The construction of nuclei for normal constituents of $B_\pi$-characters

Abstract: Let $G$ be a $\pi$-separable group for some set $\pi$ of primes, let $\chi \in B_\pi(G)$ and let $N \triangleleft G$. In this article, we explore how to construct a nucleus for an irreducible constituent of $\chi_N$ via the given nucleus $(W, y)$ for $\chi$.

Keywords: $\pi$-separable group, nucleus, Fong character, quasi-primitive character

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1 Introduction

All groups considered in this article are finite, and the terminology and notation can be found in [1,2]. Let $\pi$ be a set of primes, and let $G$ be $\pi$-separable. Gajendragadkar [3] introduced the $\pi$-special characters for $\pi$-separable groups and showed that the product of a $\pi$-special character with a $\pi'$-special character is irreducible. In [4], Isaacs defined $\chi \in \text{Irr}(G)$ to be $\pi$-factored if $\chi = a\beta$, where $a$ is $\pi$-special and $\beta$ is $\pi'$-special. Furthermore, he constructed the set $B_\pi(G)$, which is a canonical lift of $I_\pi(G)$, where $I_\pi(G)$ is the set of the irreducible $\pi$-partial characters of $G$. In particular, when $\pi = p'$ is the complement of a prime $p$, then $B_p(G)$ is exactly a lift of $\text{IBr}(G)$ (the set of irreducible Brauer characters of $G$ at the prime $p$).

The key to define $B_\pi(G)$ is the construction of a nucleus $(W, y)$ for a given $\chi \in \text{Irr}(G)$ satisfying $W \leq G$, $y$ is $\pi$-factored, and $y^G = \chi$. We briefly review it here for convenience. Let $S^*(G)$ denote the set of maximal subnormal $\pi$-factored pairs of $G$. If $\chi$ is $\pi$-factored, then we define $(W, y) = (G, \chi)$. If $\chi$ is not $\pi$-factored, we choose $(S, \phi) \in S^*(G)$ such that $(S, \phi) \leq (G, \chi)$ and observe that $G_\phi < G$. In this case, Isaacs proved that there exists a unique irreducible character $\xi \in \text{Irr}(G_\phi)$ such that $\xi^G = \chi$, where $G_\phi$ is the stabilizer of $(S, \phi)$ in $G$.

By induction, a nucleus $(W, y)$ for $\chi$ has already been constructed, and Isaacs called $(W, y)$ a nucleus for $\chi$. We mention that all nuclei for $\chi$ are uniquely determined up to $G$-conjugacy, and that the set $B_\pi(G)$ consists of those characters $\chi \in \text{Irr}(G)$ with a $\pi$-special nucleus character $y$.

Similarly, Navarro [5] constructed a normal nucleus for a given $\chi \in \text{Irr}(G)$ with a maximal normal $\pi$-factored pair of $\phi$ instead of Isaacs’ maximal subnormal $\pi$-factored pair of $\chi$, defined the set $\mathcal{N}_\pi(G)$ as the set of those members of $\text{Irr}(G)$ having $\pi$-special nucleus characters, and showed that $\mathcal{N}_\pi(G)$ is also a lift of $I_\pi(G)$. Later, Lewis [6] introduced another new nuclei from a normal $\pi$-series $\mathcal{N}_\pi(G)$, and defined the set $B_\pi(G : \mathcal{N})$, which is also a lift of $I_\pi(G)$.

In this article, we study the behavior of the Isaacs’ nucleus $(W, y)$ for $\chi \in B_\pi(G)$ with respect to normal subgroups $N$ of $G$, so the nuclei mentioned below are in the sense of Isaacs. In the case where $G/N$ is a $\pi'$-group, Isaacs showed that a nucleus for $\chi$ can be constructed by a nucleus of an irreducible constituent of $\chi_N$, see Theorem 6.2(b) of [4]. We will consider the opposite direction.
Theorem A. Let \( N < G \), where \( G \) is \( \pi \)-separable. Let \( \chi \in B_\pi(G) \), and let \((W, y)\) be a nucleus for \( \chi \). Write \( V = N \cap W \), and let \( \tau \in \Irr(V) \) lie under \( y \).

1. If \( G / N \) is a \( \pi' \)-group, then \((V, \tau)\) is a nucleus for some irreducible constituent of \( \chi_N \).
2. If \( G / N \) is a \( \pi \)-group, and assume further that some Fong character associated with \( \chi \) is quasi-primitive. Then, \((V, \tau)\) is a nucleus for some irreducible constituent of \( \chi_N \).

In the situation of Theorem A(2), let \( \alpha \in \Irr(H) \) be a quasi-primitive Fong character associated with \( \chi \), where \( H \) is a Hall \( \pi \)-subgroup of \( G \). Write \( D = N \cap H \), so that \( D \) is a Hall \( \pi \)-subgroup of \( N \), and let \( \beta \) be the unique irreducible character of \( \alpha_N \). By Theorem B of [7], we see that \( \beta \) is also a Fong character for \( N \), but it need not be quasi-primitive.

The following is immediate.

Corollary B. Let \( N < G \), where \( G \) is \( \pi \)-separable. Let \( \chi \in B_\alpha(G) \) have \( \pi' \)-degree, and let \((W, y)\) be a nucleus for \( \chi \). Write \( V = N \cap W \), and let \( \tau \in \Irr(V) \) lie under \( y \). Then, \((V, \tau)\) is a nucleus for some irreducible constituent of \( \chi_N \).

2 Preliminaries

In this section, we review some preliminary results, which we refer to [8].

Lemma 2.1. Let \((S, \varphi)\) \(\in S^*(G)\), where \( G \) is \( \pi \)-separable, and let \( N < G\), where \( G / N \) is either a \( \pi \)-group or a \( \pi' \)-group. Write \( D = S \cap N \), and let \( \delta \in \Irr(D) \) lie under \( \varphi \). Then, \((D, \delta)\) \(\in S^*(N)\).

Proof. This is precisely Lemma 4.7 of [8].

Lemma 2.2. Let \( N < G \), where \( G \) is \( \pi \)-separable, and let \( \theta \in \Irr(N) \) be \( \pi \)-special.

(a) If \( G / N \) is a \( \pi \)-group, then every member of \( \Irr(G\theta) \) is \( \pi \)-special.

(b) If \( G / N \) is a \( \pi' \)-group and \( \theta \) is invariant in \( G \), then \( \theta \) has a canonical extension \( \chi \in \Irr(G) \) and \( \chi \) is the unique \( \pi \)-special character in \( \Irr(G\theta) \). If \( \theta \) is not invariant in \( G \), then no member of \( \Irr(G\theta) \) is \( \pi \)-special.

Proof. See Theorem 2.4 of [8].

Lemma 2.3. Let \((S, \varphi)\) \(\in S^*(G)\), where \( G \) is \( \pi \)-separable. Suppose that \((D, \delta) \leq (S, \varphi)\), where \( D < S \) and \( S / D \) is either a \( \pi \)-group or a \( \pi' \)-group. Then, \( G_D \leq N_G(S) \).

Proof. This is precisely Lemma 4.16 of [8].

Lemma 2.4. Let \( G \) be \( \pi \)-separable, and suppose \((S, \varphi)\) \(\in S^*(G)\). If \( G_\varphi = G \), then \( S = G \).

Proof. This is precisely Lemma 4.10 of [8].

Lemma 2.5. Let \((S, \varphi)\) \(\in S^*(G)\), where \( G \) is \( \pi \)-separable. Then, induction defines a bijection \( \Irr(G_\varphi) \rightarrow \Irr(G) \).

Proof. This is precisely Theorem 4.9 of [8].

Lemma 2.6. Let \( \alpha \in \Irr(H) \) be quasi-primitive, where \( H \) is a Hall \( \pi \)-subgroup of a \( \pi \)-separable group \( G \), and let \( H \leq W \leq G \), where \( W \) is the largest subgroup to which \( \alpha \) extends. Let \( y \in \Irr(W) \) be the unique \( \pi \)-special extension of \( \alpha \) to \( W \), and write \( \chi = y^G \). Then, \( \chi \) is irreducible, and in fact, \( \chi \in B_\alpha(G) \). In addition, \((W, y)\) is a nucleus for \( \chi \).
Proof. See Theorem 4.30 of [8]. □

Lemma 2.7. Suppose that $G$ is $\pi$-separable, and let $H \leq G$, where $|G : H|$ is a $\pi'$-number. Then, restriction defines an injection map from $X_\pi(G)$ to $X_\pi(H)$. In particular, this holds if $H$ is a Hall $\pi$-subgroup of $G$.

Proof. This is precisely Theorem 2.10 of [8]. □

Lemma 2.8. Let $N \triangleleft G$, where $G$ is $\pi$-separable and $G/N$ is a $\pi'$-group, and let $\delta \in \text{Irr}(D)$ lie under $\phi$. Then,
(a) $G_\phi \cap N_\phi < G_\phi$;
(b) $G_\phi/(G_\phi \cap N_\phi)$ is a $\pi'$-group;
(c) $G_\phi \cap N_\phi < N_\phi$;
(d) $N_\phi/(G_\phi \cap N_\phi)$ is a $\pi'$-group.

Proof. See Lemma 4.17 of [8]. □

Lemma 2.9. Let $N \triangleleft G$, where $G$ is $\pi$-separable and $G/N$ is a $\pi'$-group. Let $\chi \in B_\pi(G)$, and suppose that $N \leq K \leq G$. Then, every irreducible constituent of $X_K$ lies in $B_\pi(K)$.

Proof. This is precisely Theorem 4.25 of [8]. □

Lemma 2.10. Let $N \triangleleft G$, where $G$ is $\pi$-separable and $G/N$ is a $\pi'$-group, and let $\psi \in \text{Irr}(N)$.
(a) If $\chi \in B_\pi(G)$ lies over $\psi$, then $\psi \in B_\pi(N)$.
(b) If $\psi \in B_\pi(N)$, then there exists a unique character $\chi \in B_\pi(G)$ that lies over $\psi$. In addition, $[\chi_N, \psi] = 1$.
(c) Suppose that $\psi$ and $\chi$ are as in (b). Let $(V, \tau)$ be a nucleus for $\psi$, and write $W = G_\tau$. Then, $W/V$ is a $\pi'$-group, and $(W, y)$ is a nucleus for $\chi$, where $y$ is the canonical extension of $\tau$ to $W$.

Proof. See Theorem 4.19 of [8]. □

3 Main results

We begin with the proof of Theorem A discussed in Section 1.

Proof of Theorem A. (1) Choose $(S, \phi) \in S(G)$ with $(S, \phi) \leq (W, \gamma)$ and $W \leq G_{\phi}$, and let $S_1 = N \cap S$. It is clear that $S_1 \leq V$. Let $\phi_1 \in \text{Irr}(S_1)$ lie under $\phi$, and observe that $(S_1, \phi_1) \in S'(N)$ by Lemma 2.1. Since $\chi \in B_\pi(G)$, we have that both $\gamma$ and $\phi$ are $\pi$-special. Note that since $W/V$ is a $\pi'$-group, it follows that $\gamma_\tau$ is irreducible, and thus $\tau = \gamma_\tau$ is $\pi$-special. This means that $\tau$ is $W$-invariant, and $Y$ is a $\pi$-special extension of $\tau$ to $W$. By Lemma 2.2(b), we have $Y$ is the unique $\pi$-special character in $\text{Irr}(W/\tau)$. Also, since $S/S_1$ is a $\pi'$-group and $\phi$ is $\pi$-special, it follows that $\phi_{S_1}$ is irreducible, and thus $\phi_{S_1} = \phi_1$ is $\pi$-special. This means that $\phi_1$ is $S$-invariant, and $\phi$ is a $\pi$-special extension of $\phi_1$ to $S$. By Lemma 2.2(b), we have $\phi$ is the unique $\pi$-special character in $\text{Irr}(S_1)$.

Since $\phi_1$ lies under $\gamma$, and $Y = \tau$ is irreducible, it follows that $\phi_1$ lies under $\tau$.

We claim that $G_{\phi_1} = G_{\phi}$. Since $G_{\phi_1}$ normalizes $S \cap N = S_1$ and stabilizes $\phi$, we see that it also stabilizes $\phi_1$ because $\phi_1 = \phi_{S_1}$, and thus $G_{\phi} \leq G_{\phi_1}$. On the other hand, by Lemma 2.3, we know that $G_{\phi_1}$ normalizes $S$. Since $G_{\phi_1}$ stabilizes $S$ and $\phi_1$, by the uniqueness of $\phi$, we know that $G_{\phi_1}$ stabilizes $\phi$, and thus $G_{\phi_1} \leq G_{\phi}$, as claimed. So we conclude that $N_{\phi_1} = G_{\phi_1} \cap N = G_{\phi} \cap N$, and thus $N_{\phi_1} \triangleleft G_{\phi}$ and $G_{\phi}/N_{\phi_1}$ is a $\pi'$-group.

We proceed by induction on $|G|$. Suppose first that $G_{\phi} = G$. Then, $S = G$ by Lemma 2.4, and thus $V = N$, and the $\pi$-special character $\tau$ is an irreducible constituent of $X_N$. Then, $(V, \tau)$ is a nucleus for $\tau$. We can assume now that $G_{\phi} < G$. By Lemma 2.5, there exists a unique irreducible character $\xi \in \text{Irr}(G_{\phi}/\phi)$ such that...
\[ \xi^G = \chi. \] Note that \( \xi \in B_\eta(G_\rho) \), and \((W, \gamma)\) is also a nucleus for \( \xi \). It follows by the inductive hypothesis applied in \( G_\rho \) that \((V, \tau)\) is a nucleus for some irreducible constituent of \( \xi_{N_\eta} \). Let \( \eta = \tau^{N_\eta} \). We have that \((V, \tau)\) is a nucleus for \( \eta \). Now \( \eta \) lies under \( \xi \) and over \( \tau \), and thus under \( \chi \) and over \( \varphi_1 \). By Lemma 2.5, we conclude that \( \psi = \eta^N \) is irreducible. It follows that \( \psi \) lies under \( \chi \), and \((V, \tau)\) is a nucleus for \( \psi \). This proves (1).

(2) Let \( \alpha \in \text{Irr}(H) \) be a quasi-primitive Fong character associated with \( \chi \), where \( H \) is a Hall \( \pi \)-subgroup of \( G \). By Lemma 2.6, it is no loss to assume that \((H, \alpha) \leq (W, \gamma)\) because the nuclei for \( \chi \) are conjugate in \( G \), and note that \( W \) is precisely the largest subgroup of \( G \) to which \( \alpha \) extends, and \( y \) is the unique \( \pi \)-special extension of \( \alpha \) to \( W \).

Since \( G / N \) is a \( \pi \)-group, it follows that \( G = NH. \) Write \( D = H \cap N \). Then, \( D \lhd H \) and \( D = H \cap V \), and thus \( D \) is a Hall \( \pi \)-subgroup of \( V \). Since \( \alpha \) is quasi-primitive, it follows that there exists a unique \( \beta \in \text{Irr}(D) \) that lies under \( \alpha \), and thus \( \beta \) is the unique irreducible constituent of \( \gamma_D \).

We claim that \( y \) is also quasi-primitive. Let \( M \lhd W \). Then, \( M \cap H \lhd H \), and \( M \cap H \) is a Hall \( \pi \)-subgroup of \( M \). Since \( \alpha \) is quasi-primitive, then there exists a unique \( \eta \in \text{Irr}(M \cap H) \) that lies under \( \alpha \). It follows that \( \zeta \) is the unique irreducible constituent of \( \gamma_{M \cap H} \). Since \( M \lhd W \) and restriction defines an injective map from \( \chi_{\eta}(M) \) into \( \chi_{\eta}(M \cap H) \) by Lemma 2.7, we have that \( \chi_{\eta} \) has the unique constituent \( \rho_1 \), and \( \rho_1 \lhd H \), as desired.

We choose \((S, \varphi) \in S'(G)\) that lies under \((W, \gamma)\). Then, \( W \lhd G_\rho \), and thus \( \varphi \) is the unique irreducible constituent of \( \chi_S \). Write \( S_1 = S \cap N \), and let \( \varphi_1 \in \text{Irr}(S_1) \) lie under \( \varphi \), and thus under \( \gamma \). It follows by Lemma 2.1 that \((S_1, \varphi_1) \in S'(N)\). Note that \( V \lhd W \) and \( S \lhd W \). Then, \( S_1 \lhd W \) because \( S_1 = S \cap N = S \cap W \cap N = S \cap V \). Since \( \gamma \) is quasi-primitive, we see that \( \varphi_1 \) is the unique irreducible constituent of \( \chi_{S_1} \), and thus \( \varphi_1 \) is the unique irreducible constituent of \( \varphi_{S_1} \). Furthermore, we have that \( \tau \) is the unique irreducible constituent of \( \gamma_V \) because \( V \lhd W \). This forces that \( \varphi_1 \) is the unique irreducible constituent of \( \tau_{S_1} \). This means that \( \varphi_1 \) is \( V \)-invariant and \( S \)-invariant. Hence, \( V \leq N_{\varphi_1} \). Since \( \gamma \) is \( \pi \)-special, it follows that \( \tau \) is \( \pi \)-special, and thus \( \tau_0 = \beta \) because \( D \) is a Hall \( \pi \)-subgroup of \( V \).

Now we prove \( G_\rho \cap N = N_{\varphi_1} \). By Lemma 2.8, we conclude that \( N_{\varphi_1} / (G_\rho \cap N_{\varphi_1}) \) is a \( \pi \)-group. Since \( |N_{\varphi_1} : N_{\varphi_1} \cap G_\rho| \) divides \( |N : D| \), and \( |N : D| = |G : H| \) is a \( \pi' \)-number, we see that \( |N_{\varphi_1} : N_{\varphi_1} \cap G_\rho| = 1 \). Hence, \( N_{\varphi_1} \leq G_\rho \), and thus \( N_{\varphi_1} \leq G_\rho \cap N \). In order to prove \( G_\rho \cap N \leq N_{\varphi_1} \), it suffices to show that \( G_\rho \cap N \) stabilizes \((S_1, \varphi_1) \). Since \( S_1 = S \cap N \) and \( N \lhd G \), it follows that \( G_\rho \cap N \) stabilizes \( S \), \( \varphi \), and \( S_1 \). Recall that \( \varphi_1 \) is the unique irreducible constituent of \( \varphi_{S_1} \), it follows that \( G_\rho \cap N \) stabilizes \( \varphi_1 \), as wanted.

Work by induction on \( |G| \). If \( G_\rho = G \), by Lemma 2.4, we conclude that \( S = G \). Then, \( V = N \) and the \( \pi \)-special character \( \tau \) is an irreducible constituent of \( \chi_S \), and thus \((V, \tau)\) is a nucleus for itself. Now we assume \( G_\rho < G \). By Lemma 2.5, there exists a unique irreducible character \( \xi \in \text{Irr}(G_\rho \varphi) \) such that \( \xi^G = \chi \). By the inductive hypothesis applied in the group \( G_\rho \) with respect to the normal group \( N_\rho \), and the character \( \xi \in \text{Irr}(G_\rho \varphi) \), we know that \((V, \tau)\) is a nucleus for some irreducible constituent of \( \chi_{G_\rho \varphi} \). Let \( \eta = \tau^{N_\eta} \). Then, we have that \((V, \tau)\) is a nucleus for \( \eta \). Now \( \eta \) lies under \( \xi \) and over \( \tau \), and thus under \( \chi \) and over \( \varphi_1 \). By Lemma 2.5, it follows that \( \eta^N = \psi \) is irreducible. We see that \( \psi \) lies under \( \chi \) and \((V, \tau)\) is a nucleus for \( \psi \), and the result follows.

**Corollary 3.1.** Let \( N \lhd G \), where \( G \) is \( \pi \)-separable and \( G / N \) is a \( \pi' \)-group and let \( N \leq K \leq G \). Let \( \chi \in B_\eta(G) \) and suppose that \( \xi \) is an arbitrary irreducible constituent of \( \chi_K \). Then, we can choose some nucleus \((W, \gamma)\) for \( \chi \) such that \((W \cap K, \tau_{W \cap K})\) is a nucleus for \( \xi \).

**Proof.** Let \( \psi \) be an arbitrary irreducible constituent of \( \xi_K \). Then, \( \xi \in B_\eta(K) \) and \( \psi \in B_\eta(N) \) by Lemma 2.9. Since \( N \lhd G \), it follows that all of the irreducible constituents of \( \chi_K \) form an orbit under the conjugation action of \( G \) on \( \text{Irr}(N) \). By Theorem A(1), we can choose a nucleus \((W, \gamma)\) for \( \chi \) such that \((W \cap N, \gamma_{W \cap N})\) is a nucleus for \( \psi \) and note that \( W \) is the stabilizer of \((W \cap N, \gamma_{W \cap N})\) in \( G \) and \( \gamma \) is the canonical extension of \( \gamma_{W \cap N} \) to \( W \). Hence, \( \gamma_{W \cap K} \) is the canonical extension of \( \gamma_{W \cap N} \) to \( W \cap K \) and \( W \cap K \) is the stabilizer of \((W \cap N, \gamma_{W \cap N}) \) in \( K \). By Lemma 2.10, we have that \((W \cap K, \gamma_{W \cap K})\) is a nucleus for \( \xi \).
Proof of Corollary B. Since $\chi = \gamma^G$, it follows that $\chi(1) = |G : W|\gamma(1)$. We know that $\chi(1)$ is a $\pi'$-number. Hence, $\gamma(1) = 1$, and $|G : W|$ is a $\pi'$-number. Then, there exists a Hall $\pi$-subgroup $H$ of $G$ with $H \leq W$. Writing $\lambda = \gamma_H$, we conclude that $\lambda \in \operatorname{Irr}(H)$ is linear and is quasi-primitive certainly.

If $|G : N| = 1$, then $(V, \tau) = (W, \gamma)$, and the result is trivial. We can therefore assume that $N < G$, and we proceed by induction on $|G : N|$. Let $M / N$ be a chief factor of $G$. Then, we have that $M / N$ is a $\pi'$-group or a $\pi$-group. Observing that $|G : M| < |G : N|$ and $M \leq G$, by the inductive hypothesis, we conclude that $(M \cap W, \gamma_M)$ is a nucleus for an irreducible constituent $\psi$ of $\chi_M$.

If $M / N$ is a $\pi'$-group, by Theorem A(1), we know that $(N \cap M \cap W, (\gamma_{M\cap W})(N\cap M\cap W)) = (N \cap W, \gamma_{W}) = (V, \tau)$ is a nucleus for an irreducible constituent $\theta$ of $\psi_N$. Obviously, $\theta$ is an irreducible constituent of $\chi_N$. If $M / N$ is a $\pi$-group and note that $\lambda_{M\cap H}$ is quasi-primitive, where $M \cap H$ is a Hall $\pi$-subgroup of $M$, by Theorem A(2), we also conclude that $(V, \tau)$ is a nucleus for an irreducible constituent of $\chi$, and the proof is complete. □

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