Research Article

Bana Al Subaiei*

On pomonoid of partial transformations of a poset

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Abstract: The main objective of this article is to study the ordered partial transformations \( \mathcal{PO}(X) \) of a poset \( X \). The findings show that the set of all partial transformations of a poset with a pointwise order is not necessarily a pomonoid. Some conditions are implemented to guarantee that \( \mathcal{PO}(X) \) is a pomonoid and this pomonoid is denoted by \( \mathcal{PO}'(X) \). Moreover, we determine the necessary conditions in order that the partial order-embedding transformations define the ordered version of the symmetric inverse monoid. The findings show that this set is an inverse pomonoid and we will denote it by \( \mathcal{IPO}'(X) \). In case the order on the poset \( X \) is total, we explore some properties of \( \mathcal{PO}'(X) \) and \( \mathcal{IPO}'(X) \), including regressive, unitary, and reversible.

Keywords: posets, pomonoids, partial transformations, inverse pomonoid

MSC 2020: 20M20, 06F05, 20M10

1 Introduction and preliminaries

Semigroups of transformations play a role in semigroup theory similar to the role of permutation groups in group theory. For a set \( X \), we let \( \mathcal{PT}(X) \) denote the monoid (under composition) of all partial transformations of \( X \) (i.e., mappings whose domain and image are the subsets of \( X \)). The submonoid of \( \mathcal{PT}(X) \) of all full transformations of \( X \) (i.e., mappings from \( X \) into \( X \)) is denoted by \( \mathcal{T}(X) \). The inverse submonoid of all full injective transformations of \( X \) is denoted by \( \mathcal{I}(X) \). These monoids are very important in the theory of semigroups since every semigroup (resp., inverse semigroup) can be embedded in some \( \mathcal{T}(X) \) (resp., \( \mathcal{I}(X) \)). This fact constitutes an analogy to Cayley’s theorem in group theory. Cayley’s theorem says that every group can be embedded in some symmetric group \( \text{Sym}(X) \) of all permutations on \( X \).

Throughout this work, we will write mappings on the right and compose them from left to right. This means that for \( f : A \to B \) and \( g : B \to C \), we will write \( x f \), instead of \( (f(x)) \), and \( x(fg) \), instead of \( (gf)(x) \). Now, suppose that \( X \) is a poset (i.e., a partially ordered set). We say that a transformation \( f \) in \( \mathcal{PT}(X) \) is order-preserving (monotone) if \( x \leq y \) implies \( xf \leq yf \), for all \( x, y \in \text{dom}(f) \). Also, we say that a transformation \( f \) in \( \mathcal{PT}(X) \) is order-embedding whenever that \( x \leq y \) if and only if \( xf \leq yf \), for all \( x, y \in \text{dom}(f) \). Note that the product (composition) of two-order-preserving transformations and two-order-embedding transformations is also order-preserving and order-embedding, respectively. As usual, the submonoid of \( \mathcal{PT}(X) \) of all partial order-preserving transformations of \( X \) will be denoted by \( \mathcal{PO}(X) \), while the monoid \( \mathcal{PO}(X) \cap \mathcal{T}(X) \) of all full transformations of \( X \) that preserve the order will be denoted by \( \mathcal{O}(X) \). These monoids have been widely studied when \( X \) is totally ordered, namely, in [1–6].

Considerable attention has been paid over the years to full, injective, partial, and partial injective transformations of a set. While there are advancements being made in the study of these concepts (cf. [7]), the ordered versions have not yet been investigated. Some researchers have studied these concepts on totally

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ordered sets and have defined them as full and partial order-preserving transformations (see, for instance, [1,2,8,9]). Also, some studies have considered the order-decreasing and increasing of full, partial, and partial one-to-one transformations of a totally ordered set (cf. [3,10–12]). However, none of these studies has looked at these concepts as partially ordered semigroups (or monoids) for any poset \( X \). Al Subaiei in [13] has only studied the full transformations of a poset \( X \) for some specific partially ordered relations on \( X \). To complete this circle of ideas, our purpose here is to focus on the partial order-preserving (monotone) transformations and partial order-embedding transformations of a poset \( X \).

Next, we recall some background. A posemigroup (resp., pomonoid) is a semigroup (resp., monoid) \( S \) partially ordered by \( \leq \), such that \( \leq \) is compatible with the semigroup (resp., monoid) operation, i.e., for all \( x, y, z \in S \), \( x \leq y \implies xz \leq yz \) and \( xz \leq yz \). For more details about monoids and pomonoids, we refer the reader to [14,15]. In [16], Sohail has considered the full transformations of a poset, when he studied the ordered representations of a pomonoid. He has stated that the sets \( O(X) \subseteq \mathcal{T}(X) \) are pomonoids with respect to the usual composition of transformations and pointwise order (i.e., for \( f, g \in O(X), f \leq g \) if and only if \( xf \leq xg \) for all \( x \in X \)). It is worth mentioning that Nasir’s work is mainly focused on the representations of a pomonoid and not the study of the transformations. As partial transformations of a poset \( X \) are not considered yet when \( X \) is not a totally ordered set, so a natural question arises: whether \( \mathcal{P}O(X) \) does constitute a pomonoid when equipped with the pointwise order? Recall that the pointwise order on \( \mathcal{P}O(X) \) is defined as follows:

\[
\forall f, g \in \mathcal{P}O(X), f \leq g \iff \text{dom}(f) \subseteq \text{dom}(g) \quad \text{and} \quad \forall x \in \text{dom}(f), xf \leq xg.
\]

It is worth noting that generalizing these concepts to the ordered case requires more conditions to guarantee the “pomonoid structure” of the set of partial transformations of a poset.

Any subset \( X \) of a poset \( Y \) is called a subposet since the partial order relation is inherited from \( Y \). Let \( X \) be a subposet of a poset \( Y \). In general, the set \( X^{\uparrow Y} = \{ y \in Y : x \leq y \quad \text{for some} \quad x \in X \} \) is known as the upper/upward closure of \( X \). It is clear that \( X \subseteq X^{\uparrow Y} \) and this inclusion relation may be strict in general. Moreover, \( X^{\uparrow Y} \) may be different from the set of upper bounds of \( X \). The following example illustrates these facts.

**Example 1.1.** Consider the poset \( Y = \{a, b, c, d, e, f\} \) equipped with the following order relation \( \leq_1 \) as in Figure 1.

Also, consider the set \( X = \{a, b, e\} \). Hence, we have \( X^{\uparrow Y} = \{a, b, e, c, d\} \) and the set of upper bounds of \( X \) is \( \{b, d\} \). Then, it is clear that \( X^{\uparrow Y} \supsetneq X \). However, if the order relation on \( Y \) is \( \leq_2 \), which is defined as in Figure 2. Then, \( X^{\uparrow Y} = \{a, b, e\} \), while the set of upper bounds of \( X \) is empty. Hence, \( X^{\uparrow Y} = X \).

Kemprasit [17] studied some properties of the partial transformation of poset \( X \), such as the idempotent elements, shift of an element and regressive. An element \( e \) in any semigroup \( S \) is called idempotent, when \( e^2 = e \) and usually \( E(S) \) denote the set of all idempotent elements in \( S \). The set of idempotents of the partial transformation monoid on poset \( X \), denoted by \( E(\mathcal{P}T(X)) \), is defined as: \( E(\mathcal{P}T(X)) = \{ f \in \mathcal{P}T(X) | \text{im}(f) \subseteq \text{dom}(f) \} \). The shift of an element \( f \in \mathcal{P}T(X) \) is the set \( S(f) = \{ x \in \text{dom}(f) | xf \neq x \} \). The element \( f \in \mathcal{P}T(X) \) is said to be regressive if for every \( x \in \text{dom}(f) \), \( xf \leq x \).

Gould and Shaheen [18] studied the concept of unitary in posemigroup, and then, Al Subaiei and Renshaw [19,20] generalized this concept further. Let \( U \) be a subpomonoid of a pomonoid \( S \) and let \( u, v \in U \) and \( s \in S \).

![Figure 1: Partially ordered relation \( \leq_1 \) on \( Y \).](image-url)
On pomonoid of partial transformations of a poset

is said to be an upper strongly right pounitary in \( S \) when \( v \leq su \) implies \( s \in U \). Moreover, \( U \) is said to be a lower strongly right pounitary in \( S \) when \( su \leq v \) implies \( s \in U \). \( U \) is said to be a right unitary in \( S \) when \( su = v \) implies \( s \in U \). Left-sided versions are defined dually.

Reversibility was studied in the literature for a pomonoid (see, for example, [21,22]). Let \( S \) be a pomonoid, then \( S \) is right reversible, if for any \( s, s' \in S \), we have \( Ss \cap Ss' \neq \emptyset \). However, a pomonoid \( S \) is called weakly right reversible whenever for any \( t, t' \in S \), we have \( St \cap (St') \neq \emptyset \). The set \( (St) = \{ s \in S : s \leq k, k \in St \} \) is called the down-set of \( S \). Weakly left reversible is defined dually.

This article is organized as follows: In Section 2, we construct an example showing that \( P\mathcal{O}(X) \) with the pointwise order is not a pomonoid (Example 2.1). Thus, we have added some conditions, and we have considered the set \( P\mathcal{O}'(X) = \{ f \in P\mathcal{O}(X) | \text{dom}(f) = \text{dom}(f') \} \). The main result of this section is Theorem 2.4, which states that \( (P\mathcal{O}'(X), \preceq) \) is a pomonoid, when it is equipped with the pointwise order (I). In Section 3, we consider the monoid \( P\mathcal{O}E(X) \) of partial order-embedding transformations on a poset \( X \). It is shown that this monoid is not a pomonoid with respect to the pointwise order. So we restrict ourselves to the set \( P\mathcal{O}E'(X) = \{ f \in P\mathcal{O}E(X) | \text{im}(f) = \text{im}(f') \} \). We establish in Theorem 3.3 that \( IP\mathcal{O}'(X) \) is a subpomonoid of \( P\mathcal{O}'(X) \). Theorem 3.4 states that \( IP\mathcal{O}'(X) \) is an inverse pomonoid. Section 4 is devoted to the study of the pomonoids \( P\mathcal{O}'(X) \) and \( IP\mathcal{O}'(X) \) in case \( X \) is a finite toset (i.e., a totally ordered set). In Proposition 4.1, we prove that if \( X \) is a finite toset, then \( IP\mathcal{O}'(X) \subseteq E(P\mathcal{O}'(X)) \). It is shown that if \( X \) is a finite toset, then \( IP\mathcal{O}'(X) \) is right reversible, and for any \( f \in IP\mathcal{O}'(X) \), \( S(f) = \emptyset \) and \( f \) is regressive (Remark 4.3). Theorem 4.4 states that \( IP\mathcal{O}'(X) \) is a right and left unitary in \( P\mathcal{O}'(X) \), when \( X \) is a finite toset. Table 1 contains all the notations used in this article.

![Figure 2: Partially ordered relation \( \leq \) on \( Y \).](image)

### Table 1: Descriptions for notations

<table>
<thead>
<tr>
<th>Notations</th>
<th>Descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X )</td>
<td>Poset</td>
</tr>
<tr>
<td>( P\mathcal{T}(X) )</td>
<td>The monoid of all partial transformations of ( X )</td>
</tr>
<tr>
<td>( T(X) )</td>
<td>The monoid of all full transformations of ( X )</td>
</tr>
<tr>
<td>( I(X) )</td>
<td>The inverse monoid of all full injective transformations of ( X )</td>
</tr>
<tr>
<td>( P\mathcal{I}(X) )</td>
<td>The inverse monoid of all partial injective transformations of ( X )</td>
</tr>
<tr>
<td>( P\mathcal{O}(X) )</td>
<td>The set of all partial order-preserving transformations of ( X )</td>
</tr>
<tr>
<td>( O(X) )</td>
<td>The monoid of all full order-preserving transformations of ( X )</td>
</tr>
<tr>
<td>( X^1 )</td>
<td>The upper closure of ( X ).</td>
</tr>
<tr>
<td>( \text{dom}(f)' )</td>
<td>( { \text{dom}(f) } )</td>
</tr>
<tr>
<td>( \text{im}(f)' )</td>
<td>( { \text{im}(f) } )</td>
</tr>
<tr>
<td>( P\mathcal{O}'(X) )</td>
<td>( { f \in P\mathcal{O}(X)</td>
</tr>
<tr>
<td>( P\mathcal{O}E'(X) )</td>
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</tr>
<tr>
<td>( IP\mathcal{O}'(X) )</td>
<td>( { f \in IP\mathcal{O}'(X)</td>
</tr>
</tbody>
</table>

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**Notations:**

- **\( X \):** Poset
- **\( P\mathcal{T}(X) \):** The monoid of all partial transformations of \( X \)
- **\( T(X) \):** The monoid of all full transformations of \( X \)
- **\( I(X) \):** The inverse monoid of all full injective transformations of \( X \)
- **\( P\mathcal{I}(X) \):** The inverse monoid of all partial injective transformations of \( X \)
- **\( P\mathcal{O}(X) \):** The set of all partial order-preserving transformations of \( X \)
- **\( O(X) \):** The monoid of all full order-preserving transformations of \( X \)
- **\( X^1 \):** The upper closure of \( X \)
- **\( \text{dom}(f)' \):** \( \{ \text{dom}(f) \} \)
- **\( \text{im}(f)' \):** \( \{ \text{im}(f) \} \)
- **\( P\mathcal{O}'(X) \):** \( \{ f \in P\mathcal{O}(X) | \text{dom}(f)' = \text{dom}(f) \} \)
- **\( P\mathcal{O}E'(X) \):** \( \{ f \in P\mathcal{O}E(X) | \text{im}(f)' = \text{im}(f) \} \)
- **\( IP\mathcal{O}'(X) \):** \( \{ f \in IP\mathcal{O}'(X) | f \) is an order embedding, \( \text{im}(f)' = \text{im}(f) \) |
2 Ordered partial transformations of a poset

Let \((X, \leq)\) be a poset and equip \(\mathcal{P}(X)\) with the following pointwise order relation:

\[
\forall f, g \in \mathcal{P}(X), f \leq g \iff \text{dom}(f) \subseteq \text{dom}(g), \quad \text{and} \quad \forall x \in \text{dom}(f), xf \leq xg.
\]

The next example shows that \(\langle \mathcal{P}(X), \leq \rangle\) is not, in general, a pomonoid.

**Example 2.1.** Let \(X = \{a, b, c\}\) be a poset with the partial order \(\leq_3\) as in Figure 3.

Our task is to show that the monoid \(\langle \mathcal{P}(X), \leq \rangle\) is not a pomonoid. Let \(f = \begin{pmatrix} a & b & c \\ a & a & a \end{pmatrix}\), \(g = \begin{pmatrix} a \\ b \end{pmatrix}\), and \(h = \begin{pmatrix} b \\ b \end{pmatrix}\).

One can easily check that all these mappings are in \(\mathcal{P}(X)\). However, \(\leq \) is not compatible with the composition. Indeed, we have \(g \leq f\) but \(gh \not\leq fh\) since \(\text{dom}(gh) = \{a\} \not\subseteq \text{dom}(fh) = \emptyset\). Therefore, \(\mathcal{P}(X)\) with the pointwise order is not a pomonoid (Figure 3).

It is worth noting that in the previous example, we have \(\text{dom}(h)^\dagger = \{a, b\} \supseteq \text{dom}(h) = \{b\}\). So, Example 2.1 encourages us to add some conditions in order to equip the monoid \(\mathcal{P}(X)\) with a pomonoid structure. To this end, let us consider the following set:

\[\mathcal{P}'(X) = \{f \in \mathcal{P}(X)|\text{dom}(f)^\dagger = \text{dom}(f)\} .\]

The set \(\mathcal{P}'(X)\) will be called the set of *ordered partial transformations* of \(X\). It is obvious that the identity map \(1_X \in \mathcal{P}'(X)\). We start our investigation with the following straightforward lemma. We include a proof for the sake of completeness.

**Lemma 2.2.** Let \(f \in \mathcal{P}(X)\). Then, \((Yf^{-1})^\dagger \subseteq (Y')f^{-1}\) for any \(Y \subseteq \text{im}(f)\).

**Proof.** Let \(x \in (Yf^{-1})^\dagger\). Then, there exists \(a \in Yf^{-1}\) such that \(a \leq x\). As \(f\) is an order-preserving map, then \(af \leq xf\). But \(af \in Y\). This implies that \(xf \in Y^\dagger\) and so \(x \in (Y')f^{-1}\). This completes the proof. \(\Box\)

The following lemma is straightforward.

**Lemma 2.3.** Let \(f, h \in \mathcal{P}(X)\). If \(f \leq g\), then \(hf \leq hg\).

**Theorem 2.4.** \(\langle \mathcal{P}'(X), \leq \rangle\) is a pomonoid.

**Proof.** First of all, we need to show that \(\mathcal{P}'(X)\) is a monoid. To this end, let \(f, g \in \mathcal{P}'(X)\). Then, \(\text{dom}(f)^\dagger = \text{dom}(f)\) and \(\text{dom}(g)^\dagger = \text{dom}(g)\). Thus,

\[
\text{dom}(fg)^\dagger = ((\text{dom}(g) \cap \text{im}(f))f^{-1})^\dagger = (\text{dom}(g)^\dagger f^{-1} \cap (\text{im}(f)^{f^{-1}})^\dagger = (\text{dom}(g)^\dagger f^{-1} \cap (\text{im}(f)^{f^{-1}})^\dagger = \text{dom}(fg).
\]

Thus, \(\text{dom}(fg) = \text{dom}(fg)^\dagger\). This shows that \(\mathcal{P}'(X)\) is a submonoid of \(\mathcal{P}(X)\) and hence a monoid.

\[
\begin{array}{c}
\text{Figure 3: Partial ordered relation } \leq_3 \text{ on } X.
\end{array}
\]
Next, we show that $\leq$ is compatible with the composition in $\mathcal{PO}(X)$. Let $f, g, h \in \mathcal{PO}(X)$ such that $f \leq g$. Our task is to show that $fh \leq gh$ and $hf \leq hg$. Let $x \in \text{dom}(fh) = (\text{dom}(h) \cap \text{im}(f))^{-1}$. Then, $x \in \text{dom}(f)$ and $xf \in \text{dom}(h)$. As $xf \leq xg$, then $xg \in \text{dom}(h)$. But $\text{dom}(h) \subseteq \text{dom}(h)$, which yields that $xg \in \text{dom}(h)$. Hence, $xg \in (\text{dom}(h) \cap \text{im}(g))^{-1} = \text{dom}(gh)$. Therefore, $\text{dom}(fh) \subseteq \text{dom}(gh)$. Now, using the fact that $h$ is an order-preserving map and $f \leq g$, we obtain readily $x(fh) \leq x(gh)$ for any $x \in \text{dom}(fh)$. Therefore, $fh \leq gh$. For $hf \leq hg$, it follows directly from Lemma 2.3. □

### 3 Order-embedding partial transformations of a poset

In this section, we consider the following monoid:

$$\mathcal{P}I(X) = \{ f \in \mathcal{P}T(X) \mid f \text{ is injective} \} = \mathcal{P}T(X) \cap I(X).$$

We set

$$\mathcal{POE}(X) = \{ f \in \mathcal{P}T(X) \mid f \text{ is an order embedding} \}$$

and

$$\mathcal{POE}'(X) = \{ f \in \mathcal{POE}(X) \mid \text{im}(f)^{-1} = \text{im}(f) \}.$$

The subset $\mathcal{POE}(X)$ of $\mathcal{P}I(X)$ is also a monoid and this is clear from the definition of $\mathcal{POE}(X)$. Moreover, $\mathcal{POE}'(X) = \mathcal{PO}(X) \cap \mathcal{POE}'(X)$. In this section, we consider the order version of the inverse monoid of all partial injective transformations of $X$.

**Proposition 3.1.** The set $\mathcal{POE}'(X)$ is a monoid.

**Proof.** It is clear that $\mathcal{POE}'(X)$ is a subset of the monoid $\mathcal{P}I(X)$. Let $f, g \in \mathcal{POE}'(X)$. We need to show that $fg \in \mathcal{POE}'(X)$. It is clear that $fg \in \mathcal{POE}(X)$. Let $x \in \text{im}(fg)$, then $x \in \text{im}(g)^{-1}$, which yields that $x \in \text{dom}(g)^{-1}$. It is easy to check that $x \in \text{im}(g)^{-1}$. Thus, there exists $y \in (\text{im}(f) \cap \text{dom}(g))^{-1}$ such that $x \leq y$. Write $y = ag$ for some $a \in \text{im}(f) \cap \text{dom}(g)$, and let $d \in \text{dom}(f)$ such that $a = df$. As $ag \leq x$, then $x \in \text{im}(g)^{-1}$. But, by assumption, $\text{im}(g) = \text{im}(g)^{-1}$, so $x = bg$ for some $b \in \text{dom}(g)$. Now, we obtain $ag \leq bg$. Since, $g$ is an order-embedding, we infer that $a \leq b$. But, $a \in \text{im}(f)$. So $b \in \text{im}(f)^{-1}$. Hypothetically, $\text{im}(f)^{-1} = \text{im}(f)$. Thus, $b = cf$ for some $c \in \text{dom}(f)$. It follows that $x = c(fg) \in \text{im}(f) \cap \text{dom}(g)^{-1} = \text{im}(fg)$. Therefore, $\text{im}(fg) \subseteq \text{im}(fg)$. As the reverse inclusion is always true, we conclude that $\text{im}(fg)^{-1} = \text{im}(fg)$. This completes the proof. □

From the previous proposition, we can have the following result.

**Corollary 3.2.** $\mathcal{POE}'(X)$ is a submonoid of $\mathcal{POE}(X)$.

**Theorem 3.3.** The following hold true:

1. $\mathcal{P}O(X)$ is a subpomonoid of $\mathcal{PO}(X)$.
2. $\mathcal{P}O(X)$ is a submonoid of $I(X)$.

**Proof.**

1. Clearly, $\mathcal{P}O(X) \subseteq \mathcal{POE}'(X)$. Using Proposition 3.1, it is easy to show that $\mathcal{P}O(X)$ is a submonoid of $\mathcal{POE}'(X)$. Using similar argument to Theorem 2.4, we can show that $fh \leq gh$ and $hf \leq hg$ for every $f \leq g$ and $f, g, h \in \mathcal{P}O(X)$.

2. It is clear by using similar argument as in case 1. □
Theorem 3.4. \( IPO(X) \) is an inverse pomonoid.

Proof. Let \( f \in IPO(X) \). Then, \( f \) is an order-embedding map, \( \text{dom}(f) = \text{dom}(f)^\uparrow \) and \( \text{im}(f) = \text{im}(f)^\downarrow \). From Theorem 3.3, we know that \( IPOL(X) \) is a pomonoid and \( IPO(X) \) is a submonoid of \( I(X) \). Hence, there exists \( f^{-1} \in I(X) \). The proof will be completed if we show that \( f^{-1} \in IPO(X) \). It is well known that for any \( \varphi \in I(X) \), \( \text{dom}(\varphi) = \text{im}(\varphi)^\downarrow \) and \( \text{im}(\varphi) = \text{dom}(\varphi)^\uparrow \). As \( f^{-1} \in I(X) \), we obtain immediately \( \text{im}(f^{-1}) = \text{dom}(f) = \text{dom}(f)^\uparrow \) and \( \text{im}(f^{-1}) = \text{im}(f) = \text{im}(f)^\downarrow \). It obvious that \( f^{-1} \) is an order-embedding. Hence, \( f^{-1} \in IPO(X) \). The proof is complete.

Example 3.5. Consider the poset \( X = \{a, b, c\} \) as in Example 2.1. Then, we have

(1) \( PO'(X) = \{a_0 = 0, a_1 = \{a, b, c\}, a_2 = \{a, b, c\}, a_3 = \{a, b, c\}, a_4 = \{a, b, c\}, a_5 = \{a, b, c\}, a_6 = \{a, b, c\}, a_7 = \{a, b, c\}, a_8 = \{a, b, c\}, a_9 = \{a, b, c\}, a_{10} = \{a, b, c\}, a_{11} = \{a, b, c\}, a_{12} = \{a, b, c\}, a_{13} = \{a, b, c\}, a_{14} = \{a, b, c\}, a_{15} = \{a, b, c\}, a_{16} = \{a, b, c\}, a_{17} = \{a, b, c\}, a_{18} = \{a, b, c\}, a_{19} = \{a, b, c\}, a_{20} = \{a, b, c\}, a_{21} = \{a, b, c\}, a_{22} = \{a, b, c\}, a_{23} = \{a, b, c\}, a_{24} = \{a, b, c\}, a_{25} = \{a, b, c\}, a_{26} = \{a, b, c\}, a_{27} = \{a, b, c\}, a_{28} = \{a, b, c\}, a_{29} = \{a, b, c\} \} \)

(2) \( IPO'(X) = \{a_0 = 0, a_1 = \{a, b, c\}, a_2 = \{a, b, c\}, a_3 = \{a, b, c\}, a_4 = \{a, b, c\}, a_5 = \{a, b, c\}, a_6 = \{a, b, c\}, a_7 = \{a, b, c\}, a_8 = \{a, b, c\}, a_9 = \{a, b, c\}, a_{10} = \{a, b, c\}, a_{11} = \{a, b, c\}, a_{12} = \{a, b, c\}, a_{13} = \{a, b, c\}, a_{14} = \{a, b, c\}, a_{15} = \{a, b, c\}, a_{16} = \{a, b, c\}, a_{17} = \{a, b, c\}, a_{18} = \{a, b, c\}, a_{19} = \{a, b, c\}, a_{20} = \{a, b, c\}, a_{21} = \{a, b, c\}, a_{22} = \{a, b, c\}, a_{23} = \{a, b, c\}, a_{24} = \{a, b, c\}, a_{25} = \{a, b, c\}, a_{26} = \{a, b, c\}, a_{27} = \{a, b, c\}, a_{28} = \{a, b, c\}, a_{29} = \{a, b, c\} \} \)

Example 3.6. Let \( X = \{1, 2\} \) with the natural order relation. Then, one can easily check that

(1) \( PO'(X) = \{a_0 = 0, a_1 = \{1, 2\}, a_2 = \{1, 2\}, a_3 = \{1, 2\}, a_4 = \{1, 2\}, a_5 = \{1, 2\} \} \)

(2) \( IPO'(X) = \{a_0 = 0, a_1 = \{1, 2\}, a_2 = \{1, 2\}, a_3 = \{1, 2\}, a_4 = \{1, 2\}, a_5 = \{1, 2\} \} \)

Moreover, the order on \( PO'(X) \) is as in Figure 4.

Example 3.7. Let \( X = \{1, 2, 3\} \) with the natural ordered relation. Then, we have

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Figure 4: Ordered relation \( \leq \) on \( PO'(X) \).
```
4 Totally ordered poset

In this section, we study \( PO(X) \) and \( IPO(X) \) in case \( X \) is a toset (i.e., a totally ordered set). Note that even when \( X \) is a toset, the aforementioned monoids differ from the semigroup of order-preserving partial transformation which is known in the literature [1–3]. To see this, consider the poset \( X \) in Example 3.6 and note that the mappings \( \{1, 2\} \) and \( \{1, 1\} \) do not belong to \( PO(X) \). However, these maps are elements of the semigroup of order-preserving partial transformation.

**Proposition 4.1.** Let \( X \) be a finite toset of cardinality \( n \). Then, the following conditions hold true:

1. If \( f \in IPO(X) \), then, \( \text{dom}(f) = \text{im}(f) \) and \( af = a \) for any \( a \in \text{dom}(f) \).
2. \( |IPO(X)| = |X| + 1 \).

**Proof.**

1. Write \( X = \{a_1 < a_2 < \cdots < a_n\} \) as a chain and let \( f \in IPO(X) \). Since \( \text{dom}(f)^\prime = \text{dom}(f) \) and \( \text{im}(f)^\prime = \text{im}(f) \), then the domain and image of \( f \) will be one of the following sequences: \( X_1 = X, \ldots, X_i = \{a_i < a_{i+1} < \cdots < a_n\}, \ldots, X_{n-1} = \{a_{n-1} < a_n\} \) and \( X_n = \{a_n\} \). Moreover, since \( f \) is an order-embedding and \( X \) is totally ordered, \( af = a \) for any \( a \in \text{dom}(f) \). Therefore, \( \text{dom}(f) = \text{im}(f) = X_i \).

2. It follows from the first assertion that

\[
IPO(X) = \left\{ \begin{array}{c} 0 \\ (a_1 \ a_2 \ \ldots \ \ a_n) \\ \vdots \\ (a_{n-1} \ a_n) \\ (a_n) \end{array} \right\}.
\]

Therefore, \( |IPO(X)| = n + 1 \). This completes the proof. \( \square \)

We derive the following corollary.

**Corollary 4.2.** Let \( X \) be a finite toset. If \( f \in IPO(X) \) and \( g \in PO(X) \), then \( fg = gf = k \), where \( k \) satisfies the following conditions:

1. \( k \in PO(X) \backslash IPO(X) \).
2. \( \text{dom}(k) \subseteq \text{dom}(g) \).
3. \( ak = ag \) for any \( a \in \text{dom}(k) \).

**Remark 4.3.**

1. If \( X \) is a finite toset of cardinality \( n \), then \( IPO(X) \subseteq E(PO(X)) \). To see this, we know that

\[
IPO(X) = \left\{ \begin{array}{c} 0 \\ (a_1 \ a_2 \ \ldots \ \ a_n) \\ \vdots \\ (a_{n-1} \ a_n) \\ (a_n) \end{array} \right\}.
\]
Also, it follows from the definition of \(E(\mathcal{PO}'(X))\), that \(IPO'(X) \subseteq E(\mathcal{PO}'(X))\). Take \(f = \begin{bmatrix} a_{n-1} & a_n \\ a_{n-1} & a_{n-1} \end{bmatrix} \in \mathcal{PO}'(X)\). Clearly, \(f \in E(\mathcal{PO}'(X))\), but \(f \not\in IPO'(X)\). This completes the proof.

(2) If \(X\) is a finite toset, then for any \(f \in IPO'(X)\), \(S(f) = \emptyset\).

(3) If \(X\) is a finite toset, then any \(f\) in \(IPO'(X)\) is regressive.

(4) It is worth noting that not every element in \(\mathcal{PO}'(X)\) is regressive. To see this, take \(a_{13} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}\) in Example 3.7. Clearly, \(a_{13}\) is not regressive since \(2a_{13} = 3 \neq 2\).

**Theorem 4.4.** If \(X\) is a finite toset, then \(IPO'(X)\) is a right and left unitary in \(\mathcal{PO}'(X)\).

**Proof.** We know from Theorem 3.3 that \(IPO'(X)\) is a subpomonoid of the pomonoid \(\mathcal{PO}'(X)\). According to Corollary 4.2, we have that if \(gf \in IPO'(X)\) for any \(f \in IPO'(X)\) and any \(g \in \mathcal{PO}'(X)\), then \(g \in IPO'(X)\). Therefore, \(IPO'(X)\) is a right unitary in \(\mathcal{PO}'(X)\). In similar way, \(IPO'(X)\) is a left unitary in \(\mathcal{PO}'(X)\).

**Remark 4.5.**

(1) The pomonoid \(IPO'(X)\) is not an upper strongly right (or left) pounitary in \(\mathcal{PO}'(X)\). To see this, let \(X = \{1, 2, 3, 4\}\). Then,

\[
\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \leq \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}.
\]

(2) The pomonoid \(IPO'(X)\) is not a lower strongly right (or left) pounitary in \(\mathcal{PO}'(X)\). Take, for instance:

\[
\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix} \leq \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}.
\]

**Proposition 4.6.** If \(X\) is a finite toset, then \(IPO'(X)\) is right reversible.

**Proof.** As \(0 \in IPO'(X)\), then for all \(a \in IPO'(X)\), we have \(0 \in IPO'(X)\) a. Therefore, \(0 \in IPO'(X)\) a \(\neq \emptyset\). This completes the proof.

**Remark 4.7.** Consider Example 3.7. It follows from Proposition 4.6 that \(IPO'(X)\) is right reversible.

**Proposition 4.8.** If \(X\) is a finite toset, then \(IPO'(X)\) is weakly right reversible.

**Proof.** The conclusion follows directly, from \([21, \text{Theorem 4.6}]\), since the empty map 0 is both a zero and a minimum element.

**5 Conclusion**

In this article, we found the suitable partially ordered relation, which guarantees that the partial transformation monoid of a poset \(X\) is a pomoind. Also, we described the inverse pomoind from the partial transformation monoid of a poset \(X\). Then, our attention goes to study some properties when \(X\) is a toset.

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