Research Article

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Total Roman domination on the digraphs

Abstract: Let $D = (V, A)$ be a simple digraph with vertex set $V$, arc set $A$, and no isolated vertex. A total Roman dominating function (TRDF) of $D$ is a function $h : V \rightarrow \{0, 1, 2\}$, which satisfies that each vertex $x \in V$ with $h(x) = 0$ has an in-neighbour $y \in V$ with $h(y) = 2$, and that the subdigraph of $D$ induced by the set $\{x \in V : h(x) \geq 1\}$ has no isolated vertex. The weight of a TRDF $h$ is $\omega(h) = \sum_{v \in V} h(v)$. The total Roman domination number $\gamma_{tR}(D)$ of $D$ is the minimum weight of all TRDFs of $D$. The concept of TRDF on a graph $G$ was introduced by Liu and Chang [Roman domination on strongly chordal graphs, J. Comb. Optim. 26 (2013), no. 3, 608–619]. In 2019, Hao et al. [Total Roman domination in digraphs, Quaest. Math. 44 (2021), no. 3, 351–368] generalized the concept to digraph and characterized the digraphs of order $n \geq 3$ with $\gamma_{tR}(D) = 2$ and the digraphs of order $n \geq 3$ with $\gamma_{tR}(D) = 3$. In this article, we completely characterize the digraphs of order $n \geq k$ with $\gamma_{tR}(D) = k$ for all integers $k \geq 4$, which generalizes the results mentioned above.

Keywords: total Roman dominating function, total Roman domination number, digraph

MSC 2020: 05C20, 05C69

1 Introduction

In recent years, domination theory in digraphs has inspired widespread interest. Some variations have fallen within the scope of research (see [1–9]).

For the notation and terminology in this article, see [10]. Let $D = (V, A)$ be a simple digraph with vertex set $V$ and arc set $A$. In this article, if not otherwise specified, we assume that digraph $D$ has no isolated vertex. Let $x \in V$. The out-neighbourhood, $N^+(x)$, of $x$ is $\{y : (x, y) \in A\}$, and $N^-[x] = N^-(x) \cup \{x\}$ is called the closed out-neighbourhood. Similarly, the in-neighbourhood $N^-(x)$ and the closed in-neighbourhood $N_+^-[x]$ of $x$ can be defined. Furthermore, the set $N(x) = N^+(x) \cup N^-(x)$ is called the neighbourhood of $x$. We call the vertices in $N(x)$ the neighbours of $x$. For a vertex subset $W \subseteq V$, the subdigraph induced by $W$ is denoted by $D[W]$.

A total Roman dominating function (TRDF) in a digraph $D$ is a function $h : V \rightarrow \{0, 1, 2\}$ that satisfies the following conditions:

(a) each vertex $x \in V$ with $h(x) = 0$ has an in-neighbour $y \in V$ with $h(y) = 2$;
(b) the subdigraph of $D$ induced by the set $\{x \in V : h(x) \geq 1\}$ has no isolated vertex.

For the sake of simplicity, let $V_i = \{x \in V : h(x) = i\}$ for $i = 0, 1, 2$ and we also write $h = (V_0, V_1, V_2)$. For a vertex subset $W \subseteq V$, we define $h(W) = \sum_{v \in W} h(v)$ and $\omega(h) = h(V)$. The total Roman domination number of $D$ is

$$\gamma_{tR}(D) = \min\{\omega(h) : h \text{ is a TRDF of } D\}.$$
A TRDF $h$ is called a $y_{tr}(D)$-function if $\omega(h) = y_{tr}(D)$. For more results about this, see [11–15].

In 2019, Hao et al. [15] characterized the digraphs of order $n \geq 2$ with $y_{tr}(D) = 2$ and the digraphs of order $n \geq 3$ with $y_{tr}(D) = 3$. In this article, we generalize the results mentioned above to all positive integers $k \geq 4$ and characterize the digraphs of order $n \geq k$ with $y_{tr}(D) = k$. In Section 2, we discuss the case $k = 4$, and in Section 3, we proceed for $k \geq 5$.

## 2 The digraphs of order $n \geq 4$ with $y_{tr}(D) = 4$

In this section, we characterize the digraphs $D$ of order $n \geq 4$ with $y_{tr}(D) = 4$. To show the main result, we need to use the following propositions.

**Proposition 2.1.** [15] For any digraph $D$ of order $n \geq 2$ with no isolated vertex, $y_{tr}(D) = 2$ if and only if $n = 2$.

**Proposition 2.2.** [15] For any digraph $D$ of order $n \geq 3$ with no isolated vertex, $y_{tr}(D) = 3$ if and only if one of the following hold:

1. $n = 3$;
2. $n \geq 4$ and there exist two vertices $u$ and $v$ of $D$ such that $V(D) \setminus [u, v] \subseteq N^*(v)$ and $D([u, v])$ is connected in $D$.

**Theorem 2.3.** Let $D = (V, A)$ be a digraph of order $n \geq 4$ with no isolated vertex. Then, $y_{tr}(D) = 4$ if and only if one of the following is true:

1. $n = 4$ and $D$ does not contain a vertex subset $X$ such that $|X| = 2, V \setminus X \subseteq N^*(x)$ for a vertex $x \in X$, and $D[X]$ has no isolated vertex;
2. $n \geq 5$, $\Delta(D) = n - 2$,
   
   (2.a) $D[V \setminus N^*(v)]$ is not connected for any vertex $v$ with maximum out-degree of $D$, and
   
   (2.b) $D$ contains a vertex subset $X$ such that $|X| = 3, V \setminus X \subseteq N^*(x)$ for a vertex $x \in X$, and $D[X]$ has no isolated vertex;
3. $n \geq 5$, $\Delta'(D) \leq n - 3$,
   
   (3.a) $D$ contains a vertex subset $X$ such that $|X| = 3, V \setminus X \subseteq N^*(x)$ for a vertex $x \in X$, and $D[X]$ has no isolated vertex, or
   
   (3.b) $D$ contains a vertex subset $X$ such that $|X| = 2, V \setminus X \subseteq N^*(X)$, and $D[X]$ has no isolated vertex.

**Proof.** ($\Rightarrow$) Assume that $y_{tr}(D) = 4$. Let $h = (V_0, V_1, V_2)$ be a $y_{tr}(D)$-function.

We claim that $\Delta'(D) \leq n - 2$. Suppose not. Then, $\Delta'(D) = n - 1$, and there is a vertex $v \in V$ with $d^+(v) = n - 1$. Let $u$ be an out-neighbour of $v$. Define a function $g_0 : V \rightarrow \{0, 1, 2\}$ such that $g_0(v) = 2$, $g_0(u) = 1$, and $g_0(w) = 0$ otherwise. This results in a TRDF with weight $\omega(g_0) = 3 < 4 = y_{tr}(D)$, a contradiction. Hence, $\Delta'(D) \leq n - 2$.

Since $y_{tr}(D) = |V_0| + 2|V_1| = 4$ and $\Delta'(D) \leq n - 2$, we distinguish two cases: either $\Delta'(D) = n - 2$ or $\Delta'(D) \leq n - 3$.

**Case 1: $\Delta'(D) = n - 2$.**

**Subcase 1.1:** $|V_1| = 4$ and $|V_0| = 0$.

Since $|V_0| = 0$, we have $|V_1| = 0$, and then $|V| = |V_1| = 4$. For $n = 4$, it is easy to see that the condition 2.3(1) is true by Proposition 2.2 (2).

**Subcase 1.2:** $|V_1| = 2$ and $|V_0| = 1$.

First, we show that the condition (2.a) is true. Suppose to the contrary that there is a vertex $v \in V$ with $d^+(v) = \Delta'(D)$ such that $D[V \setminus N^*(v)]$ is connected. Then, $(u, v) \in A$ for the vertex $u \in V \setminus N^*(v)$, as shown in Figure 1(1). Define a function $g_1 : V \rightarrow \{0, 1, 2\}$ such that $g_1(v) = 2$, $g_1(u) = 1$ and $g_1(x) = 0$ for each vertex $x \in N^*(v)$. Then $g_1$ is a TRDF with weight $\omega(g_1) = 3 < 4 = y_{tr}(D)$, a contradiction. Therefore, the condition (2.a) holds.
Furthermore, let $V_0 = \{v_0\}$ and $X = V_1 \cup V_2$. Then, we have $V \setminus X = V_0 \subseteq N^r(v_0)$ and $D[X]$ has no isolated vertex by the definition of TRDF. This implies that the condition (2.b) holds, see Figure 1(2).

**Subcase 1.3:** $|V| = 0$ and $|V_0| = 2$.

Let $V_0 = \{v_0, v_1\}$, $V_1 = \emptyset$. Since $h$ is a $\gamma_{dtR}(D)$-function, there exists a vertex with the maximum out-degree $\Delta^r(D)$ in $V_2$. Without loss of generality, assume that $d(v_1) = \Delta^r(D)$.

If $v_1 \notin N^r(v_0)$ (as shown in Figure 1(3)), then define a function $g_1 : V \to \{0, 1, 2\}$ such that $g_1(v_0) = 2$, $g_1(v_1) = 1$, and $g_2(x) = 0$ otherwise. Then, $g_1$ is a TRDF with weight $\omega(g_1) = 3 < 4 = \gamma_{dtR}(D)$, a contradiction. Thus, $v_1 \in N^r(v_0)$. Let $V_2 = V \setminus N^r(v_0)$ (see Figure 1(4)). It is not difficult to see that $f = (V'_0, V'_1, V'_2) = (V \setminus \{v_0, v_1, v_2\}, \{v_1, v_2\}, \{v_0\})$ is a $\gamma_{dtR}(D)$-function with $|V'_1| = 2$ and $|V'_2| = 1$. Then, by the proof of Subcase 1.2, conditions (2.a) and (2.b) hold.

**Case 2:** $\Delta^r(D) \leq n - 3$.

**Subcase 2.1:** $|V| = 4$ and $|V_0| = 0$.

Since $|V_0| = 0$, we have $|V_0| = 0$, and then $|V| = |V_0| = 4$. For $n = 4$, it is easy to see that the condition (1) is true by Proposition 2.2.

**Subcase 2.2:** $|V| = 2$ and $|V_0| = 1$.

In the same manner as the proof of the condition (2.b), it can be obtained that $D$ contains a set $X$ of order 3 such that $V \setminus X \subseteq N^r(x)$ for a vertex $x \in X$ and $D[X]$ has no isolated vertex. That is, condition (3.a) holds.

**Subcase 2.3:** $|V| = 0$ and $|V_0| = 2$.

Let $X = V_2$. Since $V_2 = \emptyset$ and $h$ is a $\gamma_{dtR}(D)$-function, we have $V \setminus X = V_0 \subseteq N^r(x)$ and $D[X]$ has no isolated vertex. Therefore, condition (3.b) holds.

$(\Leftarrow)$ To show the sufficiency, assume that one of the three conditions (1), (2), and (3) holds in the statement of the theorem.

If (1) holds, we obtain $\gamma_{dtR}(D) \geq 4$ by Propositions 2.1 and 2.2(2). Furthermore, define a function $g_1 : V \to \{0, 1, 2\}$ such that $g_1(v) = 1$ for every vertex $v \in V$. Then $\gamma_{dtR}(D) \leq \omega(g_1) = 4$.

If (2) holds, since $D[V \setminus N^r(v)]$ is not connected for each vertex $v$ with maximum out-degree $\Delta^r(D) = n - 2$ by (2.a), there do not exist two vertices $u$ and $v$ of $D$ such that $V \setminus \{u, v\} \subseteq N^r(v)$ and $D[\{u, v\}]$ is connected.

![Figure 1](image-url)

**Figure 1:** (1) and (2): The illustrations of Subcase 1.2, where the edge with no direction means that both directions are possible. (3) and (4): The illustrations of Subcase 1.3.
in $D$. Then, $\gamma_{tr}(D) \geq 4$ follows trivially from Propositions 2.1 and 2.2. On the other hand, by condition (2.b), we have the function $g_5 = (V \setminus X, X \setminus \{x\}, \{x\})$ as a TRDF on $D$ with weight $\omega(g_5) = 4$, and so $\gamma_{tr}(D) \leq 4$.

If (3) holds, we first have $\Delta(D) \leq n - 3$, then there do not exist two vertices $u$ and $v$ of $D$ such that $V\setminus \{u, v\} \subseteq N^+(v)$. Furthermore, we obtain $\gamma_{tr}(D) \geq 4$ by Propositions 2.1 and 2.2. On the other hand, if (3.a) holds, then define the function $g_5 = (V \setminus X, X \setminus \{x\}, \{x\})$ as a TRDF on $D$ with weight $\omega(g_5) = 4$, and so $\gamma_{tr}(D) \leq 4$. If (3.b) holds, then the function $g_5 = (V \setminus X, \emptyset, X)$ is a TRDF on $D$ with weight $\omega(g_5) = 4$, and so $\gamma_{tr}(D) \leq 4$.

Consequently, we have $\gamma_{tr}(D) = 4$. □

3 The digraphs of order $n \geq k$ with $\gamma_{tr}(D) = k$ for any positive integer $k \geq 5$

**Definition 3.1.** Let $t \geq 2$ be a positive integer and $D = (V, A)$ a digraph. Then, $D$ has an $(X, W, t)$-structure if there exists a subset $X \subseteq V$ such that for a subset $W \subset V$ with $0 \leq |W| \leq \lfloor \frac{t}{2} \rfloor$ the following hold:

1. $W \subseteq X$, $V \setminus X \subseteq N^+(W)$, and $D[X]$ have no isolated vertex if $1 \leq |W| \leq \lfloor \frac{t}{2} \rfloor$, which includes two cases (Figure 2);
2. $X = V$ and $D[X] = D$ has no isolated vertex if $|W| = 0$.

**Theorem 3.2.** Let $D = (V, A)$ be a digraph of order $n \geq 4$ with no isolated vertex. Then, $\gamma_{tr}(D) \leq n$. Furthermore, the equality holds if and only if there does not exist an $(X, W, n-1)$-structure with $|X| \leq n-1-|W|$.

**Proof.** Define a function $h : V \rightarrow \{0, 1, 2\}$ such that $h(v) = 1$ for each vertex $v \in V$. Then, $h$ is a TRDF on $D$, and so $\gamma_{tr}(D) \leq \omega(h) = n$. In the following, we show the second assertion.

($\Rightarrow$) Assume $\gamma_{tr}(D) = n$. Suppose, however, that there is an $(X, W, n - 1)$-structure with $|X| \leq n - 1 - |W|$ in $D$. If $|W| = 0$, then $|V| = |X| \leq n - 1$, a contradiction. If $1 \leq |W| \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, then the function $f = (V \setminus X, X \setminus W, W)$ is a TRDF on $D$ of weight $\omega(f) \leq |X| - |W| + 2|W| \leq n - 1 - |W| - |W| + 2|W| = n - 1 < n = \gamma_{tr}(D)$, a contradiction.

($\Leftarrow$) It is sufficient to prove that $\gamma_{tr}(D) \geq n$. Suppose, however, that $\gamma_{tr}(D) \leq n - 1$. Let $h_1 = (V_0, V_1, V_2)$ be a $\gamma_{tr}(D)$-function. Since $\gamma_{tr}(D) = |V_1| + 2|V_2| \leq n - 1$, we have $1 \leq |V_2| \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. Furthermore, since $\gamma_{tr}(D) = |V_0| + 2|V_1| \leq n - 1 = |V_0| + |V_1| + |V_2|$, we obtain $|V_0| \geq |V_1| + 1$. Let $W = V_2$ and $X = V_0 \cup V_2$. Then, $V \setminus X \subseteq N^+(W)$ and $D[X]$ has no isolated vertex by the definition of TRDF, where $|X| = |V_0| + |V_2| = n - |V_0| \leq n - |W| - 1$, a contradiction. Consequently, $\gamma_{tr}(D) \geq n$. Hence, $\gamma_{tr}(D) = n$, as desired. □

![Figure 2](image-url)

*Figure 2:* (1): $N^+(W) \cap X \neq \emptyset$; (2): $N^+(W) \cap X = \emptyset$. 


Lemma 3.3. Let \( k \geq 5 \) be an integer and \( D = (V, A) \) a digraph of order \( n \geq k + 1 \) with \( \gamma_{RD}(D) \geq k \). If there exists a subset \( W \subset V \) with \( 1 \leq |W| \leq \left\lfloor \frac{k}{2} \right\rfloor \) such that \( D[V \setminus N^+(W)] \) has an isolated vertex, then \( |N'[W]| \leq n + 2|W| - k \).

Proof. Let \( W \subset V \) with \( 1 \leq |W| \leq \left\lfloor \frac{k}{2} \right\rfloor \) be a set such that \( D[V \setminus N^+(W)] \) has an isolated vertex. Suppose, however, that \( |N'[W]| \geq n + 2|W| - k + 1 \). Since \( D[V \setminus N^+(W)] \) has an isolated vertex, the function \( g = (N^+(W), V \setminus N'[W], W) \) is a TRDF on \( D \). Then, \( \gamma_{RD}(D) \leq \omega(g) = |V \setminus N'[W]| + 2|W| \leq n - (n + 2|W| - k + 1) + 2|W| = k - 1 \), a contradiction to \( \gamma_{RD}(D) \geq k \). \( \Box \)

Lemma 3.4. Let \( k \geq 5 \) be an integer and \( D = (V, A) \) a digraph of order \( n \geq k + 1 \) with \( \gamma_{RD}(D) \geq k \). If there exists a subset \( W \subset V \) with \( 1 \leq |W| \leq \left\lfloor \frac{k}{2} \right\rfloor \) such that \( D[V \setminus N^+(W)] \) has at least one isolated vertex, then \( D \) has an \((X, W, k)\)-structure and \( D[X] \) has no \((X', W', k - |W| - 1)\)-structure with \(|X'| + |W'| \leq k - 1 - |W|\) and \( W \subseteq X' \).

Proof. Let \( W \subset V \) with \( 1 \leq |W| \leq \left\lfloor \frac{k}{2} \right\rfloor \) be a set such that \( D[V \setminus N^+(W)] \) has at least one isolated vertex. Suppose that \( h = (V_0, V_1, V_2) \) is a TRDF on \( D \), let \( X = V_1 \cup W \). Then, we have \( V \setminus X \subseteq N^+(W) \) and \( D[X] \) has no isolated vertex by the definition of TRDF. This implies that \( D \) has an \((X, W, k)\)-structure in \( D \).

Next, we prove that \( D[X] \) does not contain an \((X', W', k - |W| - 1)\)-structure with \(|X'| + |W'| \leq k - 1 - |W|\) and \( W \subseteq X' \). By contradiction, let \( D_1 = D[X] \).

Suppose, however, that \( D_1 \) has an \((X', W', k - |W| - 1)\)-structure with \(|X'| + |W'| \leq k - 1 - |W|\) and \( W \subseteq X' \). Then, the function \( f = (X'X', X'W', W') \) is a TRDF on \( D_1 \).

Let \( h_1 = (V_0^b, V_1^b, V_2^b) \) be defined as follows: \( h_1(v) = 2 \) for each vertex \( v \in W \), \( h_1(u) = 0 \) for each vertex \( u \in V \setminus X \), and \( h_1(x) = f(x) \) for each vertex \( x \in X \setminus W \). This implies \( V_0^b = (V \setminus X) \cup (X'X') \), \( V_1^b = X' \setminus (W \cup W') \), and \( V_2^b = W \cup W' \). If \( |W'| = 0 \), then \( V_0^h = V \setminus X \), \( V_1^h = X' \setminus W \), and \( V_2^h = W \). According to the definitions of \((X, W, k)\)-structure and \((X', W', k - |W| - 1)\)-structure, we have \( V_0^h \subseteq N^+(V_2^b) \) and \( D[V_0^h \cup V_2^b] = D[X] = D[X'] \) has no isolated vertex. Thus, \( h_1 \) is a TRDF on \( D \). Similarly, if \( 1 \leq |W'| \leq \left\lfloor \frac{k - 1 - |W| - 1}{2} \right\rfloor \), then we have \( X'X' \subseteq N^+(W') \) and \( D[X'] \) has no isolated vertex by the definition of \((X', W', k - |W| - 1)\)-structure. Furthermore, \( V \setminus X \subseteq N^+(W') \) according to the definition of \((X, W, k)\)-structure. Thus, \( V_0^h = (V \setminus X) \cup (X'X') \subseteq N^+(W \cup W') = N^+(V_2^b) \) and \( D[V_0^h \cup V_2^b] = D[X] \) has no isolated vertex. That is, \( h_1 \) is a TRDF on \( D \). Hence, \( \gamma_{RD}(D) \leq \omega(h_1) = |X' \setminus (W \cup W')| + 2|W \cup W'| \leq |X'| + |W'| \leq k - 1 \), a contradiction to \( \gamma_{RD}(D) \geq k \). This completes the proof of Lemma 3.4. \( \Box \)

Theorem 3.5. Let \( k \geq 5 \) be an integer and \( D = (V, A) \) a digraph of order \( n \geq k + 1 \). Then, \( \gamma_{RD}(D) \geq k \) if and only if the following hold:

1. for any subset \( W \subset V \) with \( 1 \leq |W| \leq \left\lfloor \frac{k}{2} \right\rfloor \) such that \( D[V \setminus N^+(W)] \) has an isolated vertex, there must be \( |N'[W]| \leq n + 2|W| - k \);
2. for any subset \( W \subset V \) with \( 1 \leq |W| \leq \left\lfloor \frac{k}{2} \right\rfloor \) such that \( D[V \setminus N^+(W)] \) has at least one isolated vertex, \( D \) has an \((X, W, k)\)-structure and \( D[X] \) has no \((X', W', k - |W| - 1)\)-structure with \(|X'| + |W'| \leq k - 1 - |W|\) and \( W \subseteq X' \).

Proof. Lemmas 3.3 and 3.4 mean that necessity holds. Here, we just show sufficiency.

Let \( n = (V_0, V_1, V_2) \) be a \( \gamma_{RD}(D) \)-function. Suppose, however, that \( \gamma_{RD}(D) = |V_1| + 2|V_2| \leq k - 1 \). Then, \( 1 \leq |V_2| \leq \left\lfloor \frac{k - 1 - |V_1|}{2} \right\rfloor \leq \left\lfloor \frac{k}{2} \right\rfloor \).

If \( D[V \setminus N^+(V_2)] \) has no isolated vertex, then we have \( |N'[V_2]| \leq n + 2|V_2| - k \) by (1). This implies

\[
|N'[V_2]| + k \leq n + 2|V_2| - k \leq 2|V_2| + 2|V_2| - k \leq 2|V_2|.
\]  

Furthermore, \( \gamma_{RD}(D) = |V_1| + 2|V_2| \leq k - 1 \), then

\[
k - 2|V_2| \geq |V_2| + 1.
\]  

(3.1)
Combining the inequalities (3.1) and (3.2), we obtain \( n \geq |N^*[V_2]| + |V_1| + 1 = |V_0| + |V_2| + |V_1| + 1 \), a contradiction.

If \( D[V \setminus N^*(V_2)] \) has at least one isolated vertex, then let \( W = V_2, X' = X = V_1 \cup W, \) and \( W' = \emptyset. \) It is not difficult to see that \( W \subseteq X, V \setminus X \subseteq N^*(W), \) and \( D[X] \) has no isolated vertex. This implies that there exists an \((X, W, k)\)-structure. Furthermore, since \( y_{\text{id}}(D) = |V_2| = |X'| = |X| \leq k - 1 - |W|. \) Thus, \( |X'| + |W'| \leq k - 1 - |W|. \) On the other hand, \( W \subseteq X', D[X'] = D[X] \) has no isolated vertex, and \( |W'| = 0, \) and then there exists an \((X', W', k - |W| - 1)\)-structure with \( |X'| + |W'| \leq k - 1 - |W| \) and \( W \subseteq X' \) in \( D[X], \) a contradiction to (2). \( \square \)

**Theorem 3.6.** Let \( k \geq 5 \) be an integer and \( D = (V, A) \) a digraph of order \( n \geq k + 1. \) Then, \( y_{\text{id}}(D) = k \) if and only if \( D \) satisfies Theorem 3.5(1) and (2) and one of the following is true:

(1) there exists a subset \( W \subseteq V \) with \( 1 \leq |W| \leq \left\lfloor \frac{k}{2} \right\rfloor \) such that \( |N^*[W]| = n + 2|W| - k, \) and \( D[V \setminus N^*(W)] \) has no isolated vertex;

(2) \( D \) has an \((X, W, k)\)-structure with \( |X| = k - |W| \) and \( D[V \setminus N^*(W)] \) has at least one isolated vertex.

**Proof.** \((\Rightarrow)\) From \( y_{\text{id}}(D) = k, \) we see that \( D \) satisfies Theorem 3.5(1) and (2). Next, we prove that (1) or (2) of this theorem holds. Let \( h = (V_0, V_1, V_2) \) be a \( y_{\text{id}}(D)\)-function. Since \( |V_0| + 2|V_2| = y_{\text{id}}(D) = k, \) we may deduce that one of the following is true:

(i) \( |V_0| = 0; \)

(ii) \( 1 \leq |V_0| \leq \left\lfloor \frac{k}{2} \right\rfloor. \)

Suppose that (i) holds. Obviously, we have \( |V_0| = 0, \) and then \( |V_0| = |V_1| = k, \) a contradiction to \( n \geq k + 1. \)

We now suppose that (ii) holds and distinguish two cases as follows.

**Case 1:** \( D[V \setminus N^*(V_2)] \) has no isolated vertex.

Let \( W = V_2. \) It is easy to see that \( 1 \leq |W| \leq \left\lfloor \frac{k}{2} \right\rfloor \) and \( D[V \setminus N^*(W)] \) has no isolated vertex. By Theorem 3.5(1), we have \( |N^*[W]| \leq n + 2|W| - k. \) It follows that \( |V_1| = n - (|V_0| + |W|) = n - |N^*[W]| \geq k - 2|W| = |V_2| \) according to \( y_{\text{id}}(D) = k, \) and hence \( |N^*[W]| = n + 2|W| - k, \) (1) holds.

**Case 2:** \( D[V \setminus N^*(V_2)] \) has at least one isolated vertex.

Let \( W = V_2. \) It is easy to see that \( 1 \leq |W| \leq \left\lfloor \frac{k}{2} \right\rfloor \) and \( D[V \setminus N^*(W)] \) has at least one isolated vertex. Since \( h \) is a \( y_{\text{id}}(D)\)-function, we obtain \( V(V_1 \cup W) \subseteq N^*(W) \) and \( D[V_1 \cup W] \) has no isolated vertex. Furthermore, since \( y_{\text{id}}(D) = |V_0| + 2|W| = k, |V_1| + |W| = k - |W|, \) it implies that there is a \((V_1 \cup W, W, k)\)-structure with \( |V_1 \cup W| = k - |W| \) in \( D. \) Let \( X = V_1 \cup W, \) then (2) holds.

\((\Leftarrow)\) By Theorem 3.5, we have \( y_{\text{id}}(D) \geq k. \) Thus, it suffices for us to show that \( y_{\text{id}}(D) \leq k. \) If (1) holds, then the function \( g_0 = (N^*(W), V \setminus N^*[W], W) \) is a TRDF on \( D. \) Thus, \( y_{\text{id}}(D) \leq \omega(g_0) = |V \setminus N^*[W]| + 2|W| = n - (n + 2|W| - k) + 2|W| = k. \) If (2) holds, it is not difficult to verify that \( g_1 = (V \setminus X, X \setminus W, W) \) is a TRDF on \( D. \) Note \( |X| = k - |W|, \) then we have \( y_{\text{id}}(D) \leq \omega(g_1) = |X \setminus W| + 2|W| = k - 2|W| + 2|W| = k. \) Consequently, \( y_{\text{id}}(D) = k. \) \( \square \)

**Acknowledgment:** We would like to thank the anonymous referee for a thorough and helpful reading of the article.

**Funding information:** X. Zhang: The research is partially supported by the Fundamental Research Program of Shanxi Province (20210302123202). R. Li: The research is partially supported by the Youth Foundation of Shanxi Province (201901D211197).

**Author contributions:** All authors contributed equally to the writing of this article. All authors read and approved the final manuscript.

**Conflict of interest:** The authors state that there is no conflict of interest.
Data availability statement: The raw/processed data required to reproduce these findings cannot be shared at this time as the data also forms part of an ongoing study.

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