Research Article

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Some results on the value distribution of differential polynomials

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Abstract: In this article, we study some results on the value distribution of differential polynomials and mainly prove the following theorem: suppose that $P$ is a polynomial with $\deg P \geq 3$ and $f$ is a transcendental meromorphic function. Let $\alpha$ be a small function of $f$. If $\alpha$ is a constant, we also require that there exists a constant $A \neq a$ such that $P(z) - A$ has a zero of multiplicity at least 3. Then, for any $0 < \varepsilon < 1$, we have

$$T(r, f) \leq kN\left(r, \frac{1}{P(f) - \alpha}\right) + S(r, f),$$

where if $P'(z)$ has only one zero, then $k = \frac{1}{\deg P - 2}$; if $P'(z)$ has two distinct zeros $a$ and $b$ with $P(a) \neq P(b)$ and $\alpha$ is nonconstant, then $k = \frac{1}{1-\varepsilon}$; otherwise $k = 1$.

Keywords: meromorphic function, small function, deficiency value

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1 Introduction and main results

In this article, a meromorphic function always means meromorphic in the whole complex plane. We use the following standard notations in value distribution theory, see [1–3]: $T(r, f), N(r, f), m(r, f)$, . . . .

We denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$, which is outside of an exceptional set $E$ with finite measure. A meromorphic function $\alpha$ is said to be a small function of $f$ if it satisfies $T(r, \alpha) = S(r, f)$.

Let $f$ be a nonconstant meromorphic function. Define

$$\Theta(\infty, f) = 1 - \lim_{r \to \infty} \frac{N(r, f)}{T(r, f)},$$

and

$$\delta(\infty, f) = 1 - \lim_{r \to \infty} \frac{N(r, f)}{T(r, f)}.$$  

Let $f$ be a nonconstant meromorphic function, let $j$ be a positive integer, let $n_{ij}$, $n_{ij}$, . . . , $n_{ij}$ be non-negative integers, and let $a_j(z)$ be a small function of $f$. We call that $M_j(f) = a_j(z)(f(z))^{n_{ij}}(f'(z))^{n_{ij}} . . . (f^{(k)}(z))^{n_{ij}}$  

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is a differential monomial of \( f \). We define \( d(M_j) = \sum_{i=0}^{k} n_i \) as the degree of \( M_j(f) \). Next, a differential polynomial in \( f \) is a finite sum of such monomials, i.e.,

\[
\varphi(f) = \sum_{j=0}^{n} M_j(f).
\]

(1)

We define \( d(\varphi) = \max_{0 \leq j \leq n} \{d(M_j)\} \) and \( d(\varphi) = \min_{0 \leq j \leq n} \{d(M_j)\} \) as the degree and lower degree of \( \varphi(f) \), respectively.

In 1952, Rosenbloom [4] proved the following theorem.

**Theorem A.** Let \( P \) be a polynomial with \( \deg P \geq 2 \), and let \( f \) be a transcendental entire function. Then,

\[
\lim_{r \to \infty} \frac{N(r, \frac{1}{P(f) - z})}{T(r, f)} \geq 1.
\]

From Theorem A, we obtain

**Theorem B.** Let \( P \) be a polynomial with \( \deg P \geq 2 \), and let \( f \) be a transcendental entire function. Then, \( P(f) \) has infinitely many fix-points.


**Theorem C.** Let \( P \) be a polynomial with \( \deg P \geq 3 \), and let \( f \) be a transcendental meromorphic function. Then, \( P(f) \) has infinitely many fix-points.


**Theorem D.** Suppose that \( P \) is a polynomial with \( \deg P \geq 2 \) and \( f \) is a transcendental entire function. Let \( \alpha \) be a nonconstant small function of \( f \). Then,

\[
T(r, f) \leq kN\left(r, \frac{1}{P(f) - \alpha}\right) + S(r, f),
\]

where \( k = \frac{2}{\deg P - 1} \) if \( P(z) \) has only one zero; otherwise \( k = 2 \).

In 2000, Fang and Yuan [7] improved the above results and proved the following theorem.

**Theorem E.** Suppose that \( P \) is a polynomial with \( \deg P \geq 2 \) and \( f \) is a transcendental entire function. Let \( \alpha \) be a small function of \( f \). If \( \alpha \) is a constant, we also require that there exists a constant \( A \neq \alpha \) such that \( P(z) - A \) has a zero of multiplicity at least 2. Then,

\[
T(r, f) \leq kN\left(r, \frac{1}{P(f) - \alpha}\right) + S(r, f),
\]

where \( k = \frac{1}{\deg P - 1} \) if \( P(z) \) has only one zero; otherwise \( k = 1 \).

Naturally, we pose the following problem.

**Problem 1.** Whether the entire functions of Theorem E can be replaced by meromorphic functions?

In this article, we study this problem and prove the following result.
Theorem 1. Suppose that $P$ is a polynomial with $\deg P \geq 3$ and $f$ is a transcendental meromorphic function. Let $a$ be a small function of $f$. If $a$ is a constant, we also require that there exists a constant $A \neq a$ such that $P(z) - A$ has a zero of multiplicity at least 3. Then, for any $0 < \varepsilon < 1$, we have

$$T(r, f) \leq kN\left(r, \frac{1}{P(f) - A}\right) + S(r, f),$$

where if $P'(z)$ has only one zero, then $k = \frac{1}{\deg P - 2}$; if $P'(z)$ has two distinct zeros $a$ and $b$ with $P(a) \neq P(b)$, and $a$ is nonconstant, then $k = \frac{1}{1 - \varepsilon}$; otherwise $k = 1$.

The following example shows that the condition $\deg P \geq 3$ in Theorem 1 is sharp.

Example 1. Let $P(z) = z^2, f(z) = e^{z}$ and $a(z) = z^2$, then we have $f(z) = \frac{ze^{z} + z}{1 - e^{z}}$ and $P(f) - a = (f - z)(f + z) \neq 0$.

Hence, we obtain

$$N\left(r, \frac{1}{P(f) - a}\right) = 0.$$

The following example shows that the condition $A \neq a$ in Theorem 1 is necessary.

Example 2. Let $P(z) = (z - 1)^3(z - 2) + 1, f(z) = \frac{2e^z - 1}{e^z - 1}$, and $a = A = P(1) = 1$. Hence, we obtain

$$N\left(r, \frac{1}{P(f) - a}\right) = 0.$$

The following example shows that it is necessary that $P(z) - A$ has a zero of multiplicity at least 3 in Theorem 1.

Example 3. Let $P(z) = z^3 - 6z^2 + 9z + 1, f(z) = \frac{4e^z - 1}{e^z - 1}, a = P(1) = 5$, and $A = P(3) = 1$. Hence, we obtain

$$N\left(r, \frac{1}{P(f) - a}\right) = 0.$$

In 1959, Hayman [8] proved the following result.

Theorem F. Let $f$ be a transcendental entire function and $n(\geq 2)$ be a positive integer. Then, $f^n(z)f'(z)$ assumes all finite nonzero values infinitely often.

In 1967, Clunie [9] proved that Theorem F was true for $n = 1$.

In 1969, Sons [10] generalized Theorem F and obtained the following result.

Theorem G. Let $f$ be a transcendental entire function, and let

$$\phi(f(z)) = (f(z))^n(f'(z))^{n_1}(f^{(k)}(z))^{n_k}. \quad (2)$$

(i) If $\phi$ has the form (2), where $n_0 \geq 2, n_k \geq 1$, and $n_i \geq 0$ for $i \neq 0, k$, then $\delta(a, \phi) < 1$ for all $a \neq 0, \infty$.

(ii) If $f$ satisfies $N_0(r, \frac{1}{f}) = o(T(r, f))$, where $N_0(r, \frac{1}{f})$ is the counting function of simple zeros of $f$, and $\phi$ has the form (2), where $n_0 \geq 1, n_k \geq 1$, and $n_i \geq 0$ for $i \neq 0, k$, then $\delta(a, \phi) < 1$ for all $a \neq 0, \infty$.

Yang proved the following theorems.
Theorem H. [11] Suppose that $f$ is a transcendental meromorphic function with
\[ N(r, f) + N\left(r, \frac{1}{f}\right) = S(r, f). \]

Let $\varphi(f)$ be a differential polynomial in $f$ of the form (1) with no constant term. Furthermore, we assume that the degree $n(\geq 2)$ of $\varphi(f)$ and $l_0 < n, 0 \leq l \leq n$ for all $i \neq 0$. Then, $\delta(a, \varphi(f)) < 1$ for all $a \neq 0, \infty$.

Theorem I. [12] Suppose that $f$ is a transcendental meromorphic function with
\[ N(r, f) + N\left(r, \frac{1}{f}\right) = S(r, f). \]

Let $\varphi(f)$ be a differential polynomial in $f$ of the form (1) of degree $n \geq 2$ such that all terms in $\varphi(f)$ have a degree at least two. Then, if $\varphi(f)$ consists of terms of different degrees, i.e., $\varphi(f)$ is not homogeneous, we have
\[ \delta(a, \varphi(f)) \leq 1 - \frac{1}{2n} \]
for all $a \neq \infty$.

In 2008, Bhoosnurmath et al. [13] proved the following result.

Theorem J. Suppose that $f$ is a transcendental meromorphic function with
\[ N(r, f) + N\left(r, \frac{1}{f}\right) = S(r, f). \]

Let $\varphi(f)$ be a differential polynomial in $f$ of the form (1) of degree $d(\varphi)$ and the lower degree $d(\varphi)$. Assume that $\varphi(f)$ does not reduce to a constant.

(i) If $\varphi(f)$ is a homogeneous differential polynomial, then we have
\[ \delta(a, \varphi(f)) = 0 \]
for any $a \neq 0$, i.e., $\varphi(f)$ assumes all finite nonzero complex values infinitely often.

(ii) If $\varphi(f)$ is a non-homogeneous differential polynomial with $2d(\varphi) > d(\varphi)$, then we have
\[ \delta(a, \varphi(f)) \leq 1 - \frac{2d(\varphi) - d(\varphi)}{d(\varphi)} < 1 \]
for any $a \neq 0$, i.e., $\varphi(f)$ assumes all finite nonzero complex values infinitely often.

Naturally, we pose the following problem.

Problem 2. Whether the results of Theorems H–J can be improved or not?

In this article, we study Problem 2 and prove the following theorem.

Theorem 2. Suppose that $f$ is a transcendental meromorphic function with
\[ N(r, f) + N\left(r, \frac{1}{f}\right) = S(r, f). \]

Let $\varphi(f)$ be a differential polynomial in $f$ of the form (1) of degree $d(\varphi) \geq 2$ and the lower degree $d(\varphi) \geq 1$. Assume that $\varphi(f)$ does not reduce to a constant.

(i) If $\varphi(f)$ is a homogeneous differential polynomial, then we have
\[ \Theta(a, \varphi(f)) = 0, \]
where $a(\neq 0)$ is a small function of $f$. 

(ii) If \( \varphi(f) \) is a non-homogeneous differential polynomial, then we have

\[
\Theta(a, \varphi(f)) \leq 1 - \frac{d(\varphi)}{d(\varphi)},
\]

where \( a(\neq 0) \) is a small function of \( f \).

The following example shows that the bound \( 1 - \frac{d(\varphi)}{d(\varphi)} \) in (4) is sharp.

**Example 4.** Let \( f = e^x, \varphi(f) = f^4 - 2ff' \), and \( a = -1 \), then we have

\[
N\left(r, \frac{1}{\varphi(f) - a}\right) = N\left(r, \frac{1}{e^{2x} - 1}\right) \leq 2T(r, e^x) + S(r, f).
\]

By Lemma 1, we obtain

\[
2T(r, e^x) = T(r, e^{2x}) \leq N(r, e^{2x}) + N\left(r, \frac{1}{e^{2x}}\right) + N\left(r, \frac{1}{e^{2x} - 1}\right) + S(r, f) \leq N\left(r, \frac{1}{e^{2x} - 1}\right) + S(r, f).
\]

It follows from (5) and (6) that

\[
N\left(r, \frac{1}{\varphi(f) - a}\right) = 2T(r, e^x) + S(r, f).
\]

Hence, by Lemma 3 and (7), we obtain

\[
\Theta(a, \varphi(f)) = 1 - \frac{d(\varphi)}{d(\varphi)} = \frac{1}{2}.
\]

\section{2 Lemmas}

For the proof of our results, we need the following lemmas.

**Lemma 1.** [1–3] Let \( f \) be a nonconstant meromorphic function, and let \( a_i(i = 1, 2) \) be two distinct small functions of \( f \). Then,

\[
T(r, f) \leq N(r, f) + N\left(r, \frac{1}{f - a_1}\right) + N\left(r, \frac{1}{f - a_2}\right) + S(r, f).
\]

**Lemma 2.** [14] Let \( f \) be a nonconstant meromorphic function, and let \( a_i(i = 1, 2, 3) \) be three distinct small functions of \( f \). Then, for any \( 0 < \varepsilon < 1 \), we have

\[
2T(r, f) \leq N(r, f) + \sum_{i=1}^{3} N\left(r, \frac{1}{f - a_i}\right) + \varepsilon T(r, f) + S(r, f).
\]

**Lemma 3.** [12] Let \( P(z) = a_0z^n + a_{n-1}z^{n-1} + \cdots + a_2z + a_0 \), where \( a_0(\neq 0), a_{n-1}, \ldots, a_0 \) are constants. If \( f \) is a nonconstant meromorphic function, then

\[
T(r, P(f)) = nT(r, f) + S(r, f).
\]

**Lemma 4.** [1–3] Let \( f \) be a nonconstant meromorphic function, and let \( k \) be a positive integer. Then,

\[
m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f).
\]
3 Proof of Theorem 1

Set \( n = \deg P \geq 3 \). We consider two cases.

Case 1. \( \alpha \) is a nonconstant small function of \( f \). Next, we consider two subcases.

Case 1.1. \( P'\) has only a zero \( a \). Set \( A = P(a) \), then we have
\[
P(z) - A = c_1(z - a)^m,
\]
where \( c_1 \) is a nonzero constant.

By Lemma 1 and (8), we obtain
\[
\begin{align*}
T(r,P(f)) &\leq N(r,P(f)) + N\left(r,\frac{1}{P(f) - A}\right) + N\left(r,\frac{1}{P(f) - a}\right) + S(r,P(f)) \\
&\leq N(r,f) + N\left(r,\frac{1}{f - a}\right) + N\left(r,\frac{1}{P(f) - a}\right) + S(r,f) \\
&\leq 2T(r,f) + N\left(r,\frac{1}{P(f) - a}\right) + S(r,f).
\end{align*}
\]

It follows from Lemma 3 and (9) that
\[
(n - 2)T(r,f) \leq N\left(r,\frac{1}{P(f) - a}\right) + S(r,f).
\]

That is,
\[
T(r,f) \leq \frac{1}{\deg P - 2}N\left(r,\frac{1}{P(f) - a}\right) + S(r,f).
\]

Case 1.2. \( P'\) has at least two zeros \( a \), \( b \) (\( a \neq b \)), where the multiplicity of \( a \), \( b \) is \( m_1 \), \( m_2 \), respectively, and \( m_1, m_2 \geq 1 \). Set \( A = P(a) \) and \( B = P(b) \). In the following, we consider two subcases.

Case 1.2.1. \( A \neq B \). We have
\[
P(z) - A = c_2(z - a)^{m_1+1}\phi_1(z), \quad P(z) - B = c_3(z - b)^{m_2+1}\phi_2(z),
\]
where \( c_2 \) and \( c_3 \) are two nonzero constants and \( \phi_1(z) \), \( \phi_2(z) \) are two polynomials with \( \deg \phi_1 = n - m_1 - 1 \) and \( \deg \phi_2 = n - m_2 - 1 \).

By Lemma 2 and (10), we obtain
\[
\begin{align*}
2T(r,P(f)) &\leq N(r,P(f)) + N\left(r,\frac{1}{P(f) - A}\right) + N\left(r,\frac{1}{P(f) - B}\right) + N\left(r,\frac{1}{P(f) - a}\right) + \frac{\varepsilon}{n}T(r,P(f)) \\
&\quad + S(r,P(f)) \\
&\leq N(r,f) + N\left(r,\frac{1}{f - a}\right) + (n - m_1 - 1)T(r,f) + N\left(r,\frac{1}{f - b}\right) + (n - m_2 - 1)T(r,f) \\
&\quad + N\left(r,\frac{1}{P(f) - a}\right) + \varepsilon T(r,f) + S(r,f) \\
&\leq (2n - m_1 - m_2 + 1 + \varepsilon)T(r,f) + N\left(r,\frac{1}{P(f) - a}\right) + S(r,f) \\
&\leq (2n - 1 + \varepsilon)T(r,f) + N\left(r,\frac{1}{P(f) - a}\right) + S(r,f).
\end{align*}
\]

It follows from Lemma 3 and (11) that
\[
(1 - \varepsilon)T(r,f) \leq N\left(r,\frac{1}{P(f) - a}\right) + S(r,f).
\]
That is,

\[ T(r,f) \leq \frac{1}{1 - \varepsilon} N\left( r, \frac{1}{P(f) - a} \right) + S(r,f). \]

Case 1.2.2. \( A = B \). We have

\[ P(z) - A = c_d(z - a)^{m+1}(z - b)^{m+1}\phi_3(z), \]  

where \( c_d \) is a nonzero constant and \( \phi_3(z) \) is a polynomial with \( \deg \phi_3 = n - m_1 - m_2 - 2 \).

By Lemma 1 and (12), we obtain

\[
\begin{align*}
T(r, P(f)) &\leq N(r, P(f)) + N\left( r, \frac{1}{P(f) - A} \right) + N\left( r, \frac{1}{P(f) - a} \right) + S(r, P(f)) \\
&\leq N(r, f) + N\left( r, \frac{1}{f - a} \right) + N\left( r, \frac{1}{f - b} \right) + (n - m_1 - m_2 - 2)T(r, f) + N\left( r, \frac{1}{P(f) - a} \right) \\
&\quad + S(r, f) \\
&\leq (n - m_1 - m_2 + 1)T(r, f) + N\left( r, \frac{1}{P(f) - a} \right) + S(r, f) \\
&\leq (n - 1)T(r, f) + N\left( r, \frac{1}{P(f) - a} \right) + S(r, f).
\end{align*}
\]

It follows from Lemma 3 and (13) that

\[ T(r, f) \leq N\left( r, \frac{1}{P(f) - a} \right) + S(r, f). \]

Case 2. \( \alpha \) is a constant function. Then there exists a constant \( A \neq \alpha \) such that \( P(z) - A \) has a zero of multiplicity at least 3. Hence, we have

\[ P(z) - A = c_3(z - a)^{m}\phi_3(z), \]  

where \( m \geq 3 \), \( c_3 \) is a nonzero constant and \( \phi_3(z) \) is a polynomial with \( \deg \phi_3 = n - m \).

By Lemma 1 and (14), we obtain

\[
\begin{align*}
T(r, P(f)) &\leq N(r, P(f)) + N\left( r, \frac{1}{P(f) - A} \right) + N\left( r, \frac{1}{P(f) - a} \right) + S(r, P(f)) \\
&\leq N(r, f) + N\left( r, \frac{1}{f - a} \right) + (n - m)T(r, f) + N\left( r, \frac{1}{P(f) - a} \right) + S(r, f) \\
&\leq (n - m + 2)T(r, f) + N\left( r, \frac{1}{P(f) - a} \right) + S(r, f) \\
&\leq (n - 1)T(r, f) + N\left( r, \frac{1}{P(f) - a} \right) + S(r, f).
\end{align*}
\]

It follows from Lemma 3 and (15) that

\[ T(r, f) \leq N\left( r, \frac{1}{P(f) - a} \right) + S(r, f). \]

This completes the proof of Theorem 1.

### 3.1 Proof of Theorem 2

From the assumption and (1), we know that \( q(f) \neq 0 \).

In the following, we consider two cases.
Case 1. \( \varphi(f) \) is a homogeneous differential polynomial. Then, by (1), we have

\[
\varphi(f) = \sum_{j=0}^{n} a_j(z)(f(z))^{n_0}(f'(z))^{n_1} \cdots (f^{(k)}(z))^{n_k} \\
= \sum_{j=0}^{n} a_j(z)(f(z))^{n_0 + \cdots + n_k} \left( \frac{f'(z)}{f(z)} \right)^{n_1} \cdots \left( \frac{f^{(k)}(z)}{f(z)} \right)^{n_k} \\
= \sum_{j=0}^{n} b_j(z)(f(z))^{d_\varphi} = B(z)(f(z))^{d_\varphi},
\]

where \( b_j(z) = a_j(z) \left( \frac{f'(z)}{f(z)} \right)^{n_1} \cdots \left( \frac{f^{(k)}(z)}{f(z)} \right)^{n_k} \) and \( B(z) = \sum_{j=0}^{n} b_j(z) \).

Since \( a_j(z) \) is a small function of \( f \), by Lemma 4 and (3), we obtain

\[
m(r, b_j) = m\left(r, a_j \left( \frac{f'}{f} \right)^{n_1} \cdots \left( \frac{f^{(k)}}{f} \right)^{n_k} \right) \leq m(r, a_j) + m\left(r, \left( \frac{f'}{f} \right)^{n_1} \cdots \left( \frac{f^{(k)}}{f} \right)^{n_k} \right) = S(r, f),
\]

and

\[
N(r, b_j) = N\left(r, a_j \left( \frac{f'}{f} \right)^{n_1} \cdots \left( \frac{f^{(k)}}{f} \right)^{n_k} \right) \\
\leq N(r, a_j) + N\left(r, \left( \frac{f'}{f} \right)^{n_1} \cdots \left( \frac{f^{(k)}}{f} \right)^{n_k} \right) \\
\leq O\left( N(r, f) + N\left(r, \frac{1}{f} \right) \right) + S(r, f) \\
\leq O\left( N(r, f) + N\left(r, \frac{1}{f} \right) \right) + S(r, f) = S(r, f).
\]

Hence, we obtain

\[
T(r, b_j) = m(r, b_j) + N(r, b_j) = S(r, f), \quad T(r, B) = T \left(r, \sum_{j=0}^{n} b_j \right) = S(r, f). \tag{17}
\]

By (3), (16), (17), and Lemma 1, we have

\[
T(r, \varphi(f)) \leq N(r, \varphi(f)) + N\left(r, \frac{1}{\varphi(f)} \right) + N\left(r, \frac{1}{\varphi(f) - a} \right) + S(r, \varphi(f)) \\
\leq \underbrace{N(r, Bf^{d_\varphi})}_{\text{By (16)}} + N\left(r, \frac{1}{Bf^{d_\varphi}} \right) + N\left(r, \frac{1}{\varphi(f) - a} \right) + S(r, \varphi(f)) \tag{18}
\]

It follows from (18) that

\[
1 \leq \lim_{r \to \infty} \frac{N\left(r, \frac{1}{\varphi(f) - a} \right)}{T(r, \varphi(f))} + \lim_{r \to \infty} \frac{S(r, \varphi(f))}{T(r, \varphi(f))} = \lim_{r \to \infty} \frac{N\left(r, \frac{1}{\varphi(f) - a} \right)}{T(r, \varphi(f))}.
\]

Hence, we obtain \( \Theta(a, \varphi(f)) = 0 \).
Case 2. \( \varphi(f) \) is a non-homogeneous differential polynomial. Then by (1), we have
\[
\varphi(f) = \sum_{j=0}^{n} a_j(f(z))^{n_0} (f'(z))^{n_1} \cdots (f^{(k)}(z))^{n_k} \\
= \sum_{j=0}^{n} a_j(f(z))^{n_0 + n_j} \left( \frac{f'(z)}{f(z)} \right)^{n_1} \cdots \left( \frac{f^{(k)}(z)}{f(z)} \right)^{n_k} \\
= \sum_{j=0}^{n} b_j(f(z)) f(z) = \sum_{i=0}^{s} d_i(f(z))^{m_i},
\]
where \( s \leq n, \ b_j(z) = a_j(z) \left( \frac{f'(z)}{f(z)} \right)^{n_1} \cdots \left( \frac{f^{(k)}(z)}{f(z)} \right)^{n_k} \), \( l_j = n_{j_0} + n_{j_1} + \cdots + n_{j_k}, \ d(z) \neq 0 \), and \( m_0, m_1, \ldots, m_s \) are positive integers with \( d(\varphi) = m_0 < m_1 < \cdots < m_s = d(\varphi) \).

By using the same argument as used in Case 1, we know that \( b_j(z) \) and \( d(z) \) are small functions of \( f \).

By Lemma 3, we obtain
\[
T(r, \varphi(f)) = d(\varphi) T(r, f) + S(r, f).
\]

It follows from (19), (20), (3), and Lemma 1 that
\[
T(r, \varphi(f)) \leq N(r, \varphi(f)) + N\left(r, \frac{1}{\varphi(f)} \right) + N\left(r, \frac{1}{\varphi(f) - a} \right) + S(r, \varphi(f)) \\
\leq N\left(r, \frac{1}{d \varphi f^{m_0} + \cdots + d \varphi f^{m_s}} \right) + N\left(r, \frac{1}{\varphi(f) - a} \right) + S(r, \varphi(f)) \\
\leq N\left(r, \frac{1}{m_s - m_0} \right) T(r, f) + N\left(r, \frac{1}{\varphi(f) - a} \right) + S(r, \varphi(f)) \\
= (d(\varphi) - d(\varphi)) T(r, f) + N\left(r, \frac{1}{\varphi(f) - a} \right) + S(r, \varphi(f)) \\
= \frac{d(\varphi) - d(\varphi)}{d(\varphi)} T(r, \varphi(f)) + N\left(r, \frac{1}{\varphi(f) - a} \right) + S(r, \varphi(f)).
\]

By (21), we have
\[
\frac{d(\varphi)}{d(\varphi)} T(r, \varphi(f)) \leq N\left(r, \frac{1}{\varphi(f) - a} \right) + S(r, \varphi(f)).
\]

It follows that
\[
\frac{d(\varphi)}{d(\varphi)} \leq \lim_{r \to \infty} \frac{N\left(r, \frac{1}{\varphi(f) - a} \right)}{T(r, \varphi(f))} + \lim_{r \to \infty} \frac{S(r, \varphi(f))}{T(r, \varphi(f))} = \lim_{r \to \infty} \frac{N\left(r, \frac{1}{\varphi(f) - a} \right)}{T(r, \varphi(f))}.
\]

Hence, we have \( \Theta(a, \varphi(f)) \leq 1 - \frac{d(\varphi)}{d(\varphi)} \).

This completes the proof of Theorem 2.

**Remark 1.** In Theorem 2, if \( a(z) \equiv 0 \), we have the following results.

If \( \varphi(f) \) is a homogeneous differential polynomial. Then, by (3), (16), and (17), we have
\[
N\left(r, \frac{1}{\varphi(f)} \right) = N\left(r, \frac{1}{B \varphi f} \right) = S(r, f).
\]
If \( \varphi(f) \) is a non-homogeneous differential polynomial. Then, by (3) and (19), we obtain

\[
T(r,f) \leq N\left(r, \frac{1}{\varphi(f)} \right) + S(r,f)
\]

\[
= N\left(r, \frac{1}{d_0 f^{m_0} + d_1 f^{m_1} + \ldots + d_s f^{m_s}} \right) + S(r,f)
\]

\[
\leq (m_s - m_0) T(r,f) + S(r,f)
\]

\[
= (d(\varphi) - d(\varphi)) T(r,f) + S(r,f).
\]

That is,

\[
T(r,f) \leq N\left(r, \frac{1}{\varphi(f)} \right) + S(r,f) \leq (d(\varphi) - d(\varphi)) T(r,f) + S(r,f).
\]  \hfill (22)

**Example 5.** Let \( f = e^z \) and \( \varphi(f) = f^4 - 2f^3 \). Hence, we obtain

\[
N\left(r, \frac{1}{\varphi(f)} \right) = N\left(r, \frac{1}{e^{2z} - 2} \right) \leq 2T(r, e^z) + S(r,f).
\]  \hfill (23)

By Nevanlinna’s second fundamental theorem, we have

\[
2T(r, e^z) = T(r, e^{2z}) \leq N\left(r, e^{2z} \right) + N\left(r, \frac{1}{e^{2z} - 2} \right) + S(r,f) \leq N\left(r, \frac{1}{e^{2z} - 2} \right) + S(r,f).
\]  \hfill (24)

It follows from (23) and (24) that

\[
N\left(r, \frac{1}{\varphi(f)} \right) = 2T(r, e^z) + S(r,f) = (d(\varphi) - d(\varphi)) T(r,f) + S(r,f).
\]

This example shows that the bound \((d(\varphi) - d(\varphi)) T(r,f) + S(r,f)\) in (22) is sharp.

**Example 6.** Let \( f = e^z \) and \( \varphi(f) = f^3 - f^2 \). Then, by using the same argument as used in Example 5, we know that \( N\left(r, \frac{1}{\varphi(f)} \right) = T(r,f) + S(r,f) \). Hence, the bound \( T(r,f) + S(r,f) \) in (22) is sharp.

**Remark 2.** Condition (3) is replaced by \( N\left(r, \frac{1}{f} \right) + N\left(r, \frac{1}{f} \right) = S(r,f), \) Theorem 2 still holds.

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**References**


