Non-solid cone $b$-metric spaces over Banach algebras and fixed point results of contractions with vector-valued coefficients

Abstract: In this article, without requiring solidness of the underlying cone, a kind of new convergence for sequences in cone $b$-metric spaces over Banach algebras and a new kind of completeness for such spaces, namely, wrtn-completeness, are introduced. Under the condition that the cone $b$-metric spaces are wrtn-complete and the underlying cones are normal, we establish a common fixed point theorem of contractive conditions with vector-valued coefficients in the non-solid cone $b$-metric spaces over Banach algebras, where the coefficients $s \geq 1$. As consequences, we obtain a number of fixed point theorems of contractions with vector-valued coefficients, especially the versions of Banach contraction principle, Kannan’s and Chatterjea’s fixed point theorems in non-solid cone $b$-metric spaces over Banach algebras. Moreover, some valid examples are presented to support our main results.

Keywords: non-solid cone $b$-metric spaces over Banach algebras, contractions with vector-valued coefficients, coincidence point, fixed point theorems, wrtn-convergence and wrtn-completeness

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1 Introduction

As is known to all, the fixed point theory with regard to modern metric was developed from the classical Banach contraction principle (see [1]), which is important and useful in almost all fields of applied mathematical analysis. Afterwards, Kannan [2] and Chatterjea [3] proved the fixed point theorems as follows.

Theorem. Let $(X, d)$ be a complete metric space and $T : X \rightarrow X$ be a mapping such that there exists $\gamma \in \left[0, \frac{1}{2}\right]$ satisfying

$$d(Tx, Ty) \leq \gamma (d(x, Tx) + d(y, Ty)) \text{ for all } x, y \in X,$$

or

$$d(Tx, Ty) \leq \gamma (d(x, Ty) + d(y, Tx)) \text{ for all } x, y \in X.$$

Then $T$ has a unique fixed point in $X$. 
Some interesting fixed point theorems in $K$-metric and $K$-normed linear spaces were obtained by Perov [4], Vandergraft [5], Zabrejko [6], and relevant literature therein. In 2007, these spaces was reintroduced by Huang and Zhang [7] under the definition of cone metric spaces and some fixed point theorems in such spaces were discussed. Since then, the cone metric fixed point theory is prompted to investigate by lots of authors; for detail, see [8–27] and the references therein.

In the cone metric spaces, the distance between $x$ and $y$ is defined by a vector in an ordered Banach space $E$, quite different from that which is defined by a non-negative real number in usual metric spaces. They indicated the corresponding version of Banach contraction principle and some preliminary properties in cone metric spaces. Later on, by deleting the normality of the cone in [7], Rezapour and Hamlbarani [8] discussed some fixed point results, which are generalizations of the correlative results in [7]. Since then, a lot of authors have been attracted to the field of fixed point theory in cone metric spaces and more general ones-cone $b$-metric spaces (see [7–28]), while the concept of cone $b$-metric spaces was defined by Hussain and Shah [9] and Shah et al. [10], which is an extension of cone metric spaces and $b$-metric spaces. The authors also investigated some meaningful topological properties and fixed point theorems in their work. In 2013, Huang and Xu [11] established some fixed point results and gave the relevant application in cone $b$-metric spaces. Marian and Branga [12] obtained some common fixed point results for a pair of mappings in $b$-cone metric spaces, but the main methods in their work relied strongly on a nonlinear scalarization function $\xi : Y \to \mathbb{R}$. In [13], Shi and Xu also obtained common fixed point theorems in non-normal but solid cone $b$-metric spaces. However, these results were dependent on the solidness of the underlying cones.

Recently, Liu and Xu [14] and Han and Xu [15] continued to further investigate cone metric spaces by means of Banach algebras, instead of Banach spaces and considering the contractive constants to be vectors. They presented some fixed point theorems of generalized Lipschitz contractions with the assumption that the underlying cones are solid. Since 1997, in general, almost all the results obtained in the present literature except [16] have been showed in the setting of cone metric spaces (or cone $b$-metric spaces) with the assumption that the underlying cones are solid. In fact, solidness of the cone is an important (sometime, crucial) condition, since without it the interior points could not be used to define the convergence of the sequences in a natural way. Nevertheless, some results were obtained by Kunze et al. in the case when the cone was not solid in [16]. However, as Janković et al. [17] indicated, the approach appearing in [16] is quite restrictive since some strong assumptions (separability and reflexivity of the space) are used. Because there are many examples of cone metric spaces with empty interiors of the cones, a corresponding fixed point theory would be welcome.

In this article, inspired by [16] and [17], we try to establish fixed point theory for contractions with vector-valued coefficients in the setting of cone $b$-metric spaces (and as their special cases, cone metric spaces) over Banach algebras by deleting the solidness of the underlying cones. Furthermore, we apply some main results to solve the existence and uniqueness of the solution for nonlinear integral equations.

## 2 Preliminaries

Firstly, let us recall some preliminary concepts of Banach algebras and cone $b$-metric spaces.

Let $\mathcal{A}$ be a Banach algebra over $K = \mathbb{R}$ or $\mathbb{C}$. That is, $\mathcal{A}$ is a Banach space in which an operation of multiplication is defined, subject to the following properties for all $x, y, z \in \mathcal{A}$, $a \in K$:

1. $(xy)z = x(yz);$  
2. $x(y + z) = xy + xz$ and $(x + y)z = xz + yz;$  
3. $a(xy) = (ax)y = x(ay);$  
4. $\|xy\| \leq \|x\| \|y\|$. 


The vector $e \in \mathcal{A}$ is called a unit (i.e., a multiplicative identity) in $\mathcal{A}$ if $ex = xe = x$ for all $x \in \mathcal{A}$. A Banach algebra is called unital if it has a unit. An element $x \in \mathcal{A}$ is said to be invertible if there is an inverse element $y \in \mathcal{A}$ such that $xy = yx = e$. The inverse of $x$ is denoted by $x^{-1}$. Here, $x^n = x \cdot x \cdot \cdots \cdot x$ is the $n$-fold product of $x$ with itself, and $x^0 = e$. It is obvious that $\|x^n\| \leq \|x\|^n$ for all $n \in \mathbb{N}$. Let $x \in \mathcal{A}$. The spectrum $\sigma(x)$ of $x$ is the set of all $\lambda \in \mathbb{K}$ such that $\lambda e - x$ is not invertible. The spectral radius $r_\mathcal{A}(x)$ of $x$ is defined by

$$r_\mathcal{A}(x) = \sup\{ |\lambda| : \lambda \in \sigma(x) \}.$$  

For more details, we refer to [18]. A well-known fact is that the spectrum is a non-empty compact subset of the complex plane. The following proposition for the spectral radius formula is well known (see [21]).

A subset $P$ of $\mathcal{A}$ is called a cone if:

1. $P$ is non-empty closed and $\{ \theta, e \} \subset P$, where $\theta$ denotes the null of the Banach algebra $\mathcal{A}$;
2. $aP + \beta P \subset P$ for all non-negative real numbers $a, \beta$;
3. $P^2 = PP \subset P$;
4. $P \cap (-P) = \{ \theta \}$.

For a given cone $P \subset \mathcal{A}$, we can define a partial ordering $\preccurlyeq$ with respect to $P$ by $x \preccurlyeq y$ if and only if $y - x \in P, x \preccurlyeq y$ will stand for $x \preccurlyeq y$ and $x \not\preccurlyeq y$, while $x \prec y$ will stand for $y - x \in \text{int}P$, where $\text{int}P$ denotes the interior of $P$.

The cone $P$ is called normal if there is a number $M > 0$ such that for all $x, y \in \mathcal{A}$,

$$\theta \preccurlyeq x \preccurlyeq y \text{ implies } \|x\| \leq M\|y\|.$$  

The least positive number satisfying above is called the normal constant of $P$.

In the following, unless otherwise specified, we always assume that $\mathcal{A}$ is a real unital Banach algebra (the term “real” means that the algebra is over $\mathbb{R}$) with a unit, $P$ is a cone in Banach algebra $\mathcal{A}$ and $\preccurlyeq$ is the partial ordering with respect to $P$.

**Definition 2.1.** (See [9,10,13]) Let $X$ be a nonempty set and $s \geq 1$ be a given real number. A mapping $d : X \times X \rightarrow \mathcal{A}$ is said to be cone $b$-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $\theta \prec d(x, y)$ with $x \neq y$ and $d(x, y) = \theta$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \preceq s[d(x, z) + d(z, y)]$.

The pair $(X, d)$ is called a cone $b$-metric space over a Banach algebra $\mathcal{A}$.

**Definition 2.2.** (See [9,10,13]) Let $(X, d)$ be a cone $b$-metric space over a Banach algebra $\mathcal{A}$ with the coefficient $s \geq 1$, $x \in X$ where the underlying cone $P$ is solid and let $\{x_n\}$ be a sequence in $X$. Then we say

(i) $\{x_n\}$ converges to $x$ with respect to the solidness of $P$ (for convenience, we say $\{x_n\}$ wrts-converges to $x$) whenever for every $\varepsilon \in \mathcal{A}$ with $\theta \prec \varepsilon$ there is a positive number $N \in \mathbb{N}$ such that $d(x_n, x) \prec \varepsilon$ for all $n \geq N$. (Note that here “wrts” means “with respect to solidness.”) We denote this by $x_n \xrightarrow{\text{wrts}} x(n \rightarrow +\infty)$.
(ii) $\{x_n\}$ is a wrts-Cauchy sequence whenever for every $\varepsilon \in \mathcal{A}$ with $\theta \prec \varepsilon$ there is a positive number $N \in \mathbb{N}$ such that $d(x_n, x_m) \prec \varepsilon$ for all $n, m \geq N$.
(iii) $(X, d)$ is wrts-complete if every wrts-Cauchy sequence is wrts-convergent.

**Definition 2.3.** Let $(X, d)$ be a cone $b$-metric space over a Banach algebra. Let $x \in X$ and $\{x_n\}$ be a sequence in $X$. Then we say

(i) $\{x_n\}$ converges to $x$ with respect to the norm of $E$ (for convenience, we may say $\{x_n\}$ wrtn-converges to $x$) if and only if $|d(x_n, x)| \rightarrow 0$ as $n \rightarrow +\infty$ (Note that here “wrtn” means “with respect to normality.”)

We denote this by $x_n \xrightarrow{\text{wrtn}} x(n \rightarrow +\infty)$.
(ii) $\{x_n\}$ is a wrtn-Cauchy sequence if and only if $|d(x_n, x_m)| \rightarrow 0$ as $n, m \rightarrow +\infty$.
(iii) $(X, d)$ is wrtn-complete if every wrtn-Cauchy sequence is wrtn-convergent.
Lemma 2.1. Let \((X, d)\) be a cone b-metric space over a Banach algebra with coefficient \(s \geq 1\). Let \(P\) be a normal cone with the normal constant \(M\). The limit of a wrtn-convergent sequence in a cone b-metric space over Banach algebra \(\mathcal{A}\) is unique. That is, if for any given sequence \(\{x_n\}_{n \in \mathbb{N}} \subseteq X\), there exist \(x, y \in X\) such that \(x_n \xrightarrow{\text{wrtn}} x\) and \(x_n \xrightarrow{\text{wrtn}} y\) as \(n \to +\infty\), then \(x = y\).

**Proof.** Since 
\[
d(x, y) \leq s[d(x_n, x) + d(x_n, y)],
\]
by the normality of \(P\), we obtain 
\[
|d(x, y)| \leq sM\|d(x_n, x) + d(x_n, y)\| \leq sM(\|d(x_n, x)\| + \|d(x_n, y)\|).
\]
So it follows from the wrtn-convergence of \(\{x_n\}\) that \(d(x, y) = 0\), i.e., \(x = y\). \(\square\)

**Remark 2.1.** Taking \(s = 1\), we can obtain the corresponding definitions and properties in non-solid cone metric spaces over Banach algebras. Similar to the ones presented earlier, they can be obtained in non-solid cone metric spaces and non-solid cone b-metric spaces. Moreover, by [7, Lemmas 1, 4], if the cone \(P\) is solid and normal, we can check that \(x_n \xrightarrow{\text{wrtn}} x(n \to +\infty)\) if and only if \(x_n \xrightarrow{\text{wrtn}} x(n \to +\infty)\); \((X, d)\) is a wrts-complete cone metric space if and only if \((X, d)\) is a wrtn-complete cone metric space.

**Lemma 2.2.** (See [18]) Let \(\mathcal{A}\) be a unital Banach algebra and \(x \in \mathcal{A}\). Then the spectral radius \(r_{\mathcal{A}}(x)\) of \(x\) satisfies
\[
r_{\mathcal{A}}(x) = \lim_{n \to +\infty} \|x^n\|^\frac{1}{n} = \inf_{n \in \mathbb{N}} \|x^n\|^\frac{1}{n}.
\]
In particular, \(r_{\mathcal{A}}(x) = r_{\mathcal{A}}(-x)\) and \(r_{\mathcal{A}}(x) \leq \|x\|\).

**Lemma 2.3.** (See [18,19]) Let \(\mathcal{A}\) be a unital Banach algebra with a unit \(e\) and \(x, y \in \mathcal{A}\). If \(x\) commutes with \(y\), then the following hold:

(i) \(r_{\mathcal{A}}(x + y) \leq r_{\mathcal{A}}(x) + r_{\mathcal{A}}(y)\);

(ii) \(r_{\mathcal{A}}(xy) \leq r_{\mathcal{A}}(x)r_{\mathcal{A}}(y)\);

(iii) \(|r_{\mathcal{A}}(x) - r_{\mathcal{A}}(y)| \leq |r_{\mathcal{A}}(x - y)|\).

In the following, we establish some useful and auxiliary well-known results on Banach algebra. Their proofs can be found in some literature, but we show them for the sake of completeness and the readers’ convenience.

**Lemma 2.4.** Let \(\mathcal{A}\) be a unital Banach algebra with a unit and \(x \in \mathcal{A}\). If the spectral radius \(r_{\mathcal{A}}(x)\) of \(x\) is less than 1, then \(e - x\) is invertible. Moreover,
\[
(e - x)^{-1} = \sum_{i=0}^{+\infty} x^i.
\]

**Proof.** Let \(y = r_{\mathcal{A}}(x) < 1\). By Lemma 2.2, \(\lim_{n \to +\infty} \|x^n\|^\frac{1}{n} = y < 1\). Then there is a real number \(\varepsilon > 0\) such that \(y + \varepsilon < 1\). For such \(\varepsilon > 0\), there exists \(N_{\varepsilon} > 0\) such that for any \(n \in \mathbb{N}\) with \(n \geq N_{\varepsilon}\), we have 
\[
\|x^n\|^\frac{1}{n} < y + \varepsilon
\]
or
\[
\|x^n\| < (y + \varepsilon)^n. \quad (2.1)
\]
For every \(n \in \mathbb{N}\), write \(C_n = \sum_{i=0}^{n} x^i\). For any \(m, n \in \mathbb{N}\) with \(m > n \geq N_{\varepsilon}\), by (2.1), we obtain 
\[
\|C_m - C_n\| = \left\| \sum_{i=n+1}^{m} x^i \right\| \leq \sum_{i=n+1}^{+\infty} \|x^i\| \leq \sum_{i=n+1}^{+\infty} (y + \varepsilon)^i = (y + \varepsilon)^{n+1}(1 - y - \varepsilon)^{-1}.
\]
This means \( \{C_n\} \) is a Cauchy sequence in \( \mathcal{A} \). By the completeness of \( \mathcal{A} \), there exists a point \( C \in \mathcal{A} \) such that \( C = C = \sum_{i=0}^{\infty} x^i \). Now we begin to prove

\[
C(e - x) = (e - x)C = e.
\]

It is easy to see that

\[
C_n(e - x) = (e - x)C_n = e - x^{n+1} \quad \text{for all } n \in \mathbb{N}.
\]

When \( n \) is large enough, (2.1) shows

\[
\|x^{n+1}\| < (y + e)^{n+1},
\]

and hence,

\[
\|x^{n+1}\| \to 0 \quad \text{as } n \to +\infty.
\]

Therefore, we prove \( C(e - x) = (e - x)C = e \). So \( e - x \) is invertible and \( (e - x)^{-1} = C = \sum_{i=0}^{\infty} x^i \). \( \square \)

It is worth mentioning that Lemma 2.4 can be rewritten as follows.

**Lemma 2.5.** Let \( \mathcal{A} \) be a unital Banach algebra with a unit \( e \) and \( x \in \mathcal{A} \). If \( r_\mathcal{A}(x) < |\lambda| \) for some non-zero complex constant \( \lambda \), then \( \lambda e - x \) is invertible. Moreover,

\[
(\lambda e - x)^{-1} = \sum_{i=0}^{\infty} \frac{x^i}{\lambda^{i+1}}.
\]

By (2.1) and the proof of Lemma 2.4, we are led to the following lemma.

**Lemma 2.6.** (See [19]) Let \( \mathcal{A} \) be a Banach algebra and \( x \in \mathcal{A} \). If \( r_\mathcal{A}(x) < 1 \), then \( \lim_{n \to \infty} \|x^n\| = 0 \).

**Lemma 2.7.** Let \( \mathcal{A} \) be a Banach algebra with a normal cone \( P \). If \( x, y \in \mathcal{A} \) satisfy \( \theta \leq x \preceq y \) and \( x \) commutes with \( y \), then the following hold:

1. \( \theta \leq x^n \preceq y^n \) for all \( n \in \mathbb{N} \);
2. \( r_\mathcal{A}(x) \preceq r_\mathcal{A}(y) \).

In particular, if \( A \) is commutative, then the spectral radius \( r_\mathcal{A} \) is monotone with respect to \( P \).

**Proof.** The proof of conclusion (1) is obtained by mathematic induction on \( n \). Clearly, conclusion (1) is true for \( n = 1 \). Since \( \theta \leq x \preceq y \), we have \( x \in P \), \( y \in P \) and \( y - x \in P \). By \( PP \subseteq P \) and \( x \) commutes with \( y \), we see that \( x^2 \in P \), \( xy - x^2 = x(y - x) \in P \) and \( y^2 - xy = y(y - x) \in P \), which imply

\[
y^2 - x^2 = (y^2 - xy) + (xy - x^2) \in P.
\]

Thus, \( \theta \leq x^2 \preceq y^2 \) and conclusion (1) is true for \( n = 2 \). Assume that conclusion (1) is true for \( n = k \in \mathbb{N} \). Then \( x^k \in P \) and \( y^k - x^k \in P \). Since

\[
x^k y - x^{k+1} = x^k (y - x) \in P
\]

and

\[
y^{k+1} - y^k x = y(y^k - x^k) \in P,
\]

we obtain

\[
y^{k+1} - x^{k+1} = (y^{k+1} - x^{k+1}) + (x^{k+1} - x^{k+1}) \in P,
\]

which means \( x^{k+1} \leq y^{k+1} \) holds. Hence, conclusion (1) is also true for \( n = k + 1 \). This means conclusion (1) is true for all \( n \in \mathbb{N} \).

To see conclusion (2), let \( M \) be the normal constant of \( P \). By conclusion (1), we have \( \|x^n\| \leq M\|y^n\| \) for all \( n \in \mathbb{N} \). So it follows that
\[ r_{\mathcal{R}}(x) = \lim_{n \to +\infty} \|x^n\| \leq \lim_{n \to +\infty} \|M\|y^n\| = \lim_{n \to +\infty} M^2 \cdot \lim_{n \to +\infty} (\|y^n\|) = r_{\mathcal{R}}(y), \]

which completes the proof. \qed

**Definition 2.4.** (See [20]) Let \( X \) be a set. The mappings \( f, g : X \to X \) are said to be weakly compatible, if for every \( x \in X \) holds \( ffx = gfx \) whenever \( fx = gx \).

**Definition 2.5.** (See [21]) Let \( f \) and \( g \) be self-maps of a set \( X \). If \( w = fx = gx \) for some \( x \in X \), then \( x \) is called a coincidence point of \( f \) and \( g \), and \( w \) is called a point of coincidence of \( f \) and \( g \).

**Lemma 2.8.** (See [21]) Let \( f \) and \( g \) be weakly compatible self-maps of a set \( X \). If \( f \) and \( g \) have a unique point of coincidence \( w = fx = gx \), then \( w \) is the unique common fixed point of \( f \) and \( g \).

### 3 Fixed point theorems in non-solid cone \( b \)-metric spaces over Banach algebras

In this section, we always suppose that \( P \) is a normal cone with normal constant \( M \geq 1 \). Under the aforementioned definitions and lemmas, some common fixed point results for two weakly compatible self-mappings in non-solid cone \( b \)-metric spaces over Banach algebras are presented.

**Theorem 3.1.** Let \((X, d)\) be a wrtn-complete cone \( b \)-metric space over a Banach algebra \( \mathcal{A} \) with the coefficient \( s \geq 1 \) and the mappings \( f, g : X \to X \). Let \( a_1, a_2, a_3, a_4, a_5 \in P \) and let \( k = a_2 + a_3 + s(a_4 + a_5) \) satisfying

\[ 2sr_{\mathcal{A}}(a_1) + (s + 1)r_{\mathcal{A}}(k) < 2. \quad (3.1) \]

Assume that

- (H1) the range of \( g \) contains the range of \( f \), that is, \( f(X) \subseteq g(X) \),
- (H2) \( g(X) \) or \( f(X) \) is a complete subspace of \( X \),
- (H3) \( a_1 \) commutes with \( k \),
- (H4) for all \( x, y \in X \), one has

\[ d(fx, fy) \leq a_1d(gx, gy) + a_2d(gx, fx) + a_3d(gy, fy) + a_4d(gx, fy) + a_5d(gy, fx). \quad (3.2) \]

Then \( f \) and \( g \) have a unique coincidence point in \( X \). Furthermore, if \( f \) and \( g \) are weakly compatible, then they have a unique common fixed point in \( X \).

**Proof.** Let \( x_0 \in X \) be given. By (H1), there exists an \( x_1 \in X \) such that \( f(x_0) = gx_1 \), then we can choose a sequence \( \{x_n\}_{n \in \mathbb{N} \cup \{0\}} \) such that \( f(x_n) = gx_{n+1} \) for \( n \in \mathbb{N} \cup \{0\} \) by induction. If for some natural number \( \hat{n} \), \( gx_{\hat{n}} = gx_{\hat{n}+1} = fx_{\hat{n}} \), then \( x_{\hat{n}} \) is a coincidence point of \( f \) and \( g \) in \( X \). We assume that \( gx_{n+1} \neq gx_n \) for all \( n \in \mathbb{N} \). For any \( n \in \mathbb{N} \), by (H4), we have

\[ d(gx_{n+1}, gx_n) = d(fx_n, fx_{n-1}) \]

\[ \leq a_1d(gx_n, gx_{n-1}) + a_2d(gx_n, fx_n) + a_3d(gx_{n-1}, fx_{n-1}) + a_4d(gx_{n+1}, fx_{n+1}) + a_5d(gx_{n+1}, fx_{n+1}) \]

\[ = a_1d(gx_n, gx_{n-1}) + a_2d(gx_n, gx_{n+1}) + a_3d(gx_{n-1}, gx_{n+1}) + a_4d(gx_{n+1}, gx_{n+1}) \]

\[ \leq (a_1 + a_2 + sa_5)d(gx_{n-1}, gx_n) + (a_2 + sa_4)d(gx_n, gx_{n+1}) \]

and

\[ d(gx_n, gx_{n+1}) = d(fx_{n-1}, fx_n) \]

\[ \leq a_1d(gx_{n-1}, gx_n) + a_2d(gx_{n-1}, fx_n) + a_3d(gx_n, fx_{n-1}) + a_4d(gx_{n+1}, fx_n) + a_5d(gx_{n+1}, fx_{n-1}) \]

\[ = a_1d(gx_{n-1}, gx_n) + a_2d(gx_{n-1}, gx_n) + a_3d(gx_{n-1}, gx_{n+1}) + a_4d(gx_{n-1}, gx_{n+1}) \]

\[ \leq (a_1 + a_2 + sa_5)d(gx_{n-1}, gx_n) + (a_2 + sa_4)d(gx_n, gx_{n+1}). \]
Hence, for any \( n \in \mathbb{N} \), we obtain
\[
2d(\mathit{gx}_n, \mathit{gx}_{n+1}) \leq (2a_1 + a_2 + a_3 + sa_4 + sa_5)\mathit{d}(\mathit{gx}_{n-1}, \mathit{gx}_n) + (a_2 + a_3 + sa_4 + sa_5)\mathit{d}(\mathit{gx}_n, \mathit{gx}_{n+1}).
\]
Since \( k = a_2 + a_3 + sa_4 + sa_5 \), it follows from the last inequality that
\[
(2e - k)\mathit{d}(\mathit{gx}_n, \mathit{gx}_{n+1}) \leq (2a_1 + k)\mathit{d}(\mathit{gx}_{n-1}, \mathit{gx}_n) \quad \text{for any } n \in \mathbb{N}. \tag{3.3}
\]
By (H3), it is easy to see that \( 2sa_1 \) commutes with \( (s + 1)k \) and \( k \) commutes with \( 2sa_1 + (s + 1)k \). Since
\[
k \leq 2sa_1 + (s + 1)k,
\]
and by Lemma 2.3 and (3.1), we obtain
\[
r_{\mathcal{A}}(k) \leq r_{\mathcal{A}}(2sa_1 + (s + 1)k) \leq 2sr_{\mathcal{A}}(a_1) + (s + 1)r_{\mathcal{A}}(k) < 2.
\]
By Lemma 2.5, \( 2e - k \) is invertible and
\[
(2e - k)^{-1} = \sum_{i=0}^{+\infty} \frac{k^i}{2^{i+1}}.
\]
We claim
\[
r_{\mathcal{A}}((2e - k)^{-1}) \leq \sum_{i=0}^{+\infty} \frac{[r_{\mathcal{A}}(k)]^i}{2^{i+1}}. \tag{3.4}
\]
In fact, for any \( n \in \mathbb{N} \), by Lemma 2.3, we have
\[
r_{\mathcal{A}}(\sum_{i=0}^{n} \frac{k^i}{2^{i+1}}) \leq \sum_{i=0}^{n} r_{\mathcal{A}}(\frac{k^i}{2^{i+1}}) \leq \sum_{i=0}^{n} \frac{[r_{\mathcal{A}}(k)]^i}{2^{i+1}}.
\]
Since \( \sum_{i=0}^{+\infty} \frac{k^i}{2^{i+1}} \) commutes with \( k \), we have \( \sum_{i=0}^{+\infty} \frac{k^i}{2^{i+1}} \) commutes with \( \sum_{i=0}^{n} \frac{k^i}{2^{i+1}} \) for any \( n \in \mathbb{N} \). So Lemmas 2.3 and 2.7 show
\[
r_{\mathcal{A}}((2e - k)^{-1}) = r_{\mathcal{A}}\left(\sum_{i=0}^{+\infty} \frac{k^i}{2^{i+1}}\right) \leq \sum_{i=0}^{+\infty} \frac{[r_{\mathcal{A}}(k)]^i}{2^{i+1}}.
\]
Let \( \lambda = (2e - k)^{-1}(2a_1 + k) \). By (3.3), we obtain
\[
d(\mathit{gx}_n, \mathit{gx}_{n+1}) \leq (2e - k)^{-1}(2a_1 + k)d(\mathit{gx}_{n-1}, \mathit{gx}_n) = \lambda d(\mathit{gx}_{n-1}, \mathit{gx}_n) \quad \text{for any } n \in \mathbb{N}. \tag{3.5}
\]
For any \( n \in \mathbb{N} \), from (3.5), we have
\[
d(\mathit{gx}_n, \mathit{gx}_{n+1}) \leq \lambda d(\mathit{gx}_{n-1}, \mathit{gx}_n) \leq \lambda^2 d(\mathit{gx}_{n-2}, \mathit{gx}_{n-1}) \leq \cdots \leq \lambda^{n-1} d(gx_1, gx_5). \tag{3.6}
\]
Since \( a_1 \) commutes with \( k \), it suffices to prove that \( (2e - k)^{-1} \) commutes with \( 2a_1 + k \). Then (3.4) and Lemma 2.3 lead to
\[
r_{\mathcal{A}}(\lambda) = r_{\mathcal{A}}((2e - k)^{-1}(2a_1 + k))
\leq r_{\mathcal{A}}(2e - k)^{-1}r_{\mathcal{A}}(2a_1 + k)
\leq \left(\sum_{i=0}^{+\infty} \frac{[r_{\mathcal{A}}(k)]^i}{2^{i+1}}\right)[2r_{\mathcal{A}}(a_1) + r_{\mathcal{A}}(k)]
\leq \frac{1}{2 - r_{\mathcal{A}}(k)}[2r_{\mathcal{A}}(a_1) + r_{\mathcal{A}}(k)].
\]
By (3.1), we have
\[
s[2r_{\mathcal{A}}(a_1) + r_{\mathcal{A}}(k)] < 2 - r_{\mathcal{A}}(k).
\]
So, it follows that
\[
r_{\mathcal{A}}(\lambda) \leq \frac{1}{2 - r_{\mathcal{A}}(k)}[2r_{\mathcal{A}}(a_1) + r_{\mathcal{A}}(k)] < \frac{1}{s} \leq 1.
\]
Hence, by Lemmas 2.5 and 2.6, \((e - \lambda)^{-1}\) exists and \(\lim_{n \to +\infty} \|\lambda^n\| = 0\). For any positive integers \(m\) and \(n\) with \(m > n\), by (3.6), we have
\[
d(g_{n_m}, g_{n}) \leq sd(g_{n_m}, g_{n+1}) + sd(g_{n+1}, g_{n}) \leq sd(g_{n_m}, g_{n+1}) + s^2d(g_{n+1}, g_{n+2}) + s^3d(g_{n+2}, g_{n}) + \cdots + s^{m-n-1}d(g_{m-2}, g_{m-1}) + s^{m-n}d(g_{m-1}, g_{m})
\]
\[
\leq sd(g_{n_m}, g_{n+1}) + s^2d(g_{n+1}, g_{n+2}) + s^3d(g_{n+2}, g_{n+3}) + \cdots + s^{m-n-1}d(g_{m-2}, g_{m-1}) + s^{m-n}d(g_{m-1}, g_{m})
\]
\[
\leq (s\lambda^{m-1} + s^2\lambda^{m-2} + \cdots + s^{m-n}\lambda^{m-2})d(g_{1}, g_{x_2})
\]
\[
\leq s\lambda^{m-1}(e + s\lambda + (s\lambda)^2 + \cdots + (s\lambda)^{m-n-1} + \cdots )d(g_{1}, g_{x_2})
\]
\[
= s\lambda^{m-1}(e - s\lambda)^{-1}d(g_{1}, g_{x_2}).
\]

Hence, by the normality of \(P\), we obtain
\[
\|d(g_{n_m}, g_{n})\| \leq M\|s\lambda^{m-1}(e - s\lambda)^{-1}d(g_{1}, g_{x_2})\| \leq Ms\|\lambda^{m-1}\|\|(e - s\lambda)^{-1}d(g_{1}, g_{x_2})\|.
\]

Therefore, it follows that \(\|d(g_{n_m}, g_{n})\| \to 0(n, m \to +\infty)\). That is, \(\{g_{n_m}\}\) is a \(w\)-Cauchy sequence in \(g(X)\).

If \(g(X)\) is \(w\)-complete, then there exist \(q \in g(X)\) and \(p \in X\) such that \(g_{n_m} \xrightarrow{w} q\) as \(n \to +\infty\) and \(gp = q\) (if \(f(X)\) is \(w\)-complete, there exists \(q \in f(X)\) such that \(f_{n_m} \xrightarrow{w} q\) as \(n \to +\infty\). Since \(f(X) \subset g(X)\), we can find \(p \in X\) such that \(f_{n_m} = q\)).

Now, we need to show that \(fp = q\). By (3.2), we see
\[
d(g_{n+1}, fp) = d(f_{n+1}, fp) \leq a_1d(g_{n+1}, q) + a_2d(g_{n+1}, g_{n+2}) + a_3d(q, fp) + a_4d(g_{n+1}, fp) + a_5d(q, g_{n+2}).
\]

Similarly,
\[
d(fp, g_{n+2}) = d(fp, g_{n+1}) \leq a_1d(q, g_{n+1}) + a_2d(q, fp) + a_3d(g_{n+1}, g_{n+2}) + a_4d(q, g_{n+2}) + a_5d(g_{n+1}, fp).
\]

Thus, we have
\[
2d(g_{n+2}, fp) \leq 2a_1d(g_{n+1}, q) + (a_2 + a_3)d(g_{n+1}, g_{n+2}) + (a_2 + a_3)d(q, fp) + (a_4 + a_5)d(g_{n+1}, fp)
\]
\[
\leq 2a_1d(g_{n+1}, q) + (a_2 + a_3)d(q, fp) + (a_4 + a_5)d(g_{n+1}, fp)
\]
\[
\leq 2(2a_1 + sa_2 + sa_3 + a_4 + a_5)d(g_{n+1}, q) + (sa_2 + sa_3 + sa_4 + sa_5)d(g_{n+1}, fp)
\]
\[
+ (sa_2 + a_3 + a_4 + sa_5)d(g_{n+1}, g_{n+2}).
\]

Note that by Lemma 2.7 and (3.1), we obtain
\[
r_B(a_2 + a_3 + a_4 + a_5) \leq r_B(a_2 + a_3 + s(a_4 + a_5)) < \frac{2}{s + 1} < \frac{2}{s},
\]
so \((2e - sa_2 - sa_3 - sa_4 - sa_5)^{-1}\) exists. Hence, we obtain
\[
d(g_{n+2}, fp) \leq (2e - sa_2 - sa_3 - sa_4 - sa_5)^{-1}(2sa_1 + sa_2 + sa_3 + a_4 + a_5)d(g_{n+1}, q)
\]
\[
+ (2e - sa_2 - sa_3 - sa_4 - sa_5)^{-1}(2sa_1 + a_2 + a_3 + sa_4 + sa_5)d(g_{n+1}, g_{n+2}).
\]

Similarly, by the normality of \(P\) and the fact that \(g_{n_m}\) is a \(w\)-Cauchy sequence and \(g_{n_m} \xrightarrow{w} q\) as \(n \to +\infty\), we easily deduce that \(d(g_{n+2}, fp) \to 0\) as \(n \to +\infty\). This is, \(g_{n_m} \xrightarrow{w} fp(n \to +\infty)\). Therefore, by Lemma 2.1, we have \(fp = q = gp\).

Next, let us check the uniqueness of the point of coincidence for the mappings \(f\) and \(g\). If there exist \(u, w \in X\) such that \(fu = gu = w\), by (3.2), we see that
\[
d(gu, gp) = d(fu, fp) \leq a_1d(gu, gp) + a_2d(fu, gu) + a_3d(fp, gp) + a_4d(fp, gu) + a_5d(fu, gp)
\]
\[
= (a_1 + a_5 + a_5)d(gu, gp).
\]
As \( r_A(a_1 + a_2 + a_3) \leq r_A(a_1) + r_A(a_4 + a_5) < 1 \), we can deduce that \( d(gu, gp) = \theta \), i.e., \( w = gu = gp = q \). Furthermore, if \( f \) and \( g \) are weakly compatible, by Lemma 2.8, we conclude that \( q \) is the unique common fixed point of \( f \) and \( g \). \( \square \)

It is sufficient to obtain the following fixed point theorems of contractions with vector-valued coefficients in the setting of non-solid cone \( b \)-metric or cone metric spaces over Banach algebras from Theorem 3.1, so we omit their proofs.

**Corollary 3.1.** Let \((X, d)\) be a wrtn-complete cone \( b \)-metric space over a Banach algebra \( \mathcal{A} \) with coefficient \( s \geq 1 \). Suppose the mapping \( T : X \to X \) satisfies the generalized Banach contractive condition
\[
d(Tx, Ty) \leq kd(x, y), \quad \text{for all } x, y \in X,
\]
where \( k \in P \) is a constant vector satisfying \( r_A(k) \in (0, \frac{1}{s}) \). Then \( T \) has a unique fixed point in \( X \). And for any \( x \in X \), iteration sequence \( \{T^n x\} \) wrtn-converges to the fixed point.

**Corollary 3.2.** Let \((X, d)\) be a wrtn-complete cone metric space over a Banach algebra \( \mathcal{A} \). Suppose the mapping \( T : X \to X \) satisfies the generalized Banach contractive condition
\[
d(Tx, Ty) \leq kd(x, y), \quad \text{for all } x, y \in X,
\]
where \( k \in P \) is a constant vector satisfying \( r_A(k) \in (0, 1) \). Then \( T \) has a unique fixed point in \( X \). And for any \( x \in X \), iteration sequence \( \{T^n x\} \) wrtn-converges to the fixed point.

**Corollary 3.3.** Let \((X, d)\) be a wrtn-complete cone metric space over a Banach algebra \( \mathcal{A} \). Suppose the mapping \( T : X \to X \) satisfies the generalized Kannan contractive condition
\[
d(Tx, Ty) \leq k(d(Tx, x) + d(Ty, y)), \quad \text{for all } x, y \in X,
\]
where \( k \in P \) is a constant vector satisfying \( r_A(k) \in (0, \frac{1}{3}) \). Then \( T \) has a unique fixed point in \( X \). And for any \( x \in X \), iteration sequence \( \{T^n x\} \) wrtn-converges to the fixed point.

**Corollary 3.4.** Let \((X, d)\) be a wrtn-complete cone metric space over a Banach algebra \( \mathcal{A} \). Suppose the mapping \( T : X \to X \) satisfies the generalized Chatterjea contractive condition
\[
d(Tx, Ty) \leq k(d(Tx, y) + d(Ty, x)), \quad \text{for all } x, y \in X,
\]
where \( k \in P \) is a constant vector satisfying \( r_A(k) \in (0, \frac{1}{2}) \). Then \( T \) has a unique fixed point in \( X \). And for any \( x \in X \), iteration sequence \( \{T^n x\} \) wrtn-converges to the fixed point.

**Remark 3.1.** We emphasize that one cannot use the existed fixed point results in \( b \)-metric spaces or metric spaces to deduce the main results presented in the setting of wrtn-complete non-solid cone \( b \)-metric spaces or cone metric spaces over Banach algebras by virtue of the method from [17].

In fact, if \((X, d)\) is a wrtn-complete cone metric space over a Banach algebra \( \mathcal{A} \) and the cone \( P \) is normal with normal constant \( M \). Suppose the mapping \( T : X \to X \) satisfies the generalized Banach contractive condition
\[
d(Tx, Ty) \leq kd(x, y), \quad \text{for all } x, y \in X,
\]
where \( k \in P \) is a constant vector satisfying \( r_A(k) \in (0, 1) \). By [17], we can suppose that \( M = 1 \). Therefore, according to (3.7), we have
\[
\|d(Tx, Ty)\| \leq \|k\| \|d(x, y)\|.
\]
Then the space \((X, D)\) with \( D(x, y) = \|d(x, y)\| \) is a complete metric space, but one cannot conclude that \( T \) has a unique fixed point in \( X \) by using the famous Banach contraction principle, since in general one cannot assert \( |k| < 1 \) though \( r_A(k) \in (0, 1) \). These results improve and extend many relevant results in metric spaces [30,31].
4 Applications

In this section, we will present some examples to show our main results are effective tools to verify the uniqueness of solutions to fixed point equalities whether the underlying cones are solid or non-solid.

Example 4.1. Consider the space $L_p(0 < p < 1)$ of all real function $x(t)$ $(t \in [0, 1])$ such that $\int_0^1 |x(t)|^p dt < +\infty$. Let $X = L_p$, $\mathcal{A} = \mathbb{R}^2$, $P = [(x, y) \in \mathcal{A}| x, y \geq 0] \subset \mathbb{R}^2$, and $d : X \times X \rightarrow \mathcal{A}$ such that

$$d(x, y) = \left(\alpha \left(\int_0^1 |x(t) - y(t)|^p dt\right)^{\frac{1}{p}}, \beta \left(\int_0^1 |x(t) - y(t)|^p dt\right)^{\frac{1}{p}}\right),$$

where $\alpha, \beta \geq 0$ are constants. Then $(X, d)$ is a normal and solid cone $b$-metric space over a Banach algebra $\mathcal{A}$ with the coefficient $s = 2^p - 1$. Define the mapping $T : X \rightarrow X$ by

$$Tx(t) = \frac{1}{4} \ln(1 + |x(t)|).$$

Considering a simple inequality $0 < \ln(1 + x) < x$, where $x > 0$, we have

$$d(Tx, Ty) = \left(\alpha \left(\int_0^1 \frac{1}{4} \ln(1 + |x(t)|) - \frac{1}{4} \ln(1 + |y(t)|)\right)^p dt\right)^{\frac{1}{p}}, \beta \left(\int_0^1 \frac{1}{4} \ln(1 + |x(t)|) - \frac{1}{4} \ln(1 + |y(t)|)\right)^p dt\right)^{\frac{1}{p}}$$

$$= \frac{1}{4} \left(\alpha \left(\int_0^1 \ln\left(\frac{1 + |x(t)|}{1 + |y(t)|}\right)\right)^p dt\right)^{\frac{1}{p}}, \beta \left(\int_0^1 \ln\left(\frac{1 + |x(t)|}{1 + |y(t)|}\right)\right)^p dt\right)^{\frac{1}{p}}$$

$$= \frac{1}{4} \left(\alpha \left(\int_0^1 \ln\left(1 + \frac{|x(t)| - |y(t)|}{1 + |y(t)|}\right)\right)^p dt\right)^{\frac{1}{p}}, \beta \left(\int_0^1 \ln\left(1 + \frac{|x(t)| - |y(t)|}{1 + |y(t)|}\right)\right)^p dt\right)^{\frac{1}{p}}$$

$$\leq \frac{1}{4} \left(\alpha \left(\int_0^1 \ln\left(1 + \frac{|x(t) - y(t)|}{1 + |y(t)|}\right)\right)^p dt\right)^{\frac{1}{p}}, \beta \left(\int_0^1 \ln\left(1 + \frac{|x(t) - y(t)|}{1 + |y(t)|}\right)\right)^p dt\right)^{\frac{1}{p}}$$

$$\leq \frac{1}{4} \left(\alpha \left(\int_0^1 |x(t) - y(t)|^p dt\right)^{\frac{1}{p}}, \beta \left(\int_0^1 |x(t) - y(t)|^p dt\right)^{\frac{1}{p}}\right)$$

$$= \frac{1}{4} d(x, y),$$

where $p = \frac{1}{2}$ and $k = \frac{1}{4} \in (0, \frac{1}{2})$ for $\frac{1}{s} = \frac{1}{2p - 1} = \frac{1}{2}$. Consequently, all conditions of Corollary 3.1 are satisfied and then $T$ has a unique fixed point in $X$.

Example 4.2. Let $X = L[0, 1]$ denote the set of all generalized real-valued Lebesgue integral functions on $[0, 1]$. Let $\mathcal{A} = L[0, 1]$. Consider the following nonlinear integral equation:

$$\int_0^1 F(t, f(s)) ds = f(t),$$

(4.1)
where $F$ satisfies:

(a) $F : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a generalized real-valued Lebesgue integral function where $\mathbb{R} = [-\infty, +\infty]$ denoting the set of all generalized real numbers;

(b) there exists a function $G(x)$ with $0 < \int_0^1 G(x) \, dx < 1$ such that for a.e. $x \in [0, 1]$, and a.e. $y_1, y_2 \in \mathbb{R}$, one has

$$|F(x, y_1) - F(x, y_2)| \leq G(x)|y_1 - y_2|.$$

Then equation (4.1) has a unique non-negative solution in $L[0, 1]$.

Now we check all the conditions of Corollary 3.2 are satisfied. Define a norm on $\mathcal{A}$ by $\|f\|_1 = \int_0^1 |f(t)| \, dt$ for $f \in \mathcal{A}$. Let $P = \{f \in L[0, 1] \mid f \geq 0\}$, for a.e. $t \in [0, 1]$. Then $P$ is a normal but non-solid cone of the real Banach algebra $\mathcal{A}$ with the operations as follows:

$$ (f + g)(t) = f(t) + g(t), \quad (cf)(t) = cf(t), \quad (fg)(t) = f(t)g(t), $$

for all $f = f(t), g = g(t) \in \mathcal{A}$ and $c \in \mathbb{R}$. Moreover, $\mathcal{A}$ owns the unit element $e = e(t) = 1$ for a.e. $t \in [0, 1]$.

Let $T$ be a self-mapping of $X$ defined by $Tf(t) = \int_0^1 f(t, f(s)) \, ds$.

It is easy shown that $(X, d)$ is a wrtn-complete cone metric space over Banach algebra $\mathcal{A}$ with the norm $\| \cdot \|_1$ where the cone metric is defined by $d(f, g) = e^{\int_0^1 |f(t) - g(t)| \, dt}$. Now let us check that $T : X \to X$ is a generalized Banach contraction with the generalized Lipschitz coefficient $L = \int_0^1 G(x) \, dx$ satisfying the spectral radius $r(L) < 1$. Indeed, by (b) together with the fact that $\|fg\|_1 \leq \|f\|_1 \|g\|_1$, we see

$$ d(Tf(x), Tg(x)) = e^{\int_0^1 |Tf(x) - Tg(x)| \, dx} $$

$$ = e^{\int_0^1 \left( \int_0^1 |F(x, f(t)) - F(x, g(t))| \, dt \right) \, dx} $$

$$ \leq e^{\int_0^1 \left( \int_0^1 |F(x, f(t)) - F(x, g(t))| \, dt \right) \, dx} $$

$$ \leq e^{\int_0^1 L|f(t) - g(t)| \, dt} $$

$$ \leq Le^{\int_0^1 |f(t) - g(t)| \, dt} $$

$$ = Ld(f(x), g(x)), $$

where the spectral radius $r(L)$ of $L$ satisfies $r(L) \leq \|L\|_1 = L \in (0, 1)$. Therefore, it follows from Corollary 3.2 that equation (4.1) has a unique non-negative solution in $L[0, 1]$.

**Remark 4.1.** Recently, many authors investigated the problem of whether all the fixed point results in cone $b$-metric spaces (cone metric spaces) are equivalent to that in $b$-metric spaces (metric spaces). The equivalent relationship was established between the fixed point results in metric and in cone metric spaces, see [22–29]. However, it is significant to point out that one is unable to show that the non-solid cone $b$-metric spaces (cone metric spaces) over Banach algebras introduced in this article are equivalent to any $b$-metric spaces (metric spaces), even though the cone is normal. This is due to the fact that all the methods to prove
such equivalence appearing in the literature strongly rely on the solidness of the cones, which shows that these methods together with the corresponding techniques are invalid.

**Remark 4.2.** The main results in this article are a valuable addition to the existing literature concerning fixed point theory in the context of metric spaces or abstract metric spaces such as references [30–34].

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**References**


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