Research Article

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A note on commutators of strongly singular Calderón-Zygmund operators

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Abstract: In this article, the authors consider the commutators of strongly singular Calderón-Zygmund operator with Lipschitz functions. A sufficient condition is given for the boundedness of the commutators from Lebesgue spaces $L^p(\mathbb{R}^n)$ to certain Campanato spaces $C^{p,\beta}(\mathbb{R}^n)$.

Keywords: strongly singular Calderón-Zygmund operator, commutator, Lipschitz space, Campanato space

MSC 2020: 42B20, 42B35, 47B47

1 Introduction and result

Let $T$ be the classical singular integral operator, and the commutator $T_b$ generated by $T$ and a locally integrable function $b$ is given by

$$T_b f = bT(f) - T(bf).$$

A well-known result by Coifman et al. [1] states that $T_b$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ when $b \in \text{BMO}(\mathbb{R}^n)$. They also gave some characterization of BMO(\mathbb{R}^n) in virtue of the $L^p$ boundedness of the aforementioned commutator (see also [2,3]).

In 1978, Janson [2] studied the boundedness of the commutator $T_b$ when $b \in \dot{A}_q(\mathbb{R}^n)$, the homogeneous Lipschitz space of order $0 < q < 1$, which is the space of all functions $b$, such that

$$\|b\|_{\dot{A}_q(\mathbb{R}^n)} = \sup_{x,y \in \mathbb{R}^n} \frac{|b(x) - b(y)|}{|x - y|^q} < \infty. \quad (1.1)$$

Janson proved that $T_b$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $1 < p < q < \infty$ if and only if $b \in \dot{A}_q(\mathbb{R}^n)$ with $q = n/(1/p - 1/q)$. In 1995, Paluszyński [4] made a further study of the problem and proved that $T_b$ is bounded from $L^p(\mathbb{R}^n)$ to some Triebel-Lizorkin spaces $F^p,\infty(\mathbb{R}^n)$ if and only if $b \in \dot{A}_p(\mathbb{R}^n)$, for $1 < p < \infty$ and $0 < q < 1$.

In 2015, Zhang et al. [5] gave another kind of interesting results for $T_b$ when $b$ belongs to Lipschitz spaces. They proved that $T_b$ is bounded from $L^p(\mathbb{R}^n)$ to $C^{p,\beta}(\mathbb{R}^n)$ if and only if $b \in \dot{A}_p(\mathbb{R}^n)$, for $1 < p < \infty$, $-n/p \leq \beta < 0$ and $0 < \gamma \leq \beta + n/p < 1$, where $C^{p,\beta}(\mathbb{R}^n)$ is Campanato space (see Definition 1.2).

On the other hand, motivated by the study of multiplier operator with symbol given by $e^{i(x-y)}$, away from the origin $(0 < \alpha < 1, \beta > 0)$, Alvarez and Milman [6] introduced the following strongly singular Calderón-Zygmund operator.

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Definition 1.1. [6] Let $T : S \to S'$ be a bounded linear operator. $T$ is called a strongly singular Calderón-Zygmund operator if the following conditions are fulfilled.

1. $T$ can be extended to a continuous operator from $L^2$ into itself.
2. There exists a continuous function $K(x,y)$ on $\{(x,y) : x \neq y\}$ such that
   \[ |K(x,y) - K(x,z)| + |K(y,x) - K(z,x)| \leq C \frac{|y - z|^{\delta}}{|x - z|^{n+\delta/a}}, \]
   if $2|y - z|^\alpha \leq |x - z|$ for some $0 < \delta \leq 1$ and $0 < \alpha < 1$, and,
   \[ \langle Tf, g \rangle = \int K(x,y)f(y)g(x)dydx, \]
   for $f, g \in S$ with disjoint supports.
3. For some $(1 - a)n/2 \leq \eta < n/2$, both $T$ and its conjugate operator $T^*$ can be extended to continuous operators from $L^q(R^n)$ into $L^2(R^n)$, where $1/q = 1/2 + \eta/n$.

In 1986, Alvarez and Milman studied the boundedness of strongly singular Calderón-Zygmund operator on Lebesgue spaces and Hardy spaces in [6,7]. Later on, there are many authors discussed the mapping properties of strongly singular Calderón-Zygmund operators in various spaces. See, for instance, [8–11]. We would like to note that, as stated in [6,7], the strongly singular Calderón-Zygmund operators include pseudo-differential operator with a symbol in the Hörmander class $S^{\eta,-\delta}_{\alpha,\eta}$, where $0 < \delta \leq \alpha < 1, (1 - a)n/2 < \eta < n/2$.

Now, we define the commutator generated by strongly singular Calderón-Zygmund operator $T$ and a locally integrable function $b$ as follows:
\[ T_b f(x) = b(x)T(f)(x) - T(bf)(x). \]

The main result of Alvarez et al. in [8] yields the boundedness of $T_b$ on $L^p(R^n), 1 < p < \infty$, when $b \in \text{BMO}(R^n)$. Afterward, the mapping properties of $T_b$ when $b$ belongs to BMO space or Lipschitz space, on Lebesgue spaces, Morrey spaces, Herz type spaces, and Hardy spaces have been studied by several authors. See [10–15] for example.

In this article, we will continue the study of the commutator of strongly singular Calderón-Zygmund operator when the symbol $b$ belongs to Lipschitz space. The aim is to extend some of the results in [5] to a strongly singular Calderón-Zygmund operator.

As usual, let $B = B(x_0, r)$ denote the ball centered at $x_0$ with radius $r$. For $a > 0$, $aB$ stands for the ball concentric with $B$ having $a$ times its radius, that is, $aB = B(x_0, ar)$. Denote by $|B|$ the Lebesgue measure of $B$ and by $\chi_B$ its characteristic function. For $f \in L^1_{\text{loc}}(R^n)$, we write
\[ f_B = \frac{1}{|B|}\int_B f(x)dx. \]

Definition 1.2. Let $1 \leq p < \infty, -n/p \leq \beta < 1$, the Campanato space $C^{p,\beta}(R^n)$ is given by
\[ C^{p,\beta}(R^n) = \{ f \in L^p_{\text{loc}}(R^n), \|f\|_{C^{p,\beta}(R^n)} < \infty \}, \]
where
\[ \|f\|_{C^{p,\beta}(R^n)} := \sup_B \frac{1}{|B|^{n/p}} \left( \frac{1}{|B|} \int_B |f(x) - f_B|^pdx \right)^{1/p}, \]
and the supremum is taken over all balls $B$ in $R^n$.

Our result can be stated as follows.
Theorem 1.1. Let $T$ be a strongly singular Calderón-Zygmund operator, and $\alpha, \eta, \text{ and } \delta$ be as in Definition 1.1. Suppose that $\frac{m(1-\alpha)+2n}{2q} < p < \infty$, $-n/p \leq \beta < 0$, and $0 < \gamma = \beta + n/p < 1$. If $b \in \dot{A}_\gamma(\mathbb{R}^n)$, then $T_b$ is bounded from $L^p(\mathbb{R}^n)$ to $C^{1,\gamma}(\mathbb{R}^n)$, that is, there exists a constant $C > 0$ such that for all $f \in L^p(\mathbb{R}^n)$,

$$\|T_b(f)\|_{C^{1,\gamma}(\mathbb{R}^n)} \leq C\|b\|_{\dot{A}_\gamma(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}.$$ 

Remark 1.1. Theorem 1.1 gives a new kind of boundedness for commutator $T_b$ when $b$ belongs to certain Lipschitz spaces, compared with the $(L^p, L^q)$-boundedness and the $(M^{p,\beta}, M^{q,\gamma})$-boundedness of $T_b$, when $\frac{m(1-\alpha)+2n}{2q} < p < q < \infty$ and $0 < \gamma = \frac{1}{p} - \frac{1}{q} < 1$, obtained in [12, Corollary 1] and [10, Theorem 2.2], respectively.

2 Proof of Theorem 1.1

To prove Theorem 1.1, we need some known results. The first one is due to DeVore and Sharpley [16] and Janson et al. [17] (see also Paluszyński [4], Lemma 1.5).

Lemma 2.1. Let $0 < \gamma < 1$ and $b \in \dot{A}_\gamma(\mathbb{R}^n)$, then for all $1 \leq p < \infty$,

$$\|b\|_{\dot{A}_\gamma(\mathbb{R}^n)} \approx \sup_B \frac{1}{|B|^{1/p}} \left( \frac{1}{|B|} \int_B |b(x) - b_B|^p \, dx \right)^{1/p} \approx \sup_B \frac{1}{|B|^{1/p}} \|b - b_B\|_{L^p(B)}.$$ 

The next result is easy to check by using (1.1). See also DeVore and Sharpley [16], page 14.

Lemma 2.2. [16] Let $0 < \gamma < 1$, $b \in \dot{A}_\gamma(\mathbb{R}^n)$, and $B$ and $B'$ be balls in $\mathbb{R}^n$. If $B' \subset B$, then

$$|b_{B'} - b_B| \leq C\|b\|_{\dot{A}_\gamma(\mathbb{R}^n)}|B|^{1/n}.$$ 

Now we recall the boundedness of strongly singular Calderón-Zygmund operator $T$ on Lebesgue spaces. Let us observe that $T$ is bounded from $L^{\infty}$ to BMO ([16], Theorem 2.1), from $L^1$ to $L^{1,\infty}$ ([7], Theorem 4.1), and from $H^1$ to $L^1$ ([12], Lemma 2), and note the assumption (3) in Definition 1.1, and by interpolation between these estimates, we achieve the following $L^p$-boundedness of $T$. We refer to [12] (page 1052) and [11] (pages 42 and 43), for details.

Lemma 2.3. Let $T$ be a strongly singular Calderón-Zygmund operator, and $\alpha, \eta, \text{ and } \delta$ be the same as in Definition 1.1.

(i) If $1 < p < \infty$, then $T$ is bounded from $L^p(\mathbb{R}^n)$ to itself.

(ii) If $\frac{m(1-\alpha)+2n}{2q} \leq u < \infty$, then there is a positive number $v$ satisfying $0 < u/v \leq \alpha$, such that $T$ is bounded from $L^u(\mathbb{R}^n)$ to $L^v(\mathbb{R}^n)$.

Furthermore, the index $v$ can be chosen as $v = \frac{uq'}{2q' - uq' + 2n - 2}$ when $\frac{m(1-\alpha)+2n}{2q} \leq u \leq 2$ and $v = \frac{uq'}{2}$ when $2 \leq u < \infty$, where $q$ is given in Definition 1.1 and $q'$ is its conjugate index.

Now, let us prove Theorem 1.1.

Proof of Theorem 1.1. For any $f \in L^p(\mathbb{R}^n)$, it suffices to prove
for all balls $B$ in $\mathbb{R}^n$.

For any ball $B = B(x_0, r)$ centered at $x_0$ with radius $r$, we divide the proof into two cases.

**Case 1. The case when $r > 1$.** Denote by $B' = 8B(x_0, 8r)$ the ball with the same center as $B$ and 8 times the radius. Let $f_1 = f_{B'}$ and $f_2 = f - f_1$. For any real number $c$, by Minkowski’s inequality and Hölder’s inequality, we have

$$
\frac{1}{|B|^{\frac{1}{p}}} \left( \frac{1}{|B|} \int_B |T_B f(y) - (T_B f)_B|^p dy \right)^{1/p} \leq \frac{1}{|B|^{\frac{1}{p}}} \left( \frac{1}{|B|} \int_B |T_B f(y) - c|^p dy \right)^{1/p} + \frac{1}{|B|^{\frac{1}{p}}} \left( \frac{1}{|B|} \int_B |(T_B f)_B - c|^p dy \right)^{1/p} \\
\leq \frac{1}{|B|^{\frac{1}{p}}} \left( \frac{1}{|B|} \int_B |T_B f(y) - c|^p dy \right)^{1/p} + \frac{1}{|B|^{\frac{1}{p}}} \left( \frac{1}{|B|} \int_B |(T_B f)_B - c|^p dy \right)^{1/p} \\
\leq \frac{1}{|B|^{\frac{1}{p}}} \left( \frac{1}{|B|} \int_B |T_B f(y) - c|^p dy \right)^{1/p} + \frac{1}{|B|^{\frac{1}{p}}} \left( \frac{1}{|B|} \int_B |(T_B f)_B - c|^p dy \right)^{1/p} \\
\leq \frac{2}{|B|^{\frac{1}{p}}} \left( \frac{1}{|B|} \int_B |T_B f(y) - c|^p dy \right)^{1/p} + \frac{1}{|B|^{\frac{1}{p}}} \left( \frac{1}{|B|} \int_B |(T_B f)_B - c|^p dy \right)^{1/p} \\
\leq \left( \int_B |T_B f(y) - c|^p dy \right)^{1/p} + \left( \int_B |(T_B f)_B - c|^p dy \right)^{1/p}.
$$

Let $c = -(T((b - b_0)f_2))_B$ and notice that $T_B f = T_{b-b_0} f$, one has

$$
\frac{1}{|B|^{\frac{1}{p}}} \left( \frac{1}{|B|} \int_B |T_B f(y) - (T_B f)_B|^p dy \right)^{1/p} \leq \frac{2}{|B|^{\frac{1}{p}}} \left( \frac{1}{|B|} \int_B |T_{b-b_0} f(y) + (T((b - b_0)f_2))_B|^p dy \right)^{1/p} \\
\leq \frac{2}{|B|^{\frac{1}{p}}} \left( \frac{1}{|B|} \int_B |(b(y) - b_0)TF(y)|^p dy \right)^{1/p} + \frac{2}{|B|^{\frac{1}{p}}} \left( \frac{1}{|B|} \int_B |T((b - b_0)f_1(y)|^p dy \right)^{1/p} \\
+ \frac{2}{|B|^{\frac{1}{p}}} \left( \frac{1}{|B|} \int_B |T((b - b_0)f_2(y) - (T((b - b_0)f_2))_B)|^p dy \right)^{1/p} \\
= I_1 + I_2 + I_3.
$$

For $I_1$, note that $1 < p < \infty$ and $0 < \gamma = \beta + n/p < 1$, it follows from Lemmas 2.1 and 2.3 that

$$
I_1 = \frac{2}{|B|^{\frac{1}{p}}} \left( \frac{1}{|B|} \int_B |(b(y) - b_0)TF(y)|^p dy \right)^{1/p} \leq \frac{2}{|B|^{\frac{1}{p}}} \|b - b_0\|_{L^{\infty}(B)} \left( \frac{1}{|B|} \int_B |Tf(y)|^p dy \right)^{1/p}
$$
Next we estimate $I_2$. Again we note that $1 < p < \infty$ and $0 < \gamma = \beta + n/p < 1$. By Lemmas 2.1, 2.2, and 2.3, we deduce

\[
I_2 \leq \frac{2}{|B|^{\beta/n}} \|T((b - b_0)f_2)\|_{L^\gamma(B')} \\
\leq \frac{C}{|B|^{\beta/n}} \|b - b_0\|_{H^{\beta}(R^n)} \\
\leq \frac{C}{|B|^{\beta/n}} \left\{ \|b - b_0\|_{L^\gamma(B')} + \|(b - b_0)f\|_{L^\gamma(B')} \right\} \\
\leq \frac{C}{|B|^{\beta/n}} \left\{ \|b - b_0\|_{L^\gamma(B')} + |b_0| \right\} \\
\leq \frac{C}{|B|^{\beta/n}} \left\{ C|B|^{\gamma/n} \|b\|_{H^{\beta}(R^n)} + C|b\|_{H^{\beta}(R^n)} |B|^{1/n} \right\} \\
\leq C|b\|_{H^{\beta}(R^n)} |B|^{1/n} \\
\leq C|b\|_{H^{\beta}(R^n)} |B|^{1/n}.
\]

Now, let us consider $I_3$. Since for any $w, y \in B = B(x_0, r)$ and any $z \in (B')^c$ one has $2|y - w|^\alpha < |z - w|$, it follows from Definition 1.1 that

\[
|T((b - b_0)f_2)(y) - T((b - b_0)f_2)(w)| \leq \int_{\mathbb{R}^n} |K(y, z) - K(w, z)||b(z) - b_0||f(z)|dz \\
= \int_{(B')^c} |K(y, z) - K(w, z)||b(z) - b_0||f(z)|dz \\
\leq C \int_{(B')^c} \frac{|y - w|^\delta}{|z - w|^n\delta/\alpha} |b(z) - b_0||f(z)|dz \\
\leq C \sum_{k=1}^{\infty} 2^{-k\delta/\alpha} \int_{2^kB} \frac{|y - w|^\delta}{|z - w|^n\delta/\alpha} |b(z) - b_0||f(z)|dz \\
\leq C2^{-\delta/\alpha} \int_{2^kB} |b(z) - b_0||f(z)|dz.
\]

Observe that the last term of (2.3) is always independent of $w$ and $y$, for any $w, y \in B$. Then we can write

\[
I_3 = \frac{2}{|B|^{\beta/n}} \left( 1 \int_{B} \int_{B} |T((b - b_0)f_2)(y) - T((b - b_0)f_2)(w)|dy \right)^{1/p} \\
\leq \frac{2}{|B|^{\beta/n}} \left( 1 \int_{B} \int_{B} |T((b - b_0)f_2)(y) - T((b - b_0)f_2)(w)|dy \right)^{1/p} \\
\leq \frac{2}{|B|^{\beta/n}} \left( 1 \int_{B} \int_{B} \left[ C2^{-\delta/\alpha} \sum_{k=1}^{\infty} 2^{-k\delta/\alpha} \int_{2^kB} |b(z) - b_0||f(z)|dz \right]dy \right)^{1/p} \\
\leq \frac{C2^{(1-1/\alpha)} \sum_{k=1}^{\infty} 2^{-k\delta/\alpha}}{|B|^{\beta/n}} \int_{2^kB} |b(z) - b_0||f(z)|dz \\
\leq \frac{C2^{(1-1/\alpha)} \sum_{k=1}^{\infty} 2^{-k\delta/\alpha}}{|B|^{\beta/n}} \int_{2^kB} |b(z) - b_0||f(z)|dz + \frac{C2^{(1-1/\alpha)} \sum_{k=1}^{\infty} 2^{-k\delta/\alpha}}{|B|^{\beta/n}} \int_{2^kB} |b(z) - b_0||f(z)|dz \\
= I_{3,1} + I_{3,2}.
\]
Applying Hölder’s inequality, Lemma 2.1, and noting that $0 < \gamma = \beta + n/p < 1$, we obtain

\[
I_{3,1} \leq \frac{C(\delta(1-1/a))}{|B|^{\beta/n}} \sum_{k=1}^{\infty} 2^{k\delta/a} \left( \int_{2^k B^*} |b(z) - b_{2^k B^*}|^{p'} dz \right)^{1/p} \left( \int_{2^k B^*} |f(z)|^p dz \right)^{1/p}
\]

\[
\leq \frac{C(\delta(1-1/a))}{|B|^{\beta/n}} \sum_{k=1}^{\infty} 2^{k\delta/a} 2^{k B^*} |2^{k B^*}|^{p'/n} \frac{\|b\|_{L_p(\mathbb{R}^n)} \|f\|_{L_p(\mathbb{R}^n)}}{|B|^{\beta/n}} \sum_{k=1}^{\infty} 2^{2k|\beta - \delta/a|}
\]

\[
\leq C(\delta(1-1/a)) \frac{\|b\|_{L_p(\mathbb{R}^n)} \|f\|_{L_p(\mathbb{R}^n)}}{|B|^{\beta/n}} \sum_{k=1}^{\infty} 2^{2k|\beta - \delta/a|}
\]

where in the last step we made use of the fact that $r^{\delta(1-1/a)} \leq 1$ since $r > 1$ and $\delta(1 - 1/a) < 0$ and the fact that the series $\sum_{k=1}^{\infty} 2^n |\beta - \delta/a|$ is convergent since $\beta - \delta/a < 0$.

For $I_{3,2}$, noting that $0 < \gamma = \beta + n/p < 1$ and applying Lemma 2.2 and Hölder’s inequality, we have

\[
I_{3,2} = \frac{C(\delta(1-1/a))}{|B|^{\beta/n}} \sum_{k=1}^{\infty} 2^{k\delta/a} \left( \int_{2^k B^*} |b_{2^k B^*} - b_B| |f(z)| dz \right)
\]

\[
\leq \frac{C(\delta(1-1/a))}{|B|^{\beta/n}} \sum_{k=1}^{\infty} 2^{k\delta/a} \left( \int_{2^k B^*} |b_{2^k B^*}| |2^k B^*|^{p'/n} \left( \int_{2^k B^*} |f(z)|^p dz \right)^{1/p} \right)^{1/p} \left| |B|^{\beta/n} \sum_{k=1}^{\infty} 2^{2k|\beta - \delta/a|} \right|
\]

\[
\leq C(\delta(1-1/a)) |B|^{\beta/n} \sum_{k=1}^{\infty} 2^{2k|\beta - \delta/a|}
\]

where in the last step we also made use of the fact that $r^{\delta(1-1/a)} \leq 1$ and $\sum_{k=1}^{\infty} 2^n |\beta - \delta/a|$ is convergent. This, together with the estimates for $I_{3,1}$, yields

\[
I_3 \leq C(\delta(1-1/a)) \frac{\|b\|_{L_p(\mathbb{R}^n)}}{|B|^{\beta/n}} \|f\|_{L_p(\mathbb{R}^n)}
\]

Combining the estimates for $I_1$, $I_2$, and $I_3$ leads to (2.1) for the case $r > 1$.

**Case 2.** The case $0 < r \leq 1$. Set $\tilde{B} = B(x_0, r^n)$ and denote $\tilde{B} = B(x_0, 8r^n)$. Let $f_{\delta} = f_{B^*}$ and $f_{\delta} = f_{B} - f_{\delta}$. For the same reason as that in (2.2), we deduce that

\[
\frac{1}{|B|^{\beta/n}} \left( \frac{1}{|B| \int_B |T_{\tilde{B}} f(y) - (T_{\tilde{B}} f)|^{p} dy \right)^{1/p}
\]

\[
\leq \frac{2}{|B|^{\beta/n+1/p}} \left( \frac{1}{|B| \int_B |(b(y) - b_{\tilde{B}}) Tf(y)|^{p} dy \right)^{1/p} + \frac{2}{|B|^{\beta/n+1/p}} \left( \frac{1}{|B| \int_B |T((b - b_{\tilde{B}}) f_{\delta})(y)|^{p} dy \right)^{1/p}
\]

\[
+ \frac{2}{|B|^{\beta/n+1/p}} \left( \frac{1}{|B| \int_B |T((b - b_{2^k B^*}) f_{\delta})(y)|^{p} dy \right)^{1/p}
\]

\[= I_1 + I_2 + I_3.
\]
Similar to $I_1$, we have

$$J_1 = \frac{2}{|B|^{1/n}} \left( \int_B |(b(y) - b_B)Tf(y)|^p dy \right)^{1/p} \leq C\|b\|_{\mathcal{A}_0(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}.$$

To estimate $J_2$, we first observe that, by Lemma 2.3, there is an $s$ satisfying $0 < p/s \leq \alpha$ such that $T$ is bounded from $L^p$ to $L^s$ since $\frac{n(1-n)+2n}{2n} < p < \infty$. This, together with Hölder’s inequality, Lemmas 2.1 and 2.2, gives

$$J_2 = \frac{2}{|B|^{\beta/n} \cdot s} \left( \int_B |T((b-b_B)f)(y)|^p dy \right)^{1/p} \leq \frac{2}{|B|^{\beta/n} \cdot s} \|T((b-b_B)f)\|_{L^s(\mathbb{R}^n)} \leq \frac{C}{|B|^{\beta/n} \cdot s} \|b\|_{\mathcal{A}_0(\mathbb{R}^n)} \|f\|_{L^s(\mathbb{R}^n)}$$

where the last two steps follow from the condition $0 < \gamma = \beta + n/p < 1$ and the fact that $r^{(a-1)\beta + (a/p - 1/s)n} \leq 1$ since $0 < r \leq 1$ and $(a-1)\beta + (a/p - 1/s)n > 0$.

Finally, let us consider $J_3$. Since $0 < r \leq 1$ and $2|y-w| < |z-w|$ for any $w, y \in B = B(x_0, r)$ and $z \in (B')^c$, similar to (2.3), we have

$$\|T((b-b_B)f_n)(y) - T((b-b_B)f_n)(w)\| \leq \int_{\mathbb{R}^n} |K(y, z) - K(w, z)||f_n(z)|dz,$$

$$\leq C \int_{(B')^c} \frac{|y-w|^6}{|z-w|^{n+6/\beta}} |b(z) - b_B| |f(z)|dz \leq C \sum_{k=1}^{\infty} \int_{2k/2^k < |B'|} \frac{|y-w|^6}{|z-w|^{n+6/\beta}} |b(z) - b_B| |f(z)|dz \leq C \sum_{k=1}^{\infty} \frac{2^{k/\beta}}{2^k} \int_{2k/2^k} |b(z) - b_B| |f(z)|dz.$$

Reasoning as in (2.4), we can also write

$$J_3 = \frac{2}{|B|^{\beta/n}} \left( \int_B \left| T((b-b_B)f_n)(y) - (T((b-b_B)f_n))_B \right|^p dy \right)^{1/p} \leq \frac{C}{|B|^{\beta/n}} \sum_{k=1}^{\infty} \frac{2^{k/\beta}}{2^k} \int_{2k/2^k} |b(z) - b_B| |f(z)|dz \leq \frac{C}{|B|^{\beta/n}} \sum_{k=1}^{\infty} \frac{2^{k/\beta}}{2^k} \int_{2k/2^k} |b(z) - b_B| |f(z)|dz,$$

$$= J_{3,1} + J_{3,2}.$$
To estimate $J_{3,1}$, we first observe the fact that $r^{\beta (a-1)} \leq 1$ since $0 < r \leq 1$ and $\beta (a - 1) > 0$ and the series $\sum_{k=1}^{\infty} 2^{k(\beta - \delta / a)}$ is convergent since $\beta - \delta / a < 0$. Then by Hölder’s inequality and Lemma 2.1 and noticing that $0 < y = \beta + n / p < 1$, we obtain

\[
J_{3,1} = \Bigg\frac{C}{|B|^{\beta / n}} \sum_{k=1}^{\infty} 2^{-k\delta / a} \int_{2^k B^*} |b(z) - b_{2^k B^*}| |f(z)| dz \\
\leq \Bigg\frac{C}{|B|^{\beta / n}} \sum_{k=1}^{\infty} 2^{-k\delta / a} \left( \int_{2^k B^*} |b(z) - b_{2^k B^*}|^{p'} dz \right)^{1 / p'} \|f\|_{L^p(\mathbb{R}^n)} \\
\leq \Bigg\frac{C}{|B|^{\beta / n}} \sum_{k=1}^{\infty} 2^{-k\delta / a} 2^{kB^*} \left| 2^{k\delta} \right| |b|_{L^p(\mathbb{R}^n)}^n |f|_{L^p(\mathbb{R}^n)} \\
\leq \Bigg\frac{C}{|B|^{\beta / n}} \sum_{k=1}^{\infty} 2^{-k\delta / a} 2^{kB^*} \left| 2^{k\delta} \right| |b|_{L^p(\mathbb{R}^n)}^n |f|_{L^p(\mathbb{R}^n)} \\
\leq C\|b\|_{L^p(\mathbb{R}^n)}^n |f|_{L^p(\mathbb{R}^n)} r^{\beta (a-1)} \sum_{k=1}^{\infty} 2^{k(\beta - \delta / a)} \\
\leq C\|b\|_{L^p(\mathbb{R}^n)}^n |f|_{L^p(\mathbb{R}^n)}.
\]

For $J_{3,2}$, applying Lemma 2.2 and Hölder’s inequality, and observing again the fact that $r^{\beta (a-1)} \leq 1$, the series $\sum_{k=1}^{\infty} 2^{k(\beta - \delta / a)}$ is convergent and $0 < y = \beta + n / p < 1$, and we have

\[
J_{3,2} = \Bigg\frac{C}{|B|^{\beta / n}} \sum_{k=1}^{\infty} 2^{-k\delta / a} \int_{2^k B^*} |b_{2^k B^*} - b_{2^k B^*}| |f(z)| dz \\
\leq \Bigg\frac{C}{|B|^{\beta / n}} \sum_{k=1}^{\infty} 2^{-k\delta / a} \left( \int_{2^k B^*} |f(z)| dz \right) \|
\leq \Bigg\frac{C}{|B|^{\beta / n}} \sum_{k=1}^{\infty} 2^{-k\delta / a} 2^{kB^*} \left| 2^{k\delta} \right| |b|_{L^p(\mathbb{R}^n)}^n |f|_{L^p(\mathbb{R}^n)} \\
\leq \Bigg\frac{C}{|B|^{\beta / n}} \sum_{k=1}^{\infty} 2^{-k\delta / a} 2^{kB^*} \left| 2^{k\delta} \right| |b|_{L^p(\mathbb{R}^n)}^n |f|_{L^p(\mathbb{R}^n)} \\
\leq C\|b\|_{L^p(\mathbb{R}^n)}^n |f|_{L^p(\mathbb{R}^n)} r^{\beta (a-1)} \sum_{k=1}^{\infty} 2^{k(\beta - \delta / a)} \\
\leq C\|b\|_{L^p(\mathbb{R}^n)}^n |f|_{L^p(\mathbb{R}^n)}.
\]

The estimate for $J_{3,2}$, together with the ones for $J_{3,1}$, yields

\[
J_3 \leq C\|b\|_{L^p(\mathbb{R}^n)}^n |f|_{L^p(\mathbb{R}^n)}.
\]

Combining the estimates for $J_1$, $J_2$, and $J_3$, we deduce (2.1) for the case $0 < r \leq 1$.

Now, we finish the proof of Theorem 1.1.

\[\Box\]

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