Research Article

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Some identities of degenerate harmonic and degenerate hyperharmonic numbers arising from umbral calculus

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Abstract: Hyperharmonic numbers were introduced by Conway and Guy (The Book of Numbers, Copernicus, New York, 1996), whereas harmonic numbers have been studied since antiquity. Recently, the degenerate hyperharmonic and degenerate harmonic numbers were introduced as their respective degenerate versions. The aim of this article is to further investigate some properties and identities involving the degenerate hyperharmonic numbers and degenerate harmonic numbers. Especially, we derive some identities by making use of the transfer formula for associated sequences from umbral calculus.

Keywords: degenerate harmonic number, degenerate hyperharmonic number, umbral calculus

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1 Introduction

Various degenerate versions of many special numbers and polynomials have been studied recently with their regained interests (see [1–6] and references therein). This investigation began with the pioneering work of Carlitz on the degenerate Bernoulli and degenerate Euler numbers in [7]. It is worthwhile to mention that these explorations for degenerate versions are not just limited to polynomials and numbers but also extended to transcendental functions, such as gamma functions (see [5]). It is also remarkable that the $\lambda$-umbral calculus and $\lambda$-$q$-umbral calculus were introduced as degenerate versions of the umbral calculus and the $q$-umbral calculus, respectively (see [1,8]).

The aim of this article is to further investigate some properties and identities involving the degenerate hyperharmonic numbers (see (7)) and degenerate harmonic numbers (see (6)). Especially, we derive some identities by making use of the transfer formula (see (22)) for associated sequences from umbral calculus.

The outline of this article is as follows. In Section 1, we recall the degenerate exponentials and the degenerate logarithms. We remind the reader of harmonic numbers and hyperharmonic numbers together with their explicit expression due to Conway and Guy (see [9]). Then, we recall their degenerate versions, namely, the degenerate harmonic numbers, and the degenerate hyperharmonic numbers together with their explicit expression (see [2,3]). We also mention the recently introduced degenerate Stirling numbers of the first kind and of the second kind. We also remind the reader of the degenerate Bell polynomials. Next, we briefly review some facts about umbral calculus, namely, the linear functionals and the linear differential operators on the vector space of all polynomials over $C$, Sheffer sequences, and the transfer formulas for associated...
sequences. In Section 2, we obtain explicit expressions for two formal power series whose coefficients involve the hyperharmonic numbers. Then, we derive an expression of some binomial in terms of the degenerate hyperharmonic numbers and the degenerate Stirling numbers of the first kind. In Section 3, we apply the transfer formula in order to derive some identities involving the degenerate hyperharmonic numbers in Theorems 1, 5, and 8. Also, we deduce an identity involving the degenerate Stirling numbers of both kinds. Finally, we conclude this article in Section 4. In the rest of this section, we recall what are needed throughout this study.

For integers \( n \geq 0 \), the harmonic numbers are defined by:

\[
H_0 = 0, \quad H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \quad (\text{see } [10]).
\]  

(1)

For integers \( n \) and \( r \geq 0 \), we note that the hyperharmonic numbers are given by:

\[
H^{(r)}_n = \begin{cases} 
0, & \text{if } n = 0, r \geq 0, \\
\frac{1}{n}, & \text{if } r = 0, n \geq 1, \\
\sum_{i=1}^{n} H^{(r-1)}_i, & \text{if } n, r \geq 1 \quad (\text{see } [9]).
\end{cases}
\]

(2)

From (2), we have

\[
H^{(r)}_n = \left(\frac{n + r - 1}{r - 1}\right)(H_{n+r-1} - H_{r-1}), \quad (r \geq 1) \quad (\text{see } [9]).
\]

(3)

For any nonzero \( \lambda \in \mathbb{R} \), the degenerate exponential functions are defined by:

\[
e^{\lambda t}(x) = \sum_{n=0}^{\infty} \frac{(x)_n}{n!} \frac{t^n}{\lambda^n}, \quad e^{\lambda t}(x) = e_\lambda(t),
\]

(4)

where \((x)_0 = 1, (x)_n = x(x - \lambda)(x - 2\lambda)\cdots(x - (n - 1)\lambda), (n \geq 1)\) (see [11]).

Let \( \log_{\lambda}t \) be the degenerate logarithm, which is the compositional inverse of \( e_\lambda(t) \) so that it satisfies \( e_\lambda(\log_{\lambda}(e_\lambda(t))) = e_\lambda(t) \).

In [2,3], the degenerate harmonic numbers are defined by:

\[
\frac{1}{1-t} \log_{\lambda}(1-t) = \sum_{n=0}^{\infty} H_{\lambda,n} t^n \quad (\text{see } [2]).
\]

(5)

Thus, by (5), we obtain

\[
H_{0,\lambda} = 0, \quad H_{\lambda,n} = \sum_{k=1}^{n} \frac{\lambda}{\lambda(n)} (-1)^{k-1}, \quad (n \in \mathbb{N}) \quad (\text{see } [2]).
\]

(6)

Note that \( \lim_{\lambda \to 0} H_{\lambda,n} = H_n \), \( (n \geq 0) \). For \( n, r \geq 0 \), the degenerate hyperharmonic numbers \( H^{(r)}_{\lambda,n} \) of order \( r \) are defined by:

\[
H^{(r)}_{0,\lambda} = 0, \quad (r \geq 0), \quad H^{(0)}_{\lambda,n} = \frac{1}{\lambda(n)} (-1)^{n-1}, \quad (n \geq 1), \quad H^{(r)}_{\lambda,n} = \sum_{k=1}^{n} H^{(r-1)}_{\lambda,k}, \quad (n, r \geq 1) \quad (\text{see } [2]).
\]

(7)

We observe from (6) and (7) that \( H^{(0)}_{\lambda,n} = H_{\lambda,n} \). From (7), we note that

\[
- \frac{\log_{\lambda}(1-t)}{(1-t)^r} = \sum_{n=1}^{\infty} H^{(r)}_{\lambda,n} t^n
\]

and

\[
H^{(r+1)}_{\lambda,n} = \frac{(x)_n}{\lambda(n)} \binom{n+r}{r} H_{n+r,\lambda} - H_{r,\lambda}, \quad (r \geq 0) \quad (\text{see } [2]).
\]

(9)
It is well known that Stirling numbers of the first kind are defined by:

\[ (x)_n = \sum_{k=0}^{n} S_1(n, k)x^k, \quad (n \geq 0) \quad \text{(see [12])}, \]

where \( (x)_n = x(x-1)\cdots(x-n+1), \ (n \geq 1), \ (x)_0 = 1. \)

As the inversion formula of (10), the Stirling numbers of the second kind are given by:

\[ x^n = \sum_{k=0}^{n} S(n, k)(x)_k, \quad (n \geq 0) \quad \text{(see [7,10,13])}. \]

Recently, the degenerate Stirling numbers of the first kind are defined by:

\[ (x)_n = \sum_{k=0}^{n} S_{1,\lambda}(n, k)(x)^{\lambda k}, \quad (n \geq 0), \quad (\lambda \geq 0), \quad (\lambda \geq 1), \quad (x)_0 = 1. \]

and the unsigned degenerate Stirling numbers of the first kind are given by:

\[ \binom{n}{k, \lambda} = (-1)^{n-k}S_{1,\lambda}(n, k), \quad (n, k \geq 0) \quad \text{(see [4])}. \]

Also, the degenerate Stirling numbers of the second kind are defined by:

\[ (x)_{n,k} = \sum_{k=0}^{n} S_{2,\lambda}(n, k)(x)^{\lambda k}, \quad (n \geq 0) \quad \text{(see [11])}. \]

In [11], the degenerate Bell polynomials are given by:

\[ e^{x(e^t-1)} = \sum_{n=0}^{\infty} \phi_{n,\lambda}(x) \frac{t^n}{n!}. \]

Thus, from (15), we obtain

\[ \phi_{n,\lambda}(x) = \sum_{k=0}^{n} S_{2,\lambda}(n, k)x^k, \quad (n \geq 0). \]

Note that \( \lim_{\lambda \to 0} \phi_{n,\lambda}(x) = \phi_n(x) \), where \( \phi_n(x) \) are the ordinary Bell polynomials given by:

\[ e^{x(x^\lambda-1)} = \sum_{n=0}^{\infty} \phi_n(x) \frac{t^n}{n!} \quad \text{(see [14])}. \]

Let \( \mathfrak{S} \) be the algebra of all formal power series in \( t \) over \( \mathbb{C} \) with:

\[ \mathfrak{S} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k t^k \mid a_k \in \mathbb{C} \right\}. \]

Let \( P = \mathbb{C}[x] \) be the algebra of all polynomials in \( x \) with coefficients in \( \mathbb{C} \), and let \( P^* \) denote the vector space of all linear functionals on \( P \). For \( f(t) = \sum_{k=0}^{\infty} a_k t^k \in \mathfrak{S} \), we define the linear functional on \( P \) by setting

\[ (f(t)|x^n) = a_n \quad \text{for all } n \geq 0 \quad \text{(see [14–17])}. \]

Then, we see that \( (t^k|x^n) = n! \delta_{n,k} \), where \( \delta_{n,k} \) is the Kronecker symbol. Let \( f_{L}(t) = \sum_{k=0}^{\infty} \frac{L|x^n}{k!}t^k \). Then, we see that \( (f_L(t)|x^n) = (L|x^n) \). The map \( L \mapsto f_L(t) \) is a vector space isomorphism from \( P^* \) onto \( \mathfrak{S} \). Henceforth, \( \mathfrak{S} \) is thought of as both the algebra of formal power series in \( t \) and the vector space of all linear functionals on \( P \) (see [17]). \( \mathfrak{S} \) is called the umbral algebra, and the umbral calculus is the study of umbral algebra.

The order \( o(f) \) of the power series \( f(t) \neq 0 \) is the smallest integer \( k \) for which \( a_k \) is not zero. If \( o(f) = 1 \), then \( f \) is called a delta series; if \( o(f) = 0 \), then \( f(t) \) is called an invertible series (see [17]).

Let \( f(t) \in \mathfrak{S} \) and \( p(x) \in P \). Then, we have

\[ f(t) = \sum_{k=0}^{\infty} \frac{(f(t)|x^k)}{k!}t^k \quad \text{and} \quad p(x) = \sum_{k=0}^{\infty} \frac{(t^k|p(x))}{k!}x^k \quad \text{(see [14, 17, 18])}. \]
From (19), we note that
\[ p^{(k)}(0) = \langle t^k p(x) \rangle \quad \text{and} \quad \langle 1 | p^{(k)}(x) \rangle = p^{(k)}(0), \]  
(20)
where \( p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \).

For each nonnegative integer \( k \), the linear differential operator \( t^k \) on \( P \) is defined by:
\[ t^k x^n = \begin{cases} (n)_k x^{n-k}, & \text{if } k \leq n, \\ 0, & \text{if } k > n, \end{cases} \]
and hence, we have
\[ t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \quad (k \geq 0) \quad (\text{see [14,17,19]}). \]  
(21)

For \( f(t), g(t) \in \mathcal{F} \) with \( o(f) = 1 \) and \( o(g) = 0 \), we note that there exists a unique sequence \( s_n(x) \) of polynomials such that \( (g(t)f(f(t))^k s_n(x)) = \delta_{n,k} \), for \( n, k \geq 0 \) (see [17]). The sequence \( s_n(x) \) is called the Sheffer sequence for \( (g(t), f(t)) \), which is denoted by \( s_n(x) \sim (g(t), f(t)) \) (see [17]).

For \( A_n(x) \sim (1, f(t)) \) and \( B_n(x) \sim (1, g(t)) \), the transfer formula for the associated sequences is given by:
\[ B_n(x) = x \left( \frac{f(t)}{g(t)} \right)^n \lambda^1 A_n(x), \quad (n \geq 1) \quad (\text{see [14,17]}). \]  
(22)

In this article, we investigate some properties of degenerate harmonic and hyperharmonic numbers that are derived from the generating functions of those numbers. From our investigation, we derive some new combinatorial identities involving degenerate harmonic and hyperharmonic numbers arising from transfer formula for the associated sequences.

2 Some properties of degenerate harmonic and degenerate hyperharmonic numbers

From (8), we note that
\[ \sum_{n=1}^{\infty} nH_{n,\lambda}^{(r)} t^n = \frac{d}{dt} \left( \frac{\log(1-t)}{(1-t)^r} \right) \]
\[ = (-\log(1-t))r(1-t)^{r-1} + \frac{1}{(1-t)^r} \frac{1}{1-t} (\lambda \log(1-t) + 1) \]
\[ = \frac{1}{(1-t)^{r+1}} (\lambda - r) \log(1-t) + 1). \quad (23)\]

Thus, by (23), we obtain
\[ \sum_{n=1}^{\infty} nH_{n,\lambda}^{(r)} t^n = \frac{t}{(1-t)^{r+1}} (1 + (\lambda - r) \log(1-t)). \quad (24)\]

Multiplying \( \frac{1}{1-t} \) on both sides of (24), we obtain
\[ \frac{t}{(1-t)^{r+2}} (1 + (\lambda - r) \log(1-t)) = \frac{1}{1-t} \sum_{m=1}^{\infty} mH_{m,\lambda}^{(r)} t^m \]
\[ = \left( \sum_{m=1}^{\infty} mH_{m,\lambda}^{(r)} t^m \right) \sum_{l=0}^{\infty} t^l \]
\[ = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} mH_{m,\lambda}^{(r)} t^n. \quad (25)\]
Thus, from (25), we have

$$\frac{t}{(1-t)^{r+2}}(1 + (\lambda - r)\log_{\alpha}(1-t)) = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} mH_{m,\lambda}^{(r)}\right)t^n. \quad (26)$$

From (24) with \( r = 1 \), we have

$$\sum_{k=1}^{\infty} kH_{k,\lambda}t^k = \frac{t}{(1-t)^2}((\lambda - 1)\log_{\alpha}(1-t) + 1). \quad (27)$$

From (27), we note that

$$\sum_{n=1}^{\infty} (n + 1)H_{n,\lambda}t^n = \frac{t}{(1-t)^2}((\lambda - 1)\log_{\alpha}(1-t) + 1) - \frac{1}{1-t}\log_{\alpha}(1-t) = \frac{t - \log_{\alpha}(1-t)(1-\lambda t)}{(1-t)^2}. \quad (28)$$

Hence, by (28), we obtain

$$\sum_{n=1}^{\infty} (n + 1)H_{n,\lambda}t^n = \frac{t - (1 - \lambda t)\log_{\alpha}(1-t)}{(1-t)^2}. \quad (29)$$

By (27), we obtain

$$\sum_{k=1}^{\infty} k^2H_{k,\lambda}t^{k-1} = \frac{d}{dt}\left(\frac{t(\lambda - 1)\log_{\alpha}(1-t) + t}{(1-t)^2}\right) = \frac{(\lambda - 1)\log_{\alpha}(1-t) + 1 - t(\lambda - 1)\log_{\alpha}(1-t) + 1}{(1-t)^2} + \frac{2t((\lambda - 1)\log_{\alpha}(1-t) + 1)}{(1-t)^3} \quad (30)$$

Thus, by (30), we obtain

$$\sum_{n=1}^{\infty} n^2H_{n,\lambda}t^n = \frac{t[1 + t(2 - \lambda) + (\lambda - 1)(1 + t(1 - \lambda))\log_{\alpha}(1-t)]}{(1-t)^3}. \quad (31)$$

For \( r \geq 0 \), we have

$$\sum_{n=0}^{\infty} \left(\begin{array}{c} n + r \\ n \end{array}\right) \frac{1}{(1-z)^{r+1}} = \frac{1}{(1-z)^r} + \sum_{k=0}^{\infty} (-1)_{k+r} \frac{\log_{\alpha}(1-z)^k}{k!}$$

$$= \frac{1}{(1-z)^r} + \sum_{k=1}^{\infty} (-1)_{k+1} \frac{(\log_{\alpha}(1-z) + k + 1)}{k!} \frac{\log_{\alpha}(1-z)}{(1-z)^r}$$

$$= \frac{1}{(1-z)^r} - \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)_{k+1}}{k+1} \frac{S_{\lambda}(m, k) (-1)^m z^n}{m!} \sum_{l=1}^{\infty} H_{l,\lambda}^{(r)} z^l$$

$$= \sum_{n=0}^{\infty} \left(\begin{array}{c} r + n - 1 \\ n \end{array}\right) - \sum_{m=0}^{\infty} \frac{(-1)_{k+1}}{k+1} \frac{S_{\lambda}(m, k)}{m!} (-1)^m H_{m-\lambda}^{(r)} z^n. \quad (31)$$

Thus, by comparing the coefficients on the both sides of (31), we obtain

$$\left(\begin{array}{c} n + r \\ n \end{array}\right) = \left(\begin{array}{c} n + r - 1 \\ n \end{array}\right) - \sum_{m=0}^{\infty} \frac{(-1)_{k+1}}{k+1} \frac{S_{\lambda}(m, k)}{m!} (-1)^m H_{m-\lambda}^{(r)}.$$
That is,

\[
\binom{n + r - 1}{n - 1} = \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{(-1)^{k+1} S_{1,\lambda}(m, k)}{k + 1} \frac{(-1)^{m-1} H^{(r)}_{n-m,k}}{m!},
\]

where \( n \) is a positive integer.

Thus, we have shown the following theorem.

**Theorem 1.** For \( n \geq 1 \) and \( r \geq 0 \), the following identity is valid:

\[
\binom{n + r - 1}{n - 1} = \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{(-1)^{k+1} S_{1,\lambda}(m, k)}{k + 1} \frac{(-1)^{m-1} H^{(r)}_{n-m,k}}{m!}.
\]

### 3 Some identities of degenerate hyperharmonic numbers arising from umbral calculus

Assume that

\[ B_n(x) \sim (1, t(1 - t)^r), \quad (r \geq 0). \]

Since \( x^n \sim (1, t) \), from \( (22) \), we obtain

\[
B_n(x) = x^n \sim x(1 - t)^{-rn} x^{n-1}
\]

\[
= x \sum_{k=0}^{n} \binom{n + k - 1}{k} t^k x^{n-1} = x \sum_{k=0}^{n} \binom{n + k - 1}{k} (n - 1)_k x^{n-1-k}
\]

\[
= \sum_{k=0}^{n} \binom{n + k - 1}{k} (n - 1)_k x^{n-k} = \sum_{k=1}^{n} \binom{n + n - k - 1}{n - k} (n - 1)_k x^k.
\]

By \( (15) \), we obtain

\[
\phi_{n,\lambda}(x) = \sum_{k=0}^{n} S_{2,\lambda}(n, k)x^k \sim (1, \log(1 + t))
\]

and

\[
(-1)^n\phi_{n,\lambda}(-x) \sim (1, -\log(1 - t)).
\]

On the other hand, by \( (22), (33), \) and \( (35) \), we obtain

\[
B_n(x) = x^{-\log(1 - t)} t(1 - t)^r x^{-1}(-1)^n\phi_{n,\lambda}(-x)
\]

\[
= x\left( \sum_{i=1}^{n} H^{(r)}_{i+1,\lambda} t^i \right) x^{-1}(-1)^n\phi_{n,\lambda}(-x)
\]

\[
= (-1)^n \sum_{j=1}^{n} S_{2,\lambda}(n, j)(-1)^j \left( \sum_{i=0}^{j-1} H^{(r)}_{i+1,\lambda} t^i \right) x^{j-1}
\]

\[
= (-1)^n \sum_{j=1}^{n} S_{2,\lambda}(n, j)(-1)^j \left( \sum_{i=0}^{j-1} H^{(r)}_{i+1,\lambda} \cdots H^{(r)}_{i+1,\lambda}(j - 1)_i x^{j-1} \right)
\]

\[
= (-1)^n \sum_{j=1}^{n} \sum_{i=1}^{j-1} S_{2,\lambda}(n, j)(-1)^j H^{(r)}_{i+1,\lambda} \cdots H^{(r)}_{i+1,\lambda}(j - 1)_i x^j,
\]

where \( n \) is a positive integer.
Therefore, by (34) and (36), we obtain the following theorem.

**Theorem 2.** For \( n \geq 1 \), \( r \geq 1 \), and \( 1 \leq l \leq n \), we have

\[
\left( r^n + n - l - 1 \right)_{n-l} = (-1)^n \sum_{j=1}^{n} S_{2\lambda}(n,j) (-1)^j H_{r\lambda}^{(j)} \cdots H_{r\lambda}^{(j)} (j-1)^{-l-i}.
\]

For \( n \geq 1 \), we have

\[
\left( \log_n (1 + t) \right)^n = \frac{n!}{t^n} \log_n (1 + t)^n = \frac{n!}{t^n} \sum_{l=0}^{n} S_{2\lambda}(l,n) \frac{t^l}{l!} = \frac{n!}{t^n} \sum_{l=0}^{n} S_{2\lambda}(l + n, n) \frac{t^l}{(l + n)! l!} = \sum_{l=0}^{n} \frac{S_{2\lambda}(l + n, n) t^l}{(l + n)! l!}.
\]

For \( n \geq 1 \), by (33) and (35), we obtain

\[
B_n(x) = x \left( \frac{-\log_n (1 - t)}{t(1 - t)^r} \right)^n x^{-\lambda} (-1)^n \phi_n(x) (-x)
\]

\[
= x \left( \frac{-\log_n (1 - t)}{t} \right)^n (1 - t)^{-\lambda} x^{-\lambda} (-1)^n \phi_n(-x)
\]

\[
= (-1)^n \sum_{j=1}^{n} S_{2\lambda}(n,j) (-1)^j \left( \frac{-\log_n (1 - t)}{t} \right)^n (1 - t)^{-\lambda} x^{-\lambda} = (-1)^n \sum_{j=1}^{n} S_{2\lambda}(n,j) (-1)^j \left( \frac{-\log_n (1 - t)}{t} \right)^n \sum_{l=0}^{j-1} \frac{r^n + l - 1}{l!} (j-1)^{x^{-\lambda}} = (-1)^n \sum_{j=1}^{n} S_{2\lambda}(n,j) \sum_{l=0}^{j-1} \frac{r^n + l - 1}{l!} (j-1)^{x^{-\lambda}} = (-1)^n \sum_{j=1}^{n} S_{2\lambda}(n,j) \sum_{l=0}^{j-1} \frac{r^n + l - 1}{l!} (j-1)^{x^{-\lambda}}.
\]

Therefore, by (34) and (37), we obtain the following theorem.
Theorem 3. For \( n \geq 1, r \geq 1, \) and \( 1 \leq k \leq n, \) we have
\[
\binom{rn + n - k - 1}{n - k}n! = (-1)^n \sum_{j=0}^{n-1} \sum_{k=1}^{j-k+1} \binom{r + l - 1}{j - k} S_{j,k}(n, j) S_{j,k}(j - k + n, n).
\]
Consider the following associated sequence:
\[
q_n(x) \sim (1, t(1 - t)^{r+2}).
\] (38)
From (22), for \( n \geq 1, \) we have
\[
q_n(x) = x \left( \frac{t}{t(1 - t)^{r+2}} \right)^n x^{-1} x^n = x(1 - t)^{-n(r+2)} x^{n-1} = \sum_{k=0}^{n-1} \binom{n(r + 2) + k - 1}{k} t^k x^{n-k} = \sum_{k=0}^{n} \binom{n(r + 3) - k - 1}{n - k} (n - 1)_n x^n.
\] (39)
Therefore, by (39), we obtain the following lemma.

Lemma 4. Let \( q_n(x) \sim (1, t(1 - t)^{r+2}). \) Then, we have
\[
q_n(x) = \sum_{k=0}^{n} \binom{n(r + 3) - k - 1}{n - k} (n - 1)_n x^n, \quad (n \geq 1).
\]
Assume that
\[
p_n(x) \sim (1, t(1 + (\lambda - r) \log_2(1 - t))).
\] (40)
By (22) and (40), for \( n \geq 1, \) we obtain
\[
p_n(x) = x \left( \frac{t}{t(1 + (\lambda - r) \log_2(1 - t))} \right)^n x^{-1} x^n = \sum_{l=0}^{n} \binom{n + l - 1}{l} \nu_l \sum_{j=0}^{n-1} \binom{r - \lambda}{j} S_j(j + l, n) \binom{r - \lambda}{j} S_j(j + l, n - 1) x^{n-l} (n - 1)_{n-l} x^{l-n}
\] (41)
Therefore, by (41), we obtain the following lemma.
Lemma 5. Let $p_n(x) \sim (1, t(1 + (\lambda - r) \log_\lambda(1 - t)))$. Then, we have

$$p_n(x) = \sum_{k=1}^{n} \sum_{l=0}^{n-k} (\lambda - r)! l! \left( \frac{n + l - 1}{l} \right) \left( \frac{n - 1}{l} \right) \left( \frac{n - k}{l} \right) x^k, \quad (n \geq 1).$$

For $n \geq 1$, by (22), (26), (38), and (40), we obtain

$$q_n(x) = x^{\left\lfloor \frac{1}{t(1 - t)^{t+2}} \right\rfloor} x^{-1} p_n(x)$$

$$= \sum_{j=1}^{n} \sum_{m=1}^{a} m H_{m,a}(r) \frac{1}{j} \sum_{j=1}^{n-a} \sum_{l=0}^{n-k} (\lambda - r)! l! \left( \frac{n + l - 1}{l} \right) \left( \frac{n - 1}{l} \right) \left( \frac{n - k}{l} \right) x^k$$

$$= \sum_{j=1}^{n} \sum_{m=1}^{a} m H_{m,a}(r) \frac{1}{j} \sum_{j=1}^{n-a} \sum_{l=0}^{n-k} (\lambda - r)! l! \left( \frac{n + l - 1}{l} \right) \left( \frac{n - 1}{l} \right) \left( \frac{n - k}{l} \right) x^k$$

$$= \sum_{j=1}^{n} \sum_{m=1}^{a} m H_{m,a}(r) \frac{1}{j} \sum_{j=1}^{n-a} \sum_{l=0}^{n-k} \prod_{j=0}^{k-1} (j - a - k + 1) x^k$$

Therefore, by Theorem 3 and (42), we obtain the following theorem.

Theorem 6. For $n, r \geq 1$, and $1 \leq k \leq n$, we have

$$\left( n + 3 - k - 1 \right) \left( n - 1 \right) \left( n - 1 \right) \left( n - k \right) = \frac{1}{(n - k)!} \sum_{a=1}^{n} \sum_{l=0}^{n-k} \sum_{j=0}^{k-1} \sum_{m=1}^{a} m H_{m,a}(r) \frac{1}{j} \left( \frac{n + l - 1}{l} \right) \left( \frac{n - 1}{l} \right) \left( \frac{n - k}{l} \right) x^k$$

We recall from (24) that:

$$\frac{t}{(1 - t)^{r+1}} (1 + (\lambda - r) \log_\lambda(1 - t)) = \sum_{n=1}^{m} n^\lambda t^n.$$

Let us consider the following associated sequence:

$$Q_n(x) \sim (1, t(1 - t)^{r+1}).$$

Thus, by (22) and (43), we obtain

$$Q_n(x) = x^{\left\lfloor \frac{t}{t(1 - t)^{r+1}} \right\rfloor} x^{-1} x^n = x(1 - t)^{-(r+1)n} x^{n-1} = x \sum_{k=0}^{n-1} \left( \frac{r + 1}{k} \right) x^{n-1}$$

$$= \sum_{k=0}^{n} \left( \frac{n + 1}{k} \right) x^{n-k} = \sum_{k=1}^{n} \left( \frac{n + 2 - k - 1}{n - k} \right) x^{n-k}.$$

Therefore, by (44), we obtain the following lemma.
Lemma 7. Let \( Q_n(x) \sim (1, t(1 - t)^{r+1}) \). Then, we have

\[
Q_n(x) = \sum_{k=1}^{n} \binom{n}{k} \frac{(n+1-k)}{n-k} (n-1)_k x^k, \quad (n \geq 1).
\]

Assume that

\[
R_n(x) \sim (1, t(1 + (\lambda - r) \log(1 - t))).
\]

By (22) and (45), for \( n \geq 1 \), we obtain

\[
R_n(x) = x^n \left( t \left( \frac{t(1 + (\lambda - r) \log(1 - t))}{t(1 - t)^{r+1}} \right) \right)^n x^{-1} x^n
\]

\[
= x^n \sum_{i=0}^{n} \binom{n}{n-l} \frac{(n-k-l)}{l} \left( \frac{n-1-l}{k+l} \right) \left( \frac{n-1}{k+l} \right) x^n-1
\]

\[
= x^n \sum_{l=0}^{n-l} \frac{(n-l)}{l} \left( \frac{n}{k+l} \right) \frac{(n-l)!}{(k+l+1)!} x^n
\]

\[
= x^n \sum_{l=0}^{n-l} \frac{(n-l)}{l} \left( \frac{(n-k-l)}{k+l} \right) \frac{(n-l)!}{(k+l+1)!} x^n
\]

\[
= x^n \sum_{l=0}^{n-l} \frac{(n-k-l)}{l} \left( \frac{n-1-l}{k+l} \right) \frac{(n-l)!}{(k+l+1)!} x^n
\]

\[
= x^n \sum_{l=0}^{n-l} \frac{(n-k-l)}{l} \left( \frac{n-l}{k+l} \right) \frac{(n-l)!}{(k+l+1)!} x^n
\]

Therefore, by (46), we obtain the following lemma.

Theorem 8. Let \( R_n(x) \sim (1, t(1 + (\lambda - r) \log(1 - t))) \). Then, we have

\[
R_n(x) = \sum_{k=1}^{n} \binom{n-k}{n-l} \left( \frac{(n-k-l)}{l} \right) \left( \frac{n-1}{k+l} \right) \left( \frac{1}{x} \right) x^n, \quad (n \geq 1).
\]

From (22), (23), (43), and (45), we note that

\[
Q_n(x) = x^n \left( t \left( \frac{t(1 + (\lambda - r) \log(1 - t))}{t(1 - t)^{r+1}} \right) \right)^n x^{-1} R_n(x)
\]

\[
= x^n \sum_{i=1}^{n} \binom{n}{n-a} \left( \frac{(n-1-a)}{a} \right) \frac{(n-a)}{l} \left( \frac{1}{x} \right) x^n-1
\]

\[
= \sum_{a=1}^{n-a} \binom{n}{n-a} \left( \frac{(n-1-a)}{a} \right) \frac{(n-a)}{l} \left( \frac{1}{x} \right) x^n-1
\]

\[
= \sum_{a=1}^{n-a} \binom{n}{n-a} \left( \frac{(n-1-a)}{a} \right) \frac{(n-a)}{l} \left( \frac{1}{x} \right) x^n-1
\]

\[
= \sum_{a=1}^{n-a} \binom{n}{n-a} \left( \frac{(n-1-a)}{a} \right) \frac{(n-a)}{l} \left( \frac{1}{x} \right) x^n-1
\]

\[
= \sum_{a=1}^{n-a} \binom{n}{n-a} \left( \frac{(n-1-a)}{a} \right) \frac{(n-a)}{l} \left( \frac{1}{x} \right) x^n-1
\]

\[
= \sum_{a=1}^{n-a} \binom{n}{n-a} \left( \frac{(n-1-a)}{a} \right) \frac{(n-a)}{l} \left( \frac{1}{x} \right) x^n-1
\]

\[
= \sum_{a=1}^{n-a} \binom{n}{n-a} \left( \frac{(n-1-a)}{a} \right) \frac{(n-a)}{l} \left( \frac{1}{x} \right) x^n-1
\]

\[
= \sum_{a=1}^{n-a} \binom{n}{n-a} \left( \frac{(n-1-a)}{a} \right) \frac{(n-a)}{l} \left( \frac{1}{x} \right) x^n-1
\]

\[
= \sum_{a=1}^{n-a} \binom{n}{n-a} \left( \frac{(n-1-a)}{a} \right) \frac{(n-a)}{l} \left( \frac{1}{x} \right) x^n-1
\]

\[
= \sum_{a=1}^{n-a} \binom{n}{n-a} \left( \frac{(n-1-a)}{a} \right) \frac{(n-a)}{l} \left( \frac{1}{x} \right) x^n-1
\]

\[
= \sum_{a=1}^{n-a} \binom{n}{n-a} \left( \frac{(n-1-a)}{a} \right) \frac{(n-a)}{l} \left( \frac{1}{x} \right) x^n-1
\]

\[
= \sum_{a=1}^{n-a} \binom{n}{n-a} \left( \frac{(n-1-a)}{a} \right) \frac{(n-a)}{l} \left( \frac{1}{x} \right) x^n-1
\]

\[
= \sum_{a=1}^{n-a} \binom{n}{n-a} \left( \frac{(n-1-a)}{a} \right) \frac{(n-a)}{l} \left( \frac{1}{x} \right) x^n-1
\]

\[
= \sum_{a=1}^{n-a} \binom{n}{n-a} \left( \frac{(n-1-a)}{a} \right) \frac{(n-a)}{l} \left( \frac{1}{x} \right) x^n-1
\]

\[
= \sum_{a=1}^{n-a} \binom{n}{n-a} \left( \frac{(n-1-a)}{a} \right) \frac{(n-a)}{l} \left( \frac{1}{x} \right) x^n-1
\]
Therefore, by Lemma 6 and (47), we obtain the following theorem.

**Theorem 9.** For \( n, r \geq 1, \) and \( 1 \leq k \leq n, \) we have

\[
\binom{n(r + 2) - k - 1}{n - k} \frac{1}{(n - k)!} \sum_{a=k}^{n-a} \sum_{l=0}^{l \leq \lambda} \sum_{j_{1} + \cdots + j_{k} = a-k} (j_{1} + 1) \cdots (j_{n} + 1) H_{j_{1}+1, \lambda}^{(r)} \cdots H_{j_{n}+1, \lambda}^{(r)} (a - 1)_{a-k}^{-} \times \frac{1}{(n - l)!} \sum_{l=0}^{l \leq \lambda} \sum_{j_{1} + \cdots + j_{k} = a-k} (j_{1} + 1) \cdots (j_{n} + 1) H_{j_{1}+1, \lambda}^{(r)} \cdots H_{j_{n}+1, \lambda}^{(r)} (a - 1)_{a-k}^{-}.
\]

**4 Conclusion**

Many different tools have been used in the explorations for degenerate versions of some special numbers and polynomials. In this article, we used umbral calculus in order to study the degenerate harmonic and degenerate hyperharmonic numbers. Some properties and identities relating to those numbers were derived from the transfer formula from umbral calculus.

We would like to continue to investigate various degenerate versions of certain special numbers and polynomials, including their applications to physics, science, and engineering as well as to mathematics.

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