Research Article

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On a class of stochastic differential equations driven by the generalized stochastic mixed variational inequalities

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Abstract: A new class of stochastic differential equations (SDEs) is introduced in this article, which is driven by the generalized stochastic mixed variational inequality (GS-MVI). First, the property of the solution sets of the GS-MVI is proved by Fan-Knaster-Kuratowski-Mazurkiewicz (FKKM) theorem and Aumann’s measurable selection theorem. Next, we obtain the Carathéodory property of the solution set, with which the discussed SDEs can be transformed to stochastic differential inclusions (SDIs). The solution set of the proposed SDEs is proved to be nonempty through the existence of the solutions of the corresponding SDIs by the tools of fixed point theorem.

Keywords: stochastic variational inequalities, stochastic differential equations, stochastic differential inclusions, fixed point, solution sets

MSC 2020: 65C30, 35M87

1 Introduction

Differential equation is a useful tool to characterize the dynamical phenomena in nature. The model in the form of differential variational inequality combines the differential equation and the variational inequality, which can be used to demonstrate the practical problems better as an extension, such as Nash equilibrium problems of non-cooperative games and mechanical problems driven by differential systems with inequality constraints [1].

In Liu et al. [2] and Liu and Zeng [3] introduced a differential system composed of the variational inequalities, which proved that the solution sets are nonempty, closed, and convex under certain conditions. In [4,5], the authors discussed the existence of the solutions of differential mixed variational inequalities in finite dimensional space.

The results of the aforementioned research are discussed in the certain situation. But in the real world, the state trajectory of a dynamic system is inevitably disturbed by various random interferences, which motivates the research of stochastic differential equations (SDEs) [6–10]. In recent years, the research on SDEs has been developed rapidly, which is widely used in economics [11], biology [12], physics [13], and automation [14].

Inspired by the research of SDEs and the work of Liu, we study a new class of SDEs, which is driven by a generalized stochastic mixed variational inequality (GS-MVI). Such equations can be used to solve stochastic...
control problems and can describe the influence of the uncertain factors in the system. Let $E_1$ and $E_2$ be a pair of Hilbert space, $B$ be a compact and convex subset of $E_1$ and $E_1^*$ is the dual space of $E_1$. The presented SDE is of the form as follows:

$$
\begin{aligned}
dx(t) &= A x(t) dt + \kappa(t, x(t), u(\omega)) dt + \gamma(t, x(t)) dW(t), \\
u(\omega) &\in S(B, r(t, x(t) + L(\cdot)), \varrho), \\
x(0) &= x_0,
\end{aligned}
$$

(1)

where $S(B, r(t, x(t) + L(\cdot)), \varrho)$ stands for the solution set of the following GS-MVI:

$$
\langle r(t, x(t)) + L(\omega, u(\omega)), v - u(\omega) \rangle + \varrho(\omega, v) - \varrho(\omega, u(\omega)) \geq 0 \quad \forall v \in B,
$$

(2)

where $A$ denotes the infinitesimal generator of a bounded linear operator $e^{tA}$, the mapping $\kappa : [0, T] \times E_2 \times B \rightarrow E_2$ is the drift term, and $\gamma : [0, T] \times E_2 \rightarrow E_2$ is the diffusion term. The stochastic process $\{W(t), t \in [0, T]\}$ is an $E_2$-valued Winner process (Brownian motion) with mean zero. $x_0$ is the $\mathcal{F}_0$-measurable $E_2$-valued stochastic variable, which is independent of $W(t)$. For $\forall \varphi \in B$, the mappings $r : [0, T] \times E_2 \rightarrow E_1^*$, $L : \Omega \times B \rightarrow E_1^*$, and $\varrho : \Omega \times B \rightarrow (-\infty, +\infty]$ are the stochastic operators and $\varrho \neq +\infty$. The stochastic process $x(t) : [0, T] \rightarrow E_2$ is a mild solution of equation (1) if and only if it satisfies

$$
x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A} \kappa(s, x(s), u(\omega)) ds + \int_0^t e^{(t-s)A} \gamma(s, x(s)) dW(s),
$$

(3)

where $u(\omega)$ is the solution of the GS-MVI (2). In this article, we mainly discuss the existence conditions of the mild solution $x(t)$ as defined earlier, using the properties of the solution set of the GS-MVI.

## 2 Preliminaries

In this section, we mainly focus on some fundamentals, prerequisites, and assumptions that will be applied to in the next part.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Here, $\Omega$ denotes a sample space, $P : \mathcal{F} \rightarrow [0, 1]$ denotes the probability measure, and $P(\Omega) = 1$. $\mathcal{F}$ is a $\sigma$-algebra, and $\mathcal{F}_t, t \in [0, T]$ is a complete family of the right continuously increasing sub-$\sigma$-algebras satisfying $\mathcal{F}_t \subset \mathcal{F}$. For convenience, we use the following notes [15] to denote the subset families of $E_1$ in this article:

$$
\mathcal{P}(E_1) = \{B \subset E_1 : B \text{ is nonempty}\}, \\
\mathcal{P}_{c(\mathcal{P})}(E_1) = \{B \subset E_1 : B \text{ is nonempty, convex, (compact), (bounded)}\}.
$$

**Definition 2.1.** [16] Let $G : E_2 \rightarrow \mathcal{P}(E_2)$ be a set-valued mapping in Hilbert space $E_2$:

1. if $G(x)$ has convex (closed) values such that this is sure for all $x \in E_2$, then we say that mapping $G$ is convex (closed);
2. if $G(C) = \bigcup_{x \in C} G(x)$ is bounded in $E_2$ such that this is sure for any bounded set $C \subset E_2$, then we say that mapping $G$ is bounded in bounded sets;
3. if $G(C)$ is relatively compact such that this is sure for every bounded subset $C \subset E_2$, then we say the mapping $G$ is completely continuous.

**Definition 2.2.** [16] For a pair of separable Hilbert spaces $E_1$ and $E_2$, if the set-valued mapping $F : [0, T] \times E_2 \rightarrow \mathcal{P}(E_1)$ is Carathéodory when it satisfies:

1. $F(\cdot, x) : t \rightarrow \mathcal{P}(E_1)$ is a measurable mapping for each $x \in E_2$;
2. $F(t, \cdot) : E_2 \rightarrow \mathcal{P}(E_1)$ is a continuous mapping for each $t \in [0, T]$.

**Theorem 2.1.** (Fan-Knaster-Kuratowski-Mazurkiewicz (FKKM) theorem, [17]) For the Hilbert space $E_1$ with a nonempty subset $B$, assume that the mapping $G$ satisfies the following two conditions:
the mapping $G : E_1 \rightarrow \mathcal{P}(E_1)$ is Knaster-Kuratowski-Mazurkiewicz (KKM) and it is closed for each $u \in B$; 
(2) $G(v)$ is compact in $E_1$ for some $v \in B$.

Then, it holds $\bigcap_{u \in B} G(u) \neq \emptyset$.

**Lemma 2.1.** (Aumann’s measurable selection theorem, [18]) The $(\Omega, \mathcal{F}, P)$ is the space that is mentioned earlier, and $E_1$ is a separable Hilbert space, if there exists a measurable mapping $\Psi : \Omega \rightarrow \mathcal{P}(E_1)$ satisfies

$$\text{graph}(\Psi) = \{(\omega, x) \in \Omega \times E_1 : x \in \Psi(\omega)\} \in \mathcal{F} \times \mathcal{B}(E_1),$$

then we know that $\Psi(\cdot)$ is measurable, so there is a measurable selection $\delta : \Omega \rightarrow B$ of $\Psi$ for each $\omega \in \Omega$.

**Lemma 2.2.** [16] For two complete separable metric spaces $A$ and $Z$, there exists a Carathéodory mapping $\psi : [0, T] \times A \rightarrow \mathcal{P}(Z)$, then for each measurable $\varphi : [0, T] \rightarrow \mathcal{P}(Z)$, the $\psi(T, \varphi(T)) : [0, T] \rightarrow \mathcal{P}(Z)$ is a measurable mapping.

**Definition 2.3.** [19] A set of functions is said to be equicontinuous on a given closed interval $[a, b]$ if, for any $\varepsilon > 0$, there exists a $\sigma > 0$ such that if $|f(x_2) - f(x_1)| < \varepsilon$ for each function $f(x)$ of given set whenever $x_1$ and $x_2$ are in $[a, b]$ and $|x_2 - x_1| < \sigma$.

**Theorem 2.2.** (Arzela-Ascoli theorem, [20]) For any uniformly bounded and equicontinuous function $\{f(x)\}$ defined on the interval $[a, b]$, there must exist a uniformly convergent subsequence.

**Lemma 2.3.** [21] Let $A$ be a Banach space. If a multivalued map $T : A \rightarrow \mathcal{P}(A)$ is completely continuous, then $T$ is upper semicontinuous iff $T$ is a closed graph operator.

**Theorem 2.3.** [22] For Banach space $X$ that has a nonempty, bounded, closed, and convex subset $\rho$ consider that the mapping $T : \Omega \rightarrow \mathcal{P}(X)$ is upper semicontinuous with closed convex values such that $T(\Omega) \subset \Omega$ and $T(\Omega)$ is compact. Then, $T$ has a fixed point.

3 Related properties of the solution set of the GS-MVI

The $B$ is a nonempty subset of the separable Hilbert space $E_1$ and $E_1^*$ is the dual space of $E_1$. In this section, we discuss the related properties of the solution set of the GS-MVI (2), which are the tools to obtain the solution of the presented SDE (1).

**Theorem 3.1.** Let $B$ be a nonempty, compact, and convex subset of separable Hilbert space $E_1$, and $E_1^*$ be the dual space of $E_1$. Suppose that these conditions are satisfied:

(i) The stochastic mapping $L : \Omega \times B \rightarrow E_1^*$ is monotone and satisfies

$$\liminf_{\tau \to 0^+} \left( r + L(\omega, tu + (1 - \tau)v), v - u \right) \leq \left( r + L(\omega, v), v - u(\omega) \right)$$

for $\forall v \in B$ and $\forall \omega \in \Omega$;

(ii) The stochastic mapping $q : \Omega \times B \rightarrow (-\infty, +\infty)$ $(q \not= +\infty)$ is lower semicontinuous and has convex values;

(iii) The mapping $H : \Omega \times B \rightarrow \mathcal{P}(B)$ holds that:

$$H(\omega, v) = \{ u \in B : \langle r + L(\omega, u(\omega)), v - u(\omega) \rangle + q(\omega, v) - q(\omega, u(\omega)) \geq 0 \},$$

for each $r \in E_1^*$, which has a measurable graph.

Then, problem (2) has a nonempty, compact, and convex solution set.
Proof. First, by Assumption (i), we know that if $u \in B$ is the solution of (2) when it satisfies the mapping $H(\omega, u)$:

$$
(r + L(\omega, v(\omega)), v - u(\omega)) + q(\omega, v) - q(\omega, u(\omega)) \geq 0.
$$

(4)

By the monotonicity of $L$, we can obtain that $H(\omega, u)$ is convex. Correspondingly, we show that the mapping $H(\omega, u)$ is closed in $E_1$ for $\forall v \in B$. Taking a sequence $\{u_n\} \subset H(\omega, u)$ with $u_n \to u$ in $E_1$, we can prove that $u \in H(\omega, u)$, because the $\{u_n\} \subset H(\omega, u)$, we obtain that:

$$
(r + L(\omega, v(\omega)), v - u_n(\omega)) + q(\omega, v) - q(\omega, u_n(\omega)) \geq 0.
$$

To make $n \to \infty$, based on Assumption (ii), we have

$$
(r + L(\omega, v(\omega)), v - u(\omega)) + q(\omega, v) - q(\omega, u(\omega)) \geq 0,
$$

which shows the $u \in H(\omega, u)$. $H(\omega, u)$ and $H(\omega, v)$ are equivalent, so $H(\omega, v)$ is convex and closed.

Next, we need to show that $H(\omega, v)$ is a KKM mapping and $\bigcap_{v \in B} H(\omega, v) \neq \emptyset$. Assume that $H(\omega, v)$ is not a KKM mapping, a finite set $\{v_1, v_2, ..., v_n\} \subset B$ that satisfies $c(v_1, v_2, ..., v_n) \not\subset \bigcup_{i=1}^n H(\omega, v_i)$. Based on this, there exists $\bar{v} \in c(v_1, v_2, ..., v_n)$ with $\bar{v} = \sum_{i=1}^n \tau_i v_i (\tau_i \geq 0, \sum_{i=1}^n \tau_i = 1, i = 1, 2, ..., n)$, which satisfies

$$
\bar{v} \not\in \bigcup_{i=1}^n H(\omega, v_i).
$$

Then, we have

$$
(r + L(\omega, \bar{v}), v_i - \bar{v}) + q(\omega, v_i) - q(\omega, \bar{v}) < 0,
$$

for all $i \in \{1, 2, ..., n\}$, from which we have

$$
0 = (r + L(\omega, \bar{v}), \bar{v} - \bar{v}) + q(\omega, \bar{v}) - q(\omega, \bar{v})
$$

$$
\leq \left(r + L(\omega, \bar{v}), \sum_{i=1}^n \tau_i v_i - \bar{v}\right) + q(\omega, \sum_{i=1}^n \tau_i v_i) - q(\omega, \bar{v})
$$

$$
< 0,
$$

so $H$ is a KKM mapping. Since $B$ is a compact set, so $H(\omega, v)$ is compact in $E_1$ for each $v \in B$. According to Theorem 2.1 in Section 2, $\bigcap_{v \in B} H(\omega, v) \neq \emptyset$ holds.

Define a mapping $\Psi : \Omega \to \mathcal{P}(B)$ that satisfies $\Psi(\omega) = \bigcap_{v \in B} H(\omega, v)$. Because the set $B$ is separable, we here assume that $\{v_i\}_{i=1}^\infty$ is a dense subset of $B$. Then, we aim at proving that

$$
\bigcap_{v \in B} H(\omega, v) = \bigcap_{i=1}^\infty H(\omega, v_i).
$$

Obviously, $\bigcap_{v \in B} H(\omega, v) \subset \bigcap_{i=1}^\infty H(\omega, v_i)$, so we need to prove

$$
\bigcap_{i=1}^\infty H(\omega, v_i) \subset \bigcap_{v \in B} H(\omega, v).
$$

Assume that $\bigcap_{i=1}^\infty H(\omega, v_i) \subset \bigcap_{v \in B} H(\omega, v)$, so there exist $v_0 \in \bigcap_{i=1}^\infty H(\omega, v_i)$ and $v_0 \not\in \bigcap_{v \in B} H(\omega, v)$; it follows that there exist $v_0 \in B$ and $v_0 \not\in H(\omega, v_0)$ such that

$$
(r + L(\omega, v_0), v_0 - u_0) + q(\omega, v_0) - q(\omega, u_0) < 0.
$$

Since dense subset $\{v_i\}_{i=1}^\infty$ is countable of $B$, so there has $v_{n_k} \subset \{v_i\}$ such that $\{v_{n_k}\} \to v_0$. Because $u_0 \in \bigcap_{i=1}^\infty H(\omega, v_{n_k})$, there holds

$$
(r + L(\omega, u_0), v_{n_k} - u_0) + q(\omega, v_{n_k}) - q(\omega, u_0) \geq 0
$$

for $\forall k \geq 1$. By definition, there is

$$
0 \leq (r + L(\omega, u_0), v_{n_k} - u_0) + q(\omega, v_{n_k}) - q(\omega, u_0)
$$

$$
\leq (r + L(\omega, u_0), v_0 - u_0) + q(\omega, v_0) - q(\omega, u_0)
$$

$$
< 0,
$$

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which contradicts with \( u_0 \notin H(\omega, v_0) \), so

\[
\bigcap_{i=1}^{\infty} H(\omega, v_i) \subseteq \bigcap_{v \in B} H(\omega, v).
\]

Then, we have

\[
\bigcap_{v \in B} H(\omega, v) = \bigcap_{i=1}^{\infty} H(\omega, v_i).
\]

For the multivalued mapping \( \Psi : \Omega \to \mathcal{P}(B) \), we have

\[
\text{graph}(\Psi) = \{ (\omega, u) : u \in \Psi(\omega) = \bigcap_{v \in B} H(\omega, v) \}
\]

\[
= \bigcap_{i=1}^{\infty} \{ (\omega, u) : u \in H(\omega, v_i) \} \subseteq \mathcal{F} \times \mathcal{B}(B).
\]

By Lemma 2.1, we deduce that there is a measurable selection \( \varsigma : \Omega \to B \), and it satisfies \( \varsigma(\omega) \in \bigcap_{v \in B} H(\omega, v) \). Because the subset \( B \) is compact, we can obtain that problem (2) has a compact solution set. Thus, we complete the proof of Theorem 3.1.

\[\Box\]

**Theorem 3.2.** For a pair of separable Hilbert spaces \( E_1 \) and \( E_2 \) and \( E_1 \) has a nonempty, compact, and convex subset \( B \), we assume that the mapping \( r : [0, T] \times E_2 \to E_1^* \) is continuous, and the mappings \( L \) and \( \varrho \) satisfy the conditions of Theorem 3.1. Then, we can define a new mapping \( N : [0, T] \times E_2 \to \mathcal{P}(B) \), which is of the following form:

\[
N(t, x) = \{ u \in B : (r(t, x) + L(\omega, u(\omega)), v - u) + \varrho(\omega, v) - \varrho(\omega, u(\omega)) \geq 0, \forall v \in B \},
\]

for \( \forall \omega \in \Omega \) and \( \forall [0, T] \times E_2 \); it fulfills that \( N \) is upper semicontinuous.

**Proof.** In order to obtain the conclusion, we need to prove that \( N^*(D) = \{ (t, x) \in [0, T] \times E_2 : N(t, x) \cap D \neq \emptyset \} \) is closed in \([0, T] \times E_2\), where \( D \) is the closed subset of \( B \).

Indeed, to take a sequence \( \{ t_n, x_n \} \subseteq N^*(D) \) that satisfies \( \{ t_n, x_n \} \to \{ t, x \} \in [0, T] \times E_2 \), and correspondingly there is \( u_n \in N(t_n, x_n) \), we can assume that \( u_n \) converges to \( u \); in order to obtain \( N^*(D) \) closed, then we need to prove that \( u \in N^*(D) \). Then, \( u_n \in N(t_n, x_n) \), so

\[
(r(t_n, x_n) + L(\omega, u_n), v - u_n) + \varrho(\omega, v) - \varrho(\omega, u_n) \geq 0
\]

for all \( v \in B \). From the monotonicity of \( L \), there holds

\[
(r(t_n, x_n) + L(\omega, v), v - u_n) + \varrho(\omega, v) - \varrho(\omega, u_n) \geq 0.
\]

Because the \( r \) is continuous and letting \( n \to \infty \), we obtain

\[
(r(t, x) + L(\omega, v), v - u) + \varrho(\omega, v) - \varrho(\omega, u) \geq 0.
\]

Because \( B \) is convex, it has that \( v = \lambda \varsigma + (1 - \lambda)u \in B, \forall \varsigma \in B, 0 \leq \lambda \leq 1 \)

\[
(r(t, x) + L(\omega, \lambda \varsigma + (1 - \lambda)u), \lambda \varsigma + (1 - \lambda)u - u) + \varrho(\omega, \lambda \varsigma + (1 - \lambda)u) - \varrho(\omega, u) \geq 0, \forall \varsigma \in B.
\]

From the properties of \( L \) and letting \( \lambda \to 0 \), there is

\[
(r(t, x) + L(\omega, u), \varsigma - u) + \varrho(\omega, \varsigma) - \varrho(\omega, u) \geq 0, \forall \varsigma \in B.
\]

Hence, we obtain \( u \in S(B, r(t, x) + L(u), \varrho) \). So \( N \) is upper semicontinuous.

Since \( N \) is the upper semicontinuous mapping and has compact and convex values, we obtain that \( N(t, x(t)) \) is measurable for each \( t \in [0, T] \). \( \Box \)

**4 Existence results of the SDE driven by the GS-MVI**

The existence of the solutions of problem (1) can be proved by transforming it into a corresponding stochastic differential inclusion (SDI). Given the multivalued mapping \( N \) introduced in Theorem 3.2, we can naturally define a mapping \( Q : [0, T] \times E_2 \to \mathcal{P}(E_2) \), \( Q(t, x) = \kappa(t, x, N(t, x)) \); then problem (1) can be converted to:
\begin{align*}
dx(t) & \in Ax(t)dt + Q(t, x(t))dt + y(t, x(t))dW(t), \\
x(0) & = x_0,
\end{align*}

for all $t \in [0, T]$.

**Lemma 4.1.** For a pair of separable Hilbert spaces $E_1$ and $E_2$, $B$ is a nonempty, compact and convex subset of $E_1$. Assume that $L$, $r$, and $q$ satisfy the hypotheses in Theorem 3.2. Moreover, suppose:

(i) For $\forall (x, u) \in [0, T] \times E_2$, the convex set $\mathcal{P}_u(B) \subseteq B$, the set $\kappa(t, x, \mathcal{P}_u(B))$ is convex in $E_2$;
(ii) For $\forall (x, u) \in E_2 \times B$, $\kappa(\cdot, x, u) : [0, T] \to E_2$ is measurable;
(iii) For $\forall t \in [0, T]$, $\kappa(t, \cdot, \cdot) : E_2 \times E_1 \to B$ is continuous;
(iv) For $\forall t \in [0, T]$ and $\forall u(\omega) \in B$, the mapping $\kappa$ satisfies: $E |\kappa(t, x(t), u(\omega))|^2 \leq C_1(1 + ||x(t)||^2)$ with the constant $C_1 > 0$.

Then, we have

(M1) For $\forall x \in E_2$, there exists a strongly measurable selection of $Q(\cdot; x)$;
(M2) For $\forall t \in [0, T]$, the mapping $Q(t, \cdot)$ is upper semicontinuous;
(M3) $Q$ satisfies that for $\forall t \in [0, T]$, $E |Q(t, x(t))|^2 = \sup[E|q(t)|^2 : q(t) \in Q(t, x(t))] \leq C_1(1 + ||x(t)||^2)$.

**Proof.** From assumptions (ii) and (iii), we obtain that $\kappa$ is a Carathéodory mapping. Then, we can infer that $Q(\cdot, x) = \kappa(\cdot, x, N(\cdot, x))$ is a measurable mapping for each $x \in E_2$ by applying Lemma 2.2 and Theorem 3.2. Also, $E_2$ is a separable space, so $Q(\cdot, x)$ is strongly measurable. By Theorem 3.1, we can obtain that the set-valued mapping $N$ is nonempty, compact, and convex, according to Assumptions (i) and (iii), $Q(t, x) \in \mathcal{P}_{pc}(E_2)$ for every $(t, x) \in [0, T] \times E_2$; thus, we know that there exists a strongly measurable selection of $Q(\cdot, x)$ [23].

For $t \in [0, T]$, define composite functions $Q = Q_2 \circ Q_1$, where $Q_1 = (\cdot, N(\cdot, \cdot)) : E_2 \to P(E_2 \times B)$ and $Q_2 = \kappa(t, \cdot, \cdot) : E_2 \times E_1 \to E_2$. From Theorem 3.2, $Q_1$ is upper semicontinuous. By Assumption (iii), we know that $Q_2$ is continuous. Therefore, $Q(t, \cdot)$ is upper semicontinuous.

We know from the conclusions (M1) and (M2) that there is a measurable selection $q(t) \in Q(t, x(t))$; here, the value of $N$ is compact; because $E |\kappa(t, x(t), u(\omega))|^2 \leq C_1(1 + ||x(t)||^2)$, the conclusion (M3) is established. \hfill \Box

**Remark 4.1.** In this lemma, we proved that mapping $Q(\cdot, x)$ has strongly measurable selections; for all $x \in E_2$, the conclusion $S_Q = \{q(t) \in L([0, T], X) : q(t) \in Q(t, x(t))$, and for all $t \in [0, T] \neq \emptyset$ holds, which can be used to deduce that problem (6) has a mild solution.

By using the variation of constant formula in [24], the equivalent form of SDI (6) can be written as:

\begin{align*}
x(t) & \in e^{tA}x_0 + \int_0^t e^{(t-s)A}Q(s, x(s))ds + \int_0^t e^{(t-s)A}y(s, x(s))dW(s), \\
x(0) & = x_0.
\end{align*}

(7)

To obtain the solution of problem (6), it is necessary to examine the existence of the solutions of problem (7). Based on this line of thought, we shall make the following hypotheses [25]:

(H1) $A$ is a infinitesimal generator of an analytic semigroup in $E_2$, which satisfies that $|e^{tA}| \leq C$, and the constant $C$ satisfies $C > 0$;
(H2) There exists constant $C_2 > 0$ such that \[ E |y(t, x(t))|^2 \leq C_2(1 + ||x(t)||^2) \]

for any $t \in [0, T]$, and $x(t) \in E_2$ holds.
Lemma 4.2. [26] Let $M$ be a Hilbert space, $\mathcal{L}$ denote the linear continuous mappings that map $L([0, T], M)$ to $C([0, T], M)$, and $Q$ be a Carathéodory multivalued mapping with compact and convex values; we also assume that $S_{0} = \{q(t) \in L([0, T], M) : q(t) \in Q(t, x(t)), \forall t \in [0, T]\} \neq \emptyset$ holds, and then the operator:

$$\mathcal{L} \circ S_{0} : C([0, T], M) \rightarrow \mathcal{P}_{cp}(C([0, T], M))$$

$$x \mapsto (\mathcal{L} \circ S_{0})(x) = \mathcal{L}(S_{0,x})$$

has a closed graph.

Theorem 4.1. Let $Z = C([0, T], E)$ be a Banach space and $|x_{0}|^{2} < \infty$. Under the assumptions of Lemma 4.1 and Hypotheses $(H1)$ and $(H2)$, then the solution set of problem (1) is nonempty in $Z$.

Proof. Because problem (7) is converted from problem (1), we just need to prove the existence of problem (7)'s solution. Define a multivalued mapping $\Phi : Z \rightarrow 2^{Z}$ by:

$$\Phi(x(t)) = \left\{ y \in Z : y(t) = e^{d}x_{0} + \int_{0}^{t}e^{(t-s)^{\lambda}}q(s)ds + \int_{0}^{t}e^{(t-s)^{\lambda}}p(s, x(s))dW(s) \right\},$$

where $q \in S_{0,x} = \{q \in L([0, T], E) : q(t) \in Q(t, x(t))) \}$ for $\forall t \in [0, T]$. We shall prove that $\Phi$ is a convex, bounded, compact, and upper semicontinuous multivalued mapping.

Step 1. $\Phi(x)$ is a convex mapping for each $x \in Z$. Suppose that there exist $q_{1} \in S_{0,x}$ and $q_{2} \in S_{0,x}$ such that $y_{1}$ and $y_{2}$ belong to $\Phi(x)$ for each $t \in [0, T]$, and then, we have

$$y_{1}(t) = e^{d}x_{0} + \int_{0}^{t}e^{(t-s)^{\lambda}}q_{1}(s)ds + \int_{0}^{t}e^{(t-s)^{\lambda}}p(s, x(s))dW(s)$$

and

$$y_{2}(t) = e^{d}x_{0} + \int_{0}^{t}e^{(t-s)^{\lambda}}q_{2}(s)ds + \int_{0}^{t}e^{(t-s)^{\lambda}}p(s, x(s))dW(s).$$

If $0 \leq m \leq 1$, then

$$(my_{1} + (1 - m)y_{2})(t) = e^{d}x_{0} + \int_{0}^{t}e^{(t-s)^{\lambda}}q(s, x(s))dW(s) + \int_{0}^{t}e^{(t-s)^{\lambda}}(mq_{1}(s) + (1 - m)q_{2}(s))ds$$

$S_{0,x}$ is a convex set based on the convexity of the $Q$, then we have

$$my_{1} + (1 - m)y_{2} \in \Phi(x).$$

Now, the proof is complete.

Step 2. $\Phi$ maps $B_{p}$ into $B_{p}$ in $Z$, where $B_{p}$ is a bounded set. We choose $p > 0$ that satisfies $3C(E|x_{0}|^{2} + T(p + 1)(C_{1}T + C_{2})) \leq p$, and we consider the bounded set $x \in B_{p} = \{x \in Z : ||x||^{2} \leq p\}$; then, we can tell that $B_{p}$ is a bounded, closed, and convex set in $Z$. To obtain the conclusion of Step 2, so we need to prove that the mapping $\Phi$ takes value in $B_{p}$. If $x \in B_{p}$ and $y \in \Phi(x)$, there exists a $q \in S_{0,x}$ such that

$$y(t) = e^{d}x_{0} + \int_{0}^{t}e^{(t-s)^{\lambda}}q(s)ds + \int_{0}^{t}e^{(t-s)^{\lambda}}p(s, x(s))dW(s),$$
then, we have

\[
E \ |y(t)|^2 = E \left| e^{tA}x_0 + \int_0^t e^{(t-s)A}q(s)ds + \int_0^t e^{(t-s)A}y(s, x(s))dW(s) \right|^2
\]

\[
\leq 3E|e^{tA}x_0|^2 + 3E \left| \int_0^t e^{(t-s)A}q(s)ds \right|^2 + 3E \left| \int_0^t e^{(t-s)A}y(s, x(s))dW(s) \right|^2
\]

\[
\leq 3CE|x_0|^2 + 3CT \int_0^t |q(s)|^2ds + 3C \int_0^t |y(s, x)|^2ds
\]

\[
\leq 3CE|x_0|^2 + 3TC_1 + 3M(TC_2 + C_2).
\]

Then, for each \(x \in B_p\), we obtain that:

\[
\|y\|^2 \leq 3C(E|x_0|^2 + T(p + 1)(C_1 + C_2)).
\]

This shows that \(\|y\|^2 \leq p\), so mapping \(\Phi\) satisfies Step 2.

**Step 3.** \(\Phi\) maps bounded sets into equicontinuous sets in \(Z\). For each \(x\) belonging to a bounded set \(B_p = \{x \in Z : |x|^2 \leq p\}\) with \(y \in \Phi(x)\), there exists \(q \in S_{Q,x}\) such that:

\[
y(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}q(s)ds + \int_0^t e^{(t-s)A}y(s, x(s))dW(s).
\]

For \(t_1, t_2 \in [0, T]\) with \(t_1 \leq t_2\), there holds

\[
E \ |y(t_2) - y(t_1)|^2 \leq 5E \left| e^{t_2A} - e^{t_1A} \right|^2 |x_0|^2 + 5E \left| \int_{t_1}^{t_2} (e^{(t_2-s)A} - e^{(t_1-s)A})q(s)ds \right|^2
\]

\[
+ 5E \left| \int_{t_1}^{t_2} e^{(t_2-s)A}q(s)ds \right|^2 + 5E \left| \int_{t_1}^{t_2} e^{(t_2-s)A}y(s, x(s))dW(s) \right|^2
\]

\[
+ 5E \left| \int_{t_1}^{t_2} (e^{(t_2-s)A} - e^{(t_1-s)A})y(s, x(s))dW(s) \right|^2
\]

\[
\leq 5|e^{t_2A} - e^{t_1A}|^2 E |x_0|^2 + 5T \int_{t_1}^{t_2} |e^{(t_2-s)A} - e^{(t_1-s)A}|^2 (C_1||x(t)||^2 + C_1)ds
\]

\[
+ 5(t_2 - t_1) \int_{t_1}^{t_2} |e^{(t_2-s)A}|^2 (C_1||x(t)||^2 + C_1)ds
\]

\[
+ 5 \int_{t_1}^{t_2} |e^{(t_2-s)A} - e^{(t_1-s)A}|^2 (C_2||x(t)||^2 + C_2)ds
\]

\[
+ 5 \int_{t_1}^{t_2} |e^{(t_2-s)A} - e^{(t_1-s)A}|^2 (C_2||x(t)||^2 + C_2)ds.
\]

As \(t_2 \to t_1\), the right-hand side of the aforementioned inequality tends to 0, which shows that \(\Phi(Bp)\) is compact.

According to the conclusions of Step 2, Step 3, and Theorem 2.2, it can be concluded that \(\Phi : Z \to 2^Z\) is equicontinuous.

**Step 4.** The operator \(\Phi\) is a closed graph operator. To take \(\{x_n\} \in Z\) with \(y_n \in \Phi(x_n)\), when \(n \to +\infty\), there holds \(x_n \to x\), \(y_n \to y\), and we should prove that there exists \(q_n \in S_{Q,x}\) for \(y_n \to y\), then \(y \in \Phi(x)\). To prove \(y \in \Phi(x)\), we wish to show that \(y\) satisfies the following equation:
Due to $y_n \in \Phi(x_n)$, so

$$y_n(t) = e^{iA}x_0 + \int_0^te^{(t-s)i\gamma}q(s)ds + \int_0^te^{(t-s)i\gamma}y(s, x(s))dW(s).$$

Since the mapping $Q$ is continuous, then

$$\left\| (y_n(t) - e^{iA}x_0 - \int_0^te^{(t-s)i\gamma}y(s, x(s))dW(s) - (y(t) - e^{iA}x_0 - \int_0^te^{(t-s)i\gamma}y(s, x(s))dW(s)) \right\| \to 0.$$

Consider the operator that is linear continuous:

$$L : L([0, T], E_2) \to Z,$$

$$q \to L(q)(t) = \int_0^te^{(t-s)i\gamma}q(s)ds.$$

We can obtain that $L \circ S_0$ is a closed operator from Lemma 4.1, and then we have

$$y_n(t) - e^{iA}x_0 - \int_0^te^{(t-s)i\gamma}y(s, x(s))dW(s) \in L(S_{Q, x_0}).$$

Since $x_n \to x$, for some $q \in S_{Q, x}$, there is

$$y(t) - e^{iA}x_0 - \int_0^te^{(t-s)i\gamma}y(s, x(s))dW(s) = \int_0^te^{(t-s)i\gamma}q(s)ds.$$

So $\Phi$ is a closed graph operator, and we have proved that $\Phi$ is a upper semicontinuous and compact multivalued mapping by the convex and bounded mapping $\Phi$. According to Theorem 2.3 in Section 2, we obtain that $\Phi$ satisfied the fixed point theorem, and the fixed point is also the solution of problem (7).

According to Filippov implicit function lemma [27], we can obtain that for every $x \in Z$ of SDI (7), there is measurable selection $\xi$ such that $\xi(t) \in N(t, x(t))$ is a solution of GS-MVI (2), which ensures that SDE (1) composed of GS-MVI has a mild solution. □

**Remark 4.2.** The existence results of the presented SDEs composed of the GS-MVI are an extension of the study in Liu’s work [10]. In our study, we considered the stochastic situation in the SDE (1) and provided new idea to demonstrate the existence of its solutions by proving the existence of the solutions of the corresponding SDIs.

**Example 1.** The electrical circuits with (ideal) diodes can be described as a differential variational inequality (DVI) from [28-30]. When we consider the effects of the stochastic environment in the electrical circuits with (ideal) diodes, the DVI can be considered a stochastic equation stochastic differential variational inequality as follows:

$$\begin{cases}
\{dx(t) = [Bx(t) + Cu(t) + f(t)]dt + [Dx(t) + g(t)]dBt, \\
x(0) = x_0,
\end{cases} \quad (8)

(Qx(t) + Mu(t), v - u(t)) + q(v) - q(u(t)) \geq 0, \\
\forall v \in B, \forall t \in [0, T] \text{ and } \forall \omega \in \Omega,
$$

where $x$ is the electric current across the inductor and $u$ denotes the voltage-current pairs. It follows from Theorem (4.1) that (8) admits a mild solution $(x(t), u(t))$. When $q$ equals 0, equation (8) degenerates into Model 5.2 in [31].
5 Conclusion

In this article, we focus on a class of evolutionary SDEs composed of the GS-MVIs, which is a useful tool to demonstrate the stochastic dynamics in logistics and urban traffic management, etc. Here, the obtained solution of SDE (1) is the mild solution in the weak sense of the Carathéodory condition, which is also the solution of the corresponding stochastic integral inclusion. The results of this article are the extension of the differential variational inequalities in the uncertain situations. Comparing with the certain situation, the linear growth condition and the continuity are required for $\kappa$ and $\gamma$ in (1) to ensure the nonemptyness of the solution set, instead of the Lipschitz condition, etc. Further research could focus on:

1. We are thinking about extending Brownian motion in the proposed GS-MVI to higher dimension;
2. For practical use of our result, it is necessary to design a numerical algorithm to solve the proposed GS-MVI and discuss the convergence and stability of the algorithm.

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