \in I$. This shows that $A_{kk} \neq 0$. By Lemma 4.4, $xA_{kk}y \neq 0$ for any $i, j, k, l \in I$ and nonzero elements $x \in A_{ij}, y \in A_{kl}$. Now by Theorem 3.3, $K(D, \Omega)$ is prime.

However, $K(D, \Omega)$ is prime. Similarly, we may verify the primitive case. We complete the proof.

We may now prove the following proposition.

Proposition 4.7. Let $K(D, \Omega)$ be a generalized path algebra and $i \in V$.

(i) If $D$ has no paths from $i$ to $i$, then $A_{ii}$ is prime (primitive, semiprime and semiprimitive, respectively) if and only if so is $\Omega_{ii}$.

(ii) If $D$ has paths from $i$ to $i$, then the following conditions are equivalent:

- (1) $A_{ii}$ is semiprime;
- (2) $\text{ann}_A(\Omega_{ii}) = 0$ and $\text{ann}_\Omega(\Omega_{ii}) = 0$, where $\text{ann}_A(X)$ (ann$_\Omega(X)$) is the left (right) annihilator of $X$;
- (3) $A_{ii}$ is prime.

Proof. (i). If $D$ has no paths from $i$ to $i$, then $A_{ii}' = 0$ and so $A_{ii} = [\Omega_{ii}]$. It follows that $A_{ii}$ is isomorphic to $\Omega_{ii}$, which results (i).

(ii). Assume that $D$ has paths from $i$ to $i$. We need only to verify that (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) since a prime ring is semiprime.

(1) $\Rightarrow$ (2). Suppose that $A_{ii}$ is semiprime. We assume contrariwise at least one of $\text{ann}_A(\Omega_{ii}) \neq 0$ and $\text{ann}_\Omega(\Omega_{ii}) \neq 0$ holds. Without loss of generality, we let $\text{ann}_A(\Omega_{ii}) \neq 0$ and $u \in \text{ann}_\Omega(\Omega_{ii})$, so that $u\Omega_{ii} = 0$. Consider the generalized path

$$x = a_{ii}a_{ij}a_{ij} \cdots a_{ij}a_{ii}u$$

via arrows $x_{ij}, x_{ij}, \ldots, x_{ij}, x_{ii}$, with $i_0 = i, i_n$, and for any generalized path,

$$y = b_{ij}b_{ij}b_{ij} \cdots b_{ij}b_{ii}$$

via arrows $y_{ij}, y_{ij}, \ldots, y_{ij}, y_{ii}$, with $j_0 = i, j_n$. Obviously, $b_{ij} \in \Omega_{ii}$. Therefore,

$$xy = a_{ii}a_{ij}a_{ij} \cdots a_{ij}a_{ii}(ub_{ij})b_{ij}b_{ij} \cdots b_{ii}b_{ii} = a_{ii}a_{ij}a_{ij} \cdots a_{ij}a_{ii}0 \cdots 0 = 0,$$

and hence, $xb_{ij} = 0$, thereby by the arbitrariness of $b_{ij}$, $xA_{ii} = 0$. It follows that $xA_{ii}x = 0$, contrary to the hypothesis that $A_{ii}$ is semiprime. It results (2).

(2) $\Rightarrow$ (3). Assume that (2) is satisfied. We let contrarily $w, z \in K(D, \Omega) \setminus \{0\}$ such that

$$wA_{ii}z = 0.$$  \hspace{1cm} (4.3)

Let $w = \sum_{k=1}^{r}w_k$ and $z = \sum_{k=1}^{s}z_k$, where $w_k, z_k$ are generalized paths, and

- $\ell(w_1) \geq \ell(w_2) \geq \cdots \geq \ell(w_r)$;
- $\ell(z_1) \geq \ell(z_2) \geq \cdots \geq \ell(z_s)$.
\[ - w_i \notin \Omega_{ji}w_k\Omega_{ij} \text{ for } 2 \leq k \leq r; \\
- z_l \notin \Omega_{ji}z_l\Omega_{ij} \text{ for } 2 \leq l \leq s. \]

The equality (4.3) can derive that \( w\Omega iz = 0 \), so that for any \( d \in \Omega_{ij} \),
\[ w_dz_1 + w_dz_2 + \cdots + w_dz_s = 0, \]
and hence, by (E2) and comparing the lengths of generalized paths \( w_dz_i \), \( w_dz_1 = 0 \). It follows that
\[ 0 = w_i\Omega iz_1 = a_{ij}x_{ij}a_{ij}x_{ij}a_{ij}x_{ij} \cdots x_{ij}x_{ij}(u\Omega iv)y_{h_1}b_{h_2}y_{h_3} \cdots y_{h_{2n-1}}b_{h_n}, \]
(4.4)
where \( w_i = a_{ij}x_{ij}a_{ij}x_{ij}a_{ij}x_{ij} \cdots x_{ij}x_{ij}u \) and \( z_l = y_{h_1}b_{h_2}y_{h_3} \cdots y_{h_{2n-1}}b_{h_n} \). Again by (E1), the equality (4.4) can imply that \( u\Omega iv = 0 \). We have \( u\Omega iv = 0 \) and \( \Omega iv = 0 \) by picking \( u = 1 \) or \( v = 1 \). This is contrary to (2). Therefore, \( A_{ij} \)
is prime. \( \square \)

**Lemma 4.8.** Let \( Q \) be a quiver and \( K \) a field. Then the following statements are true for the path algebra \( K(Q) \):
(i) If \( Q \) has no paths from \( i \) to \( j \), then \( A_{ij} \cong K \).
(ii) If \( Q \) has paths from \( i \) to \( j \), then \( A_{ij} \) is semiprimitive.

**Proof.** (i). If \( Q \) has no paths from \( i \) to \( j \), then \( A_{ij} = [\Omega_{ij}] = [K_{eij}] \cong K_{eij} \). But \( K_{eij} \cong K \), so \( A_{ij} \cong K \).
(ii). We assume on the contrary that \( A_{ij} \) is not semiprimitive. With notations in the proof of (2) \( \Rightarrow \) (3) in Proposition 4.7, assume that \( w \) is a nonzero element in \( rad A_{ij} \), and further, let \( z \) be a nonzero element of \( A_{ij} \) such that \( wz + w + z = 0 \). Without the loss of generality, we let \( z = \sum_{k=1}^{n} z_k \), and each \( z_k \) has the same properties as (2) \( \Rightarrow \) (3) in Proposition 4.7. So,
\[ w_iz_1 + \cdots + w_iz_j + z_1 + \cdots + z_j + w_1 + \cdots + w_r = wz + w + z = 0. \]
(4.5)
Consider that the length of \( w_iz_j \) is bigger than those of \( w_1 \) and \( z_j \), equation (4.5) derives that \( w_iz_1 + \cdots + w_iz_j = 0 \). Notice that the length of \( w_iz_j \) is maximum among all \( w_iz_j \), and this equation implies that \( w_iz_j = 0 \). It follows that \( w_1 = 0 \) or \( z_1 = 0 \) since \( w_1 \) and \( z_1 \) are both \( \Omega \)-paths from \( i \) to \( j \). Now by the maximality of \( l(w_1) \) and \( l(z_1) \), all \( w_1 \) are zero or all \( z_1 \) are zero. Therefore, \( w = 0 \) or \( z = 0 \), contrary to that \( w, z \) are neither zero elements. Consequently, each \( A_{ij} \) is semiprimitive. \( \square \)

For a path algebra \( K(Q) \), by Lemma 4.8, each \( \Omega_{ij} \) is semiprimitive and of course, semiprime. Now, the following theorem is an immediate consequence of Theorem 4.5.

**Theorem 4.9.** Let \( Q \) be a quiver and \( K \) a field. Then, the following conditions are equivalent:
(i) \( K(Q) \) is semiprime;
(ii) \( \overline{Q}^{PC} \) is the disjoint union of complete quivers;
(iii) \( K(Q) \) is semiprimitive.

By Proposition 4.7 and since a field is prime, each \( A_{ij} \) of the path algebra \( K(Q) \) is prime. Theorem 4.6 results in the following theorem.

**Theorem 4.10.** Let \( Q \) be a quiver and \( K \) a field. Then \( K(Q) \) is prime if and only \( \overline{Q}^{PC} \) is a complete quiver.

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References