Abstract: The prime objective of the approach is to give geometric classifications of \( k \)-almost Ricci solitons associated with paracontact manifolds. Let \( M^{2n+1} (\varphi, \xi, \eta, g) \) be a paracontact metric manifold, and if a \( K \)-paracontact metric \( g \) represents a \( k \)-almost Ricci soliton \((g, V, k, \lambda)\) and the potential vector field \( V \) is Jacobi field along the Reeb vector field \( \xi \), then either \( k = \lambda - 2n \), or \( g \) is a \( k \)-Ricci soliton. Next, we consider \( K \)-paracontact manifold as a \( k \)-almost Ricci soliton with the potential vector field \( V \) is collinear with \( \xi \). We have proved that if a paracontact metric as a \( k \)-almost Ricci soliton associated with the non-zero potential vector field \( V \) is collinear with \( \xi \) and the Ricci operator \( Q \) commutes with paracontact structure \( \varphi \), then it is Einstein of constant scalar curvature equals to \(-2n(2n+1)\). Finally, we have deduced that a para-Sasakian manifold admitting a gradient \( k \)-almost Ricci soliton is Einstein of constant scalar curvature equals to \(-2n(2n+1)\).

Keywords: \( k \)-almost Ricci solitons, Ricci soliton, Einstein manifold, paracontact metric manifold, infinitesimal paracontact transformation

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1 Introduction and main results

Through the last two decades, the Ricci solitons as self-similar solutions of the Ricci flow had been increasingly present in theory and in application to theoretical physics and mathematics. Nowadays, the Ricci solitons and their generalizations on pseudo-Riemannian settings are important research topics on differential geometry. On pseudo-Riemannian geometry, paracontact metric structures were originated by Kaneyuki and Williams [1], as a natural odd-dimensional analogous to para-Hermitian structures. The significance of paracontact geometry is emitted from the para-Kähler manifolds theory, while the theory of Legendre foliations is linked to the geometry of paracontact metric manifolds. Several authors investigated paracontact geometry using various meaningful geometric conditions [2–4]. For recent progress of Ricci solitons on paracontact geometry and related studies, the reader can use [5–35] as references.

In [36], Pigola et al. generalized the notion of Ricci soliton to almost Ricci soliton by imposing the soliton constant \( \lambda \) to be a deferential function and defined as follows: a pseudo-Riemannian manifold \((M^n, g)\) of
dimension $n$ is referred to as an **almost Ricci soliton** if we find a vector field $V$ on $M^n$ and a differential function $\lambda : M^n \to \mathbb{R}$ in which $\mathcal{L}_V g + 2\text{Ric} = 2\lambda g$, where $\mathcal{L}_V$ stands for the Lie-derivative in the direction of $V$ and $\text{Ric}$, $r$ are Ricci tensor, scalar curvature of $g$. If the potential vector field is Killing, then the Ricci soliton is trivial and soliton equation reduces to an Einstein metric (i.e., in case that $\dim M > 2$, the Ricci tensor is just a constant multiple of the Riemannian metric). Thus, we can say almost Ricci's soliton is a direct generality of Einstein's metric. The $k$-almost Ricci soliton is a generalization of almost Ricci solitons, Ricci solitons, Yamabe solitons, and Einstein metrics. It was first introduced by Gomes et al. [37] who defines a pseudo-Riemannian manifold $(M^n, g)$ is called a $k$-almost Ricci soliton if we find a vector field $V$ on $M^n$ and two differential functions $\lambda$, $k : M^n \to \mathbb{R}$ in which

$$
\frac{k}{2} \mathcal{L}_V g + \text{Ric} = \lambda g,
$$

(1.1)

and it is denoted by $(M^n, g, V, k, \lambda)$. It is clear that $k$-almost Ricci soliton is a natural extension of the almost Ricci soliton by Pigola et al. [36]. A $k$-almost Ricci soliton is referred to as steady, shrinking, or expanding according as $\lambda = 0$, $\lambda > 0$, or $\lambda < 0$, respectively. It is trivial (Einstein) if the potential vector field $V$ is homothetic, that is, for some constant $c$, $\mathcal{L}_V g = cg$. Otherwise, it is non-trivial. In particular, for constant $\lambda$, it is referred to as $k$-Ricci soliton, and an almost Ricci soliton is just the 1-almost Ricci soliton. A special case of soliton occurs when the potential vector field $V$ is the gradient of a differential function $u$ on $M^n$, i.e., $V = \nabla u$, and then the $k$-almost Ricci soliton is called a gradient $k$-almost Ricci soliton. Then equation (1.1) is exhibited as follows:

$$
k\nabla^2 u + \text{Ric} = \lambda g,
$$

(1.2)

where $\nabla^2 u$ is the Hessian of $u$. In this case, we denote $(M^n, g, \nabla u, k, \lambda)$ as a gradient $k$-almost Ricci soliton associated with the potential function $u$. It was proved [37] that a compact nontrivial $k$-almost Ricci soliton of dimension $\leq 3$ in which $k$ is of defined signal and constant scalar curvature and is isometric to a standard sphere of well-determined potential function. It has been shown that a non-trivial non-compact $k$-almost Ricci soliton $(M^n, g, V, k, \lambda)$ with $k$ defined signal, such that $\mathcal{L}_V r \leq 0$ and $|\text{Ric}(V)|$ lies in $L^1(M^n)$, is an Einstein manifold of non positive scalar curvature $r$, that is, isometric to $\mathbb{R}^n$ if $r = 0$. Recently, in [38], Hajar has shown that if a potential vector field $V$ is a non-zero conformal vector field, then a compact $k$-almost Ricci soliton with $n > 2$ is isometric to a Euclidean sphere $\mathbb{S}^n$. In addition, he has proved that every two-dimensional non-trivial compact gradients $k$-almost Ricci soliton $(M^n, g, \nabla u, k, \lambda)$ is isometric to a Euclidean sphere $\mathbb{S}^n(r)$ if $M^n$ is of constant scalar curvature. Faraji et al. [39] studied the $k$-almost Ricci soliton admitting concurrent potential fields and proved several classifications results on $k$-almost Ricci soliton immersed as a hypersurface in Euclidean space $\mathbb{R}^{n+1}$. On the other hand, many important Ricci solitons were investigated on contact metric structures as a natural odd-dimensional counterpart of complex structures. For example, $k$-almost Ricci solitons in contact geometry are considered in [40]. It has been proved that there is an isometric mapping between gradient $k$-almost Ricci soliton and the sphere $\mathbb{S}^{2n+1}$ if $K$-contact metric endowed with gradient $k$-almost Ricci soliton. Several results have been derived in contact geometry (see [40] and the references therein).

Recently, the interest in Ricci soliton’s study has come to light. The generality of the study on paracontact metric manifolds is a natural odd-dimensional counterpart to para-Hermitian structures. In this direction, Patra [12,13] showed that if a $K$-paracontact metric $g$ as a Ricci soliton or an almost Ricci soliton with the nontrivial potential vector field $V$ point-wise collinear with $\xi$ and the paracontact structure $\phi$ commutes with Ricci operator $Q$, then $V$, a constant multiple of $\xi$ and $g$, is Einstein with constant scalar curvature. On the other hand, Patra [12] showed that the metric $g$ of $K$-paracontact manifold is Einstein with constant scalar curvature if it is a gradient Ricci soliton. Such a finding has been generalized by Patra et al. [13] for almost Ricci soliton. Motivated by previous deployment on solitons geometry, we provide proof of the following results.

**Theorem 1.1.** Assume that $M^{2n+1}(\phi, \xi, \eta, g)$ is a $K$-paracontact manifold admitting a $k$-almost Ricci soliton with the potential vector field $V$ is being Jacobi field in direction of $\xi$. Then, either $k = \lambda - 2n$ or $g$ is a $k$-Ricci soliton.
Theorem 1.2. Assume that a K-paracontact metric $g$ represents a $k$-almost Ricci soliton and the potential vector field is an infinitesimal paracontact transformation and the paracontact structure $\varphi$ commutes with Ricci operator $Q$, then $g$ is $\eta$-Einstein and $V$ leaves $\varphi$ invariant.

Theorem 1.3. Suppose that $M^{2n+1}(\varphi, \xi, \eta, g)$ is a K-paracontact manifold admitting a $k$-almost Ricci soliton with the non-zero potential vector field $V$ is collinear with $\xi$, then $g$ is Einstein with constant scalar curvature is equal to $-2n(2n + 1)$.

To generalize the aforementioned theorem on paracontact metric manifold, we provide proof of the following result.

Theorem 1.4. Suppose that $M^{2n+1}(\varphi, \xi, \eta, g)$ is a $k$-almost Ricci soliton associated to the non-trivial potential vector field $V$, that is collinear with $\xi$, and the Ricci operator $Q$ commuting with paracontact structure $\varphi$, then $g$ is Einstein of constant scalar curvature equals to $-2n(2n + 1)$.

Next, we consider the gradient $k$-almost Ricci soliton on $K$-paracontact manifold and derive the following:

Theorem 1.5. Assume that a $K$-paracontact metric $g$ represents a gradient $k$-almost Ricci soliton and the paracontact structure $\varphi$ commutes with Ricci operator $Q$, then $g$ is Einstein of constant scalar curvature $-2n(2n + 1)$.

For the para-Sasakian manifold, we give the following interesting result.

Theorem 1.6. Assume that $M^{2n+1}(\varphi, \xi, \eta, g)$ is a para-Sasakian manifold and $g$ represents a gradient $k$-almost Ricci soliton. Then, $g$ is Einstein with constant scalar curvature equal to $-2n(2n + 1)$.

Remark 1.1. It is noticed that the results of $k$-almost Ricci solitons in paracontact geometry are different and interesting as compared to contact geometry [40]. Therefore, the paracontact metric is motivational in the context of soliton geometry.

Thus, while gradient Ricci soliton and gradient almost Ricci soliton are covered, $k$-almost Ricci soliton generalizes the generalized $m$-quasi Einstein metric. To obtain more information, see [8, 37]. Yun et al. recently investigated Bach-flat $k$-almost gradient Ricci solitons [19]. Geometric flows are a family of partial differential equations that describe the evolution of geometric objects over time. Ricci flow is one such flow that is used to study the behavior of metrics on manifolds. The $k$-almost Ricci solitons arise naturally as fixed points of certain geometric flows, and their study can shed light on the long-time behavior of these flows. On the other side, $k$-almost Ricci solitons also arise in various areas of mathematical physics, including string theory and quantum gravity. They provide a mathematical framework for studying the behavior of physical systems at the smallest scales, where the effects of quantum mechanics become important.

2 Preliminaries and notations

Here, we recall some basic conceptions and equations of paracontact metric manifolds [2, 5--7]. A $(2n + 1)$-dimensional smooth manifold $M$ (or $M^{2n+1}$) is of an almost paracontact structure $(\varphi, \xi, \eta)$ if it has a vector field $\xi$ (called Reeb vector field), a $(1, 1)$-tensor field $\varphi$ and a 1-form $\eta$ verifying the relations:

(i) $\varphi(\xi) = 0, \quad \eta(\xi) = 1, \quad \eta \circ \varphi = 0, \quad \varphi^2 = I - \eta \circ \xi$,

(2.1)

(ii) There exists a distribution $\mathcal{D} : p \in M \rightarrow \mathcal{D}_p \subset T_p M : \mathcal{D}_p = \text{Ker}(\eta) = \{x \in T_p M : \eta(x) = 0\}$ is referred to as paracontact distribution generated by $\eta$. 


Let $\mathcal{X}_M$ be the module over $C^\infty(M)$ of all vector fields on $M$. If an almost paracontact manifold has a pseudo-Riemannian metric $g$ in which

$$g(\phi X_1, \phi X_2) = -g(X_1, X_2) + \eta(X_1)\eta(X_2), \quad X_1, X_2 \in \mathcal{X}_M,$$

then $M$ admits an almost paracontact metric structure $(\phi, \xi, \eta, g)$ and $g$ is referred to as a compatible metric. Moreover, compatible metrics necessarily have signature $(n + 1, n)$. The fundamental 2-form $\Phi$ of an almost paracontact metric structure $(\phi, \xi, \eta, g)$ is given by

$$\Phi(X_1, X_2) = -\phi X_1 \cdot \phi X_2 \in \mathcal{X}_M.$$

In case that $\Phi = \lambda \eta$ for some real function $\lambda$ on $M$, then $M$ is referred to as a paracontact metric manifold and we can define two self-adjoint operators on $M$ as follows:

$$h = \frac{1}{2} L_\xi g, \quad \lambda = \frac{1}{2} L_\xi h,$$

where $L_\xi$ is the Lie-derivative in the direction of $\xi$ and $R$ is the Riemann curvature tensor of $g$ defined as follows:

$$R(X_1, X_2) = \nabla_X \nabla_Y - \nabla_Y \nabla_X + [X, Y] \otimes g,$$

where $\nabla$ is the operator of covariant differentiation of $g$.

A paracontact metric structure on $M$ is referred to as normal if the almost para-complex structure $J$ on $\mathcal{T}/\mathbb{R} \times M$ given as follows:

$$JX = \phi X + \eta(X)\xi \in \mathcal{X}_M,$$

where $\mathcal{T}$ is the tangent bundle of $M$ and the coordinate of $\mathcal{T}/\mathbb{R}$ is integrable. A normal paracontact metric manifold is referred to as para-Sasakian. Equivalently, a paracontact metric manifold is referred to as para-Sasakian if

$$\nabla_X \phi = -\phi X, \quad \nabla_X \xi = -X + \eta(X)\xi,$$

where $Q$ stands for the Ricci operator associated to the Ricci tensor defined as

$$\operatorname{Ric}(\xi, \xi) = g(Q\xi, \xi) = \tau_{\xi} h - \tau_{(\xi, \xi)} = 2n - \tau_{\xi} h,$$

where $\tau_{\xi} h$ is the mean curvature of $\xi$ and $\tau_{(\xi, \xi)}$ is the scalar curvature at a point $p \in M$.

A para-Sasakian manifold is $K$-paracontact, but the converse is true only if it is of dimension 3 [5,6]. A para-Sasakian manifold $M$ is referred to as $\eta$-Einstein if its Ricci tensor $\text{Ric}$ can be written as follows:

$$\text{Ric} = a g + b \eta \otimes \eta,$$

in which $a, b$ are differentiable functions on $M$. A vector field $X_1$ on a paracontact manifold $M$ is referred to as an infinitesimal paracontact transformation (or a para-contact vector field) if it preserves the paracontact form $\eta$, i.e., $\exists$ a differentiable function $f : M \to \mathbb{R}$, satisfies

$$\xi_X \eta = f \eta.$$

When $f = 0$ on $M$, the vector field $X_1$ is referred to as a strict.

**Lemma 2.1.** (Lemma 3.1 of [12]) On a $K$-paracontact manifold $(M, g)$, we have

$$\nabla_q (\xi) X_1 = Q\phi X_1 - \phi QX_1, \quad \nabla_q (\xi) \xi = Q\phi X_1 + 2n\phi X_1,$$

where $Q = \frac{1}{2} L_\xi g$. The Lie-derivative in the direction of $\xi$ and $R$ is the Riemann curvature tensor of $g$ defined as follows:

$$R(X_1, X_2) = \nabla_X \nabla_Y - \nabla_Y \nabla_X + [X, Y] \otimes g,$$

where $\nabla$ is the operator of covariant differentiation of $g$. Then the two operators $h$ and $l$ satisfy the following conditions [2]:

$$h = \phi h, \quad l = \xi h,$$

where $\tau_{\xi} h$ is the mean curvature of $\xi$ and $\tau_{(\xi, \xi)}$ is the scalar curvature at a point $p \in M$. A paracontact metric structure on $M$ is referred to as normal if the almost para-complex structure $J$ on $\mathcal{T}/\mathbb{R} \times M$ given as follows:

$$JX = \phi X + \eta(X)\xi \in \mathcal{X}_M,$$

where $\mathcal{T}$ is the tangent bundle of $M$ and the coordinate of $\mathcal{T}/\mathbb{R}$ is integrable. A normal paracontact metric manifold is referred to as para-Sasakian. Equivalently, a paracontact metric manifold is referred to as para-Sasakian if

$$\nabla_X \phi = -\phi X, \quad \nabla_X \xi = -X + \eta(X)\xi,$$

where $Q$ stands for the Ricci operator associated to the Ricci tensor defined as

$$\operatorname{Ric}(\xi, \xi) = g(Q\xi, \xi) = \tau_{\xi} h - \tau_{(\xi, \xi)} = 2n - \tau_{\xi} h,$$

where $\tau_{\xi} h$ is the mean curvature of $\xi$ and $\tau_{(\xi, \xi)}$ is the scalar curvature at a point $p \in M$. A para-Sasakian manifold is $K$-paracontact, but the converse is true only if it is of dimension 3 [5,6]. A para-Sasakian manifold $M$ is referred to as $\eta$-Einstein if its Ricci tensor $\text{Ric}$ can be written as follows:

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$$\xi_X \eta = f \eta.$$

When $f = 0$ on $M$, the vector field $X_1$ is referred to as a strict.
3 Proofs main outcomes

3.1 Proof of Theorem 1.1

Taking covariant derivative of (1.1) along \( X_i \in \mathcal{X}_M \) and using (2.4), we have
\[
(X_0 k)(\mathcal{L}_V g)(X_1, X_2) + k(\nabla_{X_i} \mathcal{L}_V g)(X_i, X_2) = 2(\nabla_{X_i} \text{Ric})(X_i, X_2) - 2(X_0 \lambda) g(X_i, X_2),
\]
(3.1)
for all \( X_1, X_2 \in \mathcal{X}_M \). Next, we recall the following commutation relation (see Yano [41, p.23]):
\[
(\mathcal{L}_V \nabla_X g - \nabla_X \mathcal{L}_V g)(X_1, X_2) = -g((\mathcal{L}_V \nabla)(X_1, X_2), X_2) - g((\mathcal{L}_V \nabla)(X_2, X_1), X_2).
\]
(3.2)
On pseudo-Riemannian manifold \((M, g)\), we have
\[
\nabla_X \nabla_Y g = \nabla_Y \nabla_X g + \text{Ric}(X_1, X_2) g(X_1, X_2),
\]
(3.3)
since the metric \( g \) is parallel. By utilizing the symmetry of the \((1, 2)\)-type tensor field \( L^g \), i.e.,
\[
L^g(\nabla_X \nabla_Y g) = \nabla_Y \nabla_X g - \text{Ric}(X_1, X_2) g(X_1, X_2),
\]
and interchanging the roles of \( X_1, X_2, \) and \( X_3 \) in the preceding equation, we can compute
\[
k g((\mathcal{L}_V \nabla)(X_1, X_2), X_3) = \frac{1}{2}(\nabla_X \mathcal{L}_V g)(X_1, X_3) + \frac{1}{2}(\nabla_X \mathcal{L}_V g)(X_2, X_3) + \frac{1}{2}(\nabla_X \mathcal{L}_V g)(X_3, X_3).
\]
(3.4)
Next, by combining (3.1) and (3.4), we deduce
\[
\text{L}^g((\mathcal{L}_V \nabla)(X_1, X_2), X_3) = \frac{1}{2}(\nabla_X \mathcal{L}_V g)(X_1, X_3) + \frac{1}{2}(\nabla_X \mathcal{L}_V g)(X_2, X_3) + \frac{1}{2}(\nabla_X \mathcal{L}_V g)(X_3, X_3).
\]
(3.5)
Now by substituting \( \xi \) for \( X_2 \) in (3.5) and invoking the Lemma 2.1, we obtain
\[
k(\mathcal{L}_V \nabla)(X_1, \xi) = -2Q_0 \xi - 4n \xi_2 + (\xi_1) \xi_1 + (X_0 \lambda) \xi_1 - \eta(X_1) \nabla \lambda + \frac{1}{k}[(\xi_1)(QX_1 - \lambda X_1) - (X_0 \lambda)(\xi_1) - \eta(X_1) \nabla \lambda].
\]
(3.6)
Setting \( X_2 = \xi \) in the well-known formula [41, p.23]:
\[
\nabla_X \nabla_Y V - \nabla_Y \nabla_X V - R(V, X_1)X_2 = (\mathcal{L}_V \nabla)(X_1, X_2),
\]
(3.7)
and by plugging the value of \((\mathcal{L}_V \nabla)(X_1, \xi)\) in (3.6), we obtain
\[
k\{\nabla_X \nabla_Y V - \nabla_Y \nabla_X V - R(V, X_1)\xi\}
\[
= -2Q_0 \xi - 4n \xi_2 + (\xi_1) \xi_1 + (X_0 \lambda) \xi_1 - \eta(X_1) \nabla \lambda + \frac{1}{k}[(\xi_1)(QX_1 - \lambda X_1) - (X_0 \lambda)(\xi_1) - \eta(X_1) \nabla \lambda].
\]
(3.8)
Next taking \( \xi \) instead of \( X_1 \) and using (2.8), then \( \phi \xi = 0 \) gives
\[
k^2(\nabla_X \nabla_Y V - R(V, \xi)\xi) = (k - \lambda - 2n)(2(\xi)(\xi) - \nabla \lambda).
\]
(3.9)
Assume the potential vector \( V \) is a Jacobi field in direction of \( \xi \), That is, \( \nabla_X \nabla_Y V - R(V, \xi)\xi = 0 \). It follows from (3.9) that
\[
(k - \lambda - 2n)(\nabla \lambda - 2(\xi)(\xi)) = 0.
\]
(3.10)
Suppose \( k \neq \lambda + 2n \) on \( M \). It follows from (3.10) that \( \nabla \lambda = 2(\xi)(\xi) \). Co-variant derivative of this along \( X_1 \in \mathcal{X}_M \) and then inner product with \( X_2 \in \mathcal{X}_M \), we attain
\[
g(\nabla_X \nabla \lambda, X_2) = 2X_2(\xi)(\xi) + 2(\xi)(\xi)g(\xi X_2).
\]
(3.11)
By making use of the property \( g(\nabla_X \nabla \lambda, X_2) = g(\nabla_X \nabla \lambda, X_2) \) and the formula \( d\eta(X_1, X_2) = g(X_1, \phi X_2) \), we can conclude from (3.11) that \( (\xi)(\xi)d\eta(X_1, X_2) = 0 \) all \( X_1, X_2 \) orthogonal to \( \xi \). This gives that \( \xi = 0 \) on \( M \), as \( d\eta \) is
non-zero everywhere on \( M \). It follows from (3.10) that \( \nabla \lambda = 0 \) on \( M \), and therefore, \( \lambda \) is constant on \( M \). Hence, \( g \) is a \( k \)-Ricci soliton. The rest of the proof easily follows from (3.9).

### 3.2 Proof of Theorem 1.2

By utilizing the property of exterior product and the Lie-derivative of (2.10) in direction of \( V \), we have
\[
(\mathcal{L}_V d\eta)(X_1, X_2) = d(\mathcal{L}_V \eta)(X_1, X_2) = \frac{1}{2} [g(X_1, Vf)\eta(X_2) - g(X_2, Vf)\eta(X_1)] + f d\eta(X_1, X_2), \quad X_1, X_2 \in \mathfrak{X}_M. \tag{3.12}
\]

Now, recalling a formula on paracontact metric manifold:
\[
\eta(X_1, X_2) = g(X_1, \varphi X_2) \tag{3.13}
\]

Plugging (3.12) and \( d\eta(X_1, X_2) = g(X_1, \varphi X_2) \) in (3.13) gives us
\[
k(\mathcal{L}_V \varphi)(X_1) + 2Q\varphi X_1 + (2\lambda - kf)\varphi X_1 = \frac{k}{2} (\eta(X_1)\nabla f - g(X_1, Vf)\xi). \tag{3.14}
\]

By utilizing (2.8), the soliton equation (1.1) reduces to
\[
k(\mathcal{L}_V g)(X_1, \xi) = 2(\lambda + 2n)\eta(X_1), \quad X_1 \in \mathfrak{X}_M. \tag{3.15}
\]

Then, by taking Lie-derivative of \( \eta(X_1) = g(X_1, \xi) \) and utilizing (3.15), we obtain
\[
k g(\mathcal{L}_V \xi, X_1) = (kf - 2\lambda - 4n)\eta(X_1). \tag{3.16}
\]

As a result \( \varphi \xi = 0 \), we obtain \( (\mathcal{L}_V \varphi)\xi + \varphi(\mathcal{L}_V \xi) = 0 \), which yields from (3.16) that \( (\mathcal{L}_V \varphi)\xi = 0 \). Furthermore, if we replace \( X_1 \) by \( \xi \) in (3.14), we have
\[
2(\mathcal{L}_V \varphi)(\xi) = \nabla f - (\xi f)\xi,
\]

as \( k \) is a non-zero differentiable function on \( M \). Therefore, we must have \( \nabla f = (\xi f)\xi \). Covariantly differentiating it along \( X_1 \in \mathfrak{X}_M \) and using (2.8), we acquire
\[
\mathcal{L}_X \nabla f = X_1(\xi f)\xi - (\xi f)\varphi X_1.
\]

By utilizing this in the formula \( g(\nabla_X \nabla f, X_1) = g(\nabla_X, Vf, X_1) \), we deduce
\[
X_1(\xi f)\eta(X_1) - X_1(\xi f)\eta(X_1) = 2(\xi f)g(\varphi X_1, X_2). \tag{3.17}
\]

Now replacing \( X_1 \) by \( \varphi X_1 \) and \( X_2 \) by \( \varphi X_2 \) and recalling (2.1) and \( d\eta(X_1, X_2) = g(X_1, \varphi X_2) \), we achieve \((\xi f) d\eta(X_1, X_2) = 0\) for any vector fields \( X_1, X_2 \) on \( M \). It follows that \( \xi f = 0 \), as \( d\eta \) is non-vanishing on \( M \). As a result, \( \nabla f = 0 \) on \( M \). Thus, we can conclude that \( f \) is a constant on \( M \). Further, taking Lie-derivative of \( g(\xi, \xi) = 1 \) and using (3.15) and (3.16), we derive \( kf = \lambda + 2n \). By again taking the Lie-derivative of \( \varphi^2 X_1 = X_1 - \eta(X_1)\xi \) along \( V \), we obtain
\[
(\mathcal{L}_V \varphi)\varphi X_1 + \varphi(\mathcal{L}_V \varphi)X_1 + (\mathcal{L}_V \eta)(X_1)\xi + \eta(X_1)\mathcal{L}_V \xi = 0. \tag{3.18}
\]

By using (2.10) and (3.16) in (3.18), we arrive at
\[
(\mathcal{L}_V \varphi)\varphi X_1 + \varphi(\mathcal{L}_V \varphi)X_1 + 2(kf - \lambda - 2n)\eta(X_1)\xi = 0. \tag{3.19}
\]

As \( f \) is constant, equation (3.14) reduces to
\[
k(\mathcal{L}_V \varphi)X_1 + 2Q\varphi X_1 + (2\lambda - kf)\varphi X_1 = 0. \tag{3.20}
\]

By making use of (3.20) and \( kf = \lambda + 2n \) in (3.19), we compute
\[
Q\varphi^2 X_1 + \varphi Q\varphi X_1 + (\lambda - 2n)\varphi^2 X_1 = 0. \tag{3.21}
\]
By assumption, \( \phi \) commutes with the Ricci operator \( Q \), i.e., \( Q\phi = \phi Q \), plugging it into (3.21) and using (2.1) yields that

\[
\text{Ric} = \left\{ \frac{n - \lambda}{2} \phi \right\} + \left\{ 3n - \frac{\lambda}{2} \right\} \eta \otimes \eta.
\] (3.22)

Thus, \((M, g)\) is \( \eta \)-Einstein. Replacing \( X_i \) by \( \phi X_i \) in (3.22), we have \( 2Q\phi X_i = (2n - \lambda)\phi X_i \). By putting it into (3.20) and using \( kf = \lambda + 2n \), we obtain \( \mathcal{L}_V \phi = 0 \), and hence, \( V \) leaves \( \phi \) invariant. The proof is completed.

### 3.3 Proof of Theorem 1.3

Since the potential vector field \( V \) is collinear with \( \xi \), i.e., \( V = \sigma \xi \) for a non-zero differentiable function \( \sigma \) on \( M \). In view of its covariant derivative and equation (2.6), we can compute

\[
(L_V g)(X_i, X_j) = g(\nabla_{X_i} V, X_j) + g(\nabla_{X_j} V, X_i) = (X_i \sigma)\eta(X_j) + (X_j \sigma)\eta(X_i),
\]

where we have utilized the anti-symmetric property of \( \phi \). In view of this, equation (1.1) becomes

\[
k(X_i \sigma)\eta(X_j) + k(X_j \sigma)\eta(X_i) + 2\text{Ric}(X_i, X_j) = 2\lambda g(X_i, X_j)
\] (3.23)

for all \( X_i, X_j \in \mathfrak{X}_M \). At this point, replacing \( X_j \) by \( \xi \) (3.23) and recalling (2.8) yields that

\[
kX_i(\sigma) = (2\lambda + 4n - k\xi(\sigma))\eta(X_i), \quad X_i \in \mathfrak{X}_M.
\] (3.24)

Again, by substituting \( \xi \) for both \( X_i \) and \( X_j \) and noting that (2.8), we acquire \( k\xi(\sigma) = \lambda + 2n \), and hence, equation (3.24) follows that \( X_i(\sigma) = \xi(\sigma)\eta(X_i) \), as \( k \) is a non-zero differentiable function on \( M \). By the idea of the previous theorem, we deduce that \( \sigma \) is a constant on \( M \), and hence, equation (3.23) reduces to \( \text{Ric} = \lambda g \). With the addition of (2.8), it follows that \( \lambda = -2n \), consequently, \( \text{Ric} = -2ng \), which settle our claim.

### 3.4 Proof of Theorem 1.4

Since \( V \) is collinear with \( \xi \), i.e., \( V = \sigma \xi \) for a non-zero differentiable function \( \sigma \) on \( M \). By using its covariant derivative and equation (2.4), we can compute

\[
(L_V g)(X_i, X_j) = g(\nabla_{X_i} V, X_j) + g(\nabla_{X_j} V, X_i) = (X_i \sigma)\eta(X_j) + (X_j \sigma)\eta(X_i) + 2\text{sg}(\phi h X_i, X_j),
\]

by using the anti-symmetric property of \( \phi \) and \( \phi h = -h\phi \). From virtue of this, the soliton equation (1.1) reduces to the following:

\[
k(X_i \sigma)\eta(X_j) + k(X_j \sigma)\eta(X_i) + 2\text{Ric}(X_i, X_j) + 2k\text{sg}(\phi h X_i, X_j) = 2\lambda g(X_i, X_j)
\] (3.25)

for all \( X_i, X_j \in \mathfrak{X}_M \). Now, by substituting \( \xi \) for \( X_j \) in (3.25) and using \( \phi(\xi) = 0 \), we obtain

\[
k\{D\sigma + (\xi \sigma)\xi\} + 2Q\xi = 2\lambda \xi.
\] (3.26)

Again, setting \( X_i = X_j = \xi \) in (3.25) and applying (2.5), \( \phi(\xi) = 0 \), we derive

\[
k(\xi \sigma) + \text{Tr}_g l = \lambda.
\] (3.27)

By the hypothesis of the theorem: \( Q\phi = \phi Q \). Combining this with (2.5) yields that

\[
Q\xi = (\text{Tr}_g l)\xi.
\] (3.28)

From the aforementioned equation together with (3.27) inserting in (3.26), we achieve

\[
k\nabla\sigma = k(\xi \sigma)\xi.
\]

This implies that \( \nabla\sigma = (\xi \sigma)\xi \). Covariantly by differentiating it along \( X_i \in \mathfrak{X}_M \) and utilizing (2.6), we attain

\[
\nabla_X \nabla\sigma = X(\xi \sigma)\xi - (\xi \sigma)(\phi X_i + \phi h X_i).
\]
By using this in the formula \( g(V_{\xi}V_{\sigma}, Y_{1}) = g(V_{\eta}V_{\sigma}, X_{1}) \) and noting that \( h \varphi + \varphi h = 0 \), we deduce
\[
X_{1}(\xi \sigma) \eta(Y_{1}) - Y_{1}(\xi \sigma) \eta(X_{1}) = 2(\xi \sigma)g(\varphi(X_{1}), Y_{1}).
\] (3.29)
Now replacing \( X_{1} \) by \( \varphi X_{1} \) and \( Y_{1} \) by \( \varphi Y_{1} \) in (3.29) and recalling (2.1) and \( d\eta(X_{1}, Y_{1}) = g(X_{1}, \varphi Y_{1}) \), we achieve \((\xi \sigma)d\eta(X_{1}, Y_{1}) = 0, \forall X_{1}, Y_{1} \) on \( M \). It follows that \( \sigma \xi = 0 \), as \( d\eta \) is non-zero on \( M \). Consequently, \( \nabla \sigma = 0 \) on \( M \). Thus, we deduce that \( \sigma \) is a constant on \( M \). This transforms equation (3.25) into the following:
\[
k\sigma g(\varphi h X_{1}, X_{2}) + \text{Ric}(X_{1}, X_{2}) = \lambda g(X_{1}, X_{2}).
\] (3.30)
Now, by replacing \( X_{1} \) by \( \varphi X_{1} \) and \( X_{2} \) by \( \varphi X_{2} \) in (3.30) and using (2.1),
\[
\text{Ric}(X_{1}, X_{2}) - \text{Ric}(\varphi X_{1}, \varphi X_{2}) = \lambda(2g(X_{1}, X_{2}) - \eta(X_{1})\eta(X_{2})).
\] (3.31)
By combining (3.30) and (3.31), we obtain
\[
\text{Ric}(X_{1}, X_{2}) - \text{Ric}(\varphi X_{1}, \varphi X_{2}) = \lambda(2g(X_{1}, X_{2}) - \eta(X_{1})\eta(X_{2})).
\] (3.32)
Now, by inserting \( X_{1} = \varphi X_{1} \) in (3.32) and recalling (2.1), we acquire
\[
Q\varphi X_{1} + \varphi Q(X_{1} - \eta(X_{1})\xi) = 2\lambda \varphi X_{1}.
\] (3.33)
Since, the paracontact structure \( \varphi \) commutes with Ricci operator \( Q \), equation (3.33) entails that \( Q\varphi X_{1} = \lambda \varphi X_{1} \). Setting \( X_{2} = \xi \) in (3.34) and then taking scalar product with \( X_{2} \in X_{M} \) and using (2.11), (2.12), we deduce
\[
\left( \xi \right)^{2} + \frac{(\xi\eta)}{k} - \left( \frac{\xi\eta}{k} \right)^{2} = 2\lambda \eta(X_{2}).
\] (3.35)
With the aid of (2.8), by substituting \( X_{1} = \varphi X_{1} \) and \( X_{2} = \varphi X_{2} \) in (3.35), we acquire
\[
g(R(\xi, \varphi X_{1})\varphi X_{1}, \varphi X_{2}) = -g(Q\varphi X_{1}, X_{2}) - 2\eta\varphi(X_{1}, X_{2}) + \frac{(\xi\eta)}{k} g(Q\varphi X_{1}, \varphi X_{2}) + \left( \frac{\xi\eta}{k} - \frac{(\xi\eta)}{k} \right) g(\varphi X_{1}, \varphi X_{2}).
\] (3.36)
At this point, by applying (2.6) and (2.3), one can easily derive that
\[
g(R(\xi, X_{1})\xi, X_{2}) = g((\nabla_{Y_{1}}\varphi)X_{1}, X_{2}) - g((\nabla_{Y_{1}}\varphi)X_{2}, X_{1}).
\]
By using this in the Bianchi’s first identity, we obtain \( g(R(\xi, X_{1})X_{1}, X_{2}) = g((\nabla_{Y_{1}}\varphi)X_{1}, X_{2}) \) for all \( X_{1}, X_{2}, X_{3} \in X_{M} \). By using this in a formula on \( K \)-paracontact manifold (see Zamkovoy [2, Lemma 2.7]):
\[
(\nabla_{X}\varphi)\varphi X_{1} - (\nabla_{X}\varphi)X_{2} = 2g(X_{1}, X_{2})\xi - \eta(X_{1})(\xi_{1} + \eta(X_{1})\xi).
\]
The aforementioned equation reduces to the following:

### 3.5 Proof of Theorem 1.5

Gradient \( k \)-almost Ricci soliton equation (1.2) can be exhibited as
\[
kR(X_{1}, X_{2})\nabla u = (\nabla_{X}Q)X_{2} - (\nabla_{X}Q)X_{1} + (X_{2}\lambda)X_{1} - (X_{1}\lambda)X_{2} + \frac{1}{k}(X_{2}k)(QX_{1} - \lambda X_{1}) - (X_{1}k)(QX_{2} - \lambda X_{2}).
\] (3.34)
Now, setting \( X_{2} = \xi \) in (3.34) and then taking scalar product with \( X_{2} \in X_{M} \) and using (2.11), (2.12), we deduce
\[
g(R(\xi, X_{1})\varphi X_{1}, X_{2}) = g(\varphi QX_{1}, X_{2}) + 2n g(\varphi X_{1}, X_{2}) + \frac{(\xi\eta)}{k} g(QX_{1}, X_{2}) + \left( \xi - \frac{(\xi\eta)}{k} \right) g(X_{1}, X_{2})
\]
\[
+ \left( \frac{\lambda + 2n}{k}(X_{1}k) - (X_{1}\lambda) \right) \eta(X_{2}).
\] (3.35)
With the aid of (2.8), by substituting \( X_{1} = \varphi X_{1} \) and \( X_{2} = \varphi X_{2} \) in (3.35), we acquire
\[
g(R(\xi, \varphi X_{1})\varphi X_{1}, \varphi X_{2}) = -g(Q\varphi X_{1}, X_{2}) - 2\eta g(\varphi X_{1}, X_{2}) + \frac{(\xi\eta)}{k} g(Q\varphi X_{1}, \varphi X_{2}) + \left( \xi - \frac{(\xi\eta)}{k} \right) g(\varphi X_{1}, \varphi X_{2}).
\] (3.36)
At this point, by applying (2.6) and (2.3), one can easily derive that
\[
g(R(\xi, X_{1})\xi, X_{2}) = g((\nabla_{Y_{1}}\varphi)X_{1}, X_{2}) - g((\nabla_{Y_{1}}\varphi)X_{2}, X_{1}).
\]
By using this in the Bianchi’s first identity, we obtain \( g(R(\xi, X_{1})X_{1}, X_{2}) = g((\nabla_{Y_{1}}\varphi)X_{1}, X_{2}) \) for all \( X_{1}, X_{2}, X_{3} \in X_{M} \).
By plugging the value of $\nabla g R^\xi X u X, 12, \nabla g R^\xi \phi X u \phi X, 12$ together with (3.35) and (3.36) in the proceeding equation, we obtain
\[
\left\{ k(X_i u) + (X_i \lambda) + \left( \frac{\lambda + 2n}{k} \right) (X_i k) \right\} \eta(X_i) + k(\xi \lambda) \eta(X_i) \eta(X_i)
\]
\[
= \left\{ \xi \lambda - \left( \frac{\xi \xi}{k} \right) \lambda \right\} [g(\phi X_1, \phi X_2) - g(X_1, X_2)] + \frac{\xi \xi}{k} [g(Q \phi X_1, \phi X_2) - g(Q X_1, X_2)]
\+
2k(\xi \lambda) g(X_1, X_2) - g(Q \phi X_1 + \phi Q X_1, X_2) - 4n g(\phi X_1, X_2).
\]
At this point, by anti-symmetrizing of the last equation and using an anti-symmetric property of $\phi$, we obtain
\[
2g(Q \phi X_1 + \phi Q X_1, X_2) + 8n g(\phi X_1, X_2) = \left\{ k(X_i u) + (X_i \lambda) + \left( \frac{\lambda + 2n}{k} \right) (X_i k) \right\} \eta(X_i)
\-
\left\{ k(X_i u) + (X_i \lambda) + \left( \frac{\lambda + 2n}{k} \right) (X_i k) \right\} \eta(X_i),
\]
where we have utilized the symmetric aspect of the Ricci operator $Q$. Next by substituting $\phi X_1$ for $X_1$ and $\phi X_2$ for $X_2$ in (3.37) and again using the symmetric property of $Q$ and (2.1), we obtain
\[
Q \phi X_1 + \phi Q X_1 = -4n \phi X_1, \quad X_1 \in \mathfrak{X}_M.
\]
By hypothesis: $Q \phi = \phi Q$, thus, equation (3.38) gives $\phi Q X_1 = -2n \phi X_1$. Operating this by $\phi$ and using (2.1) and (2.8), we obtain $Q X_1 = -2n X_1$, and therefore, $(M, g)$ is Einstein with Einstein constant $-2n$. The proof gets completed.

### 3.6 Proof of Theorem 1.6

On the para-Sasakian manifold, the Ricci tensor satisfies (see Lemma 3.15 of [2]):
\[
\text{Ric}(\phi X_1, X_2) = -\text{Ric}(X_i, X_2) - 2n \eta(X_i) \eta(X_i), \quad X_i, X_2 \in \mathfrak{X}_M.
\]
(3.39)
Since the Ricci operator $Q$ is symmetric, replacing $X_2$ by $\phi X_2$ in (3.39) and using (2.1), (2.8), it follows that $Q \phi = \phi Q$. The rest of the proof imitates the one of Theorem 1.5.

### 4 Conclusions and discussions

For $K$-paracontact manifold admitting a gradient Ricci soliton is Einstein [12], and this holds for gradient $k$-almost Ricci soliton (in particular, gradient almost Ricci soliton, see [13]) with the additional condition: $\phi$ commutes with Ricci operator $Q$. In [40], Ghosh and Patra proved that if a complete $K$-contact (Sasakian) metric $g$ represents a gradient $k$-almost Ricci soliton, then it is isometric to a unit sphere $S^{2n+1}$. Here, we have proven if a $K$-paracontact manifold has a gradient $k$-almost Ricci soliton satisfying $Q \phi = \phi Q$ is an Einstein manifold with Einstein constant $-2n$. For more consequences, a $k$-almost Ricci soliton is a generalization of a Ricci soliton, which is a solution to a certain geometric flow equation in differential geometry. In a $k$-almost Ricci soliton, the soliton equation is modified by adding a vector field that satisfies certain properties. $k$-almost Ricci solitons can be applied in various areas of geometry and physics. Here are a few examples: An Einstein manifold is a Riemannian manifold whose Ricci tensor is proportional to the metric tensor. It is a special case of a Ricci soliton, where the soliton vector field is zero. $k$-almost Ricci solitons provide a natural generalization of Einstein manifolds, where the soliton vector field is nonzero. It is also noticed that general relativity is a theory of gravitation that describes the behavior of spacetime in the presence of matter and energy. Some $k$-almost Ricci solitons can be...
used to study the geometry of spacetime in the presence of matter and energy and provide a framework for understanding the dynamics of gravitational waves and black holes.

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**Author contributions:** DSP and AA conceived the study and the manuscript design as well as classified the contact setting of the $k$-Almost Ricci soliton equation. YL and AA constructed the results and computed the scalar curvature equations. FM constructed Ricci operators and obtained commutative law. NA and FM introduced the applications and theories of the Ricci operators equations in various fields.

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