Abstract: The article explores research findings akin to Amitsur’s theorem, asserting that any derivation within a matrix ring can be expressed as the sum of an inner derivation and a hereditary derivation. Amitsur’s theorem and semicentral idempotents play a crucial role. This article is intended for PhD students, postdocs, and researchers.

Keywords: differential algebra, derivations, matrices, matrix rings, endomorphisms, endomorphism rings, idempotents, semicentral idempotents, additive basis, skew Ore polynomials, additively idempotent semirings, matrix semirings, polynomial semirings, endomorphism semirings

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1 Introduction

This article examines results for the derivations of matrix rings related to the well-known theorem of Amitsur for representing a derivation as a sum of an inner and a hereditary derivation. We also consider semicentral idempotents created by Birkenmeier as they are a useful tool for studying derivations. Finally, we review a number of papers on the derivations of some classes of additively idempotent semirings in which Amitsur’s theorem and semicentral idempotents play a crucial role.

This article does not exhaustively explore all possible derivations of rings and semirings. Notably absent are discussions on significant topics such as Jordan derivations, derivations of Lie algebras, and local derivations. In addition, we have opted to exclude papers that merely “generalize” results from ring theory to semirings, as they fall outside the scope of our interest.

Furthermore, we have not delved into the extensive body of work by Rowen and the mathematical community surrounding him, particularly on the generalization of ideas from classical algebraic geometry to the emerging field of additively idempotent semirings. While we acknowledge the importance of numerous papers in this domain, we have selectively marked some for reference and encourage someone from this community to conduct a comprehensive review.

Our research covers aspects of ring and semiring theory, aiming to elucidate the delineations that distinguish it. We will specifically highlight the articles and books employed as references to define the scope of our study:

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• the excellent survey article on decomposition of matrices as a product of idempotents by Jain and Leroy [1];
• the review article on the derivations of rings by Ashraf et al. [2];
• the scientific monograph of Birkenmeier et al. [3];
• the survey paper of Rachev on endomorphism semirings [4];
• the book of Glazek [5] which is an extensive list of sources in semiring theory;
• the survey paper of our colleague Dimitrov [6] concerning derivations of semirings;
• the very detailed overview of semiring applications, see Golan [7].

In addition to the aforementioned key references, we also include a supplementary list of less prominent studies related to this topic for comprehensive coverage and contextual understanding: Akian et al. [8], Akian et al. [9], Chebotar and Lee [10], Chebotar et al. [11], Chebotar and Ke [12], Dimitrov [13,14], Drensky and Gupta [15], Drensky and Makar-Limanov [16], and Dangovski et al. [17,18].

2 Amitsur’s theorem

2.1 Representation of derivations of matrix rings

Throughout the discussion, unless otherwise mentioned, $R$ denotes an associative (not necessarily commutative) ring. Recall that a derivation of $R$ is an additive map $d : R \to R$ such that $d(xy) = d(x)y + xd(y)$ where $x, y \in R$.

For a fixed $a \in R$ the map $d_a(x) = [a, x] = ax - xa$ for any $x \in R$ is a derivation called inner derivation of $R$ determined by $a$.

As a consequence of Skolem-Noether theorem, see the study by Herstein ([19], p. 99), an arbitrary derivation of the ring of square matrices over a field is inner.

Let $M$ be a ring of matrices over $R$. It is a well-known fact that the map $\delta : M \to M$ such that $\delta(A) = (\delta(a_{ij}))$ for any matrix $A = (a_{ij}) \in M$, where $\delta$ is a derivation of $R$, is a derivation called a hereditary derivation generated by $\delta$.

In 1982, Amitsur [20, Theorem 2] proved that an arbitrary derivation of the ring of square $n \times n$ matrices $M_n(R)$ over an associative ring $R$ with identity is a sum of an inner derivation and a hereditary derivation.

In 1983, Nowicki [21] showed a similar result for the so-called special subrings of rings of square $n \times n$ matrices over ring.

A particular case of Amitsur’s result, when $R$ is an algebra over a field $\Phi$, with $\text{char}(\Phi) \neq 2, 3$, appears in Benkert and Osborn [22].

In 1993, Coelho and Milies [23] proved the following result similar to those in by Amitsur [20] for the ring of upper triangular matrices $T_n(R)$:

**Theorem 2.1.** Let $R$ be a ring with unity and let $d : T_n(R) \to T_n(R)$ be a derivation. Then there exists a derivation $\delta : R \to R$ and a matrix $A \in T_n(R)$ such that $d = \delta + d_A$.

This result appeared in 1982 in the last example of Mathis [24].

In 1990, Kezlan [25] obtained the analogous result for an automorphism, which states the following:

**Theorem 2.2.** If $R$ is any commutative ring with unity, then every $R$-algebra automorphism of $T_n(R)$ is inner.

In 1995, Jondrup [26] generalized this theorem to rings in which all idempotents are central, and by using the method of generalization, he re-proved the results of Mathis and of Coelho and Milies.

In 1951, Dubisch and Perlis [27] studied a new class of matrix rings $NT_n(F)$ consisting of matrices over a field $F$ whose entries are zeroes on and over the main diagonal. They proved that any automorphism on $NT_n(F)$ is a product of certain diagonal automorphism, inner automorphism, and nil automorphism.
Kuzucuoğlu and Levchuk [28, 29] have investigated the ideals and automorphisms of the ring $R_0(K, J) = N_{T_0}(K) + M_0(J)$, where $K$ is an associative ring and $J$ is an ideal of $K$.

In 2006, Chun and Park [30] defined the following derivations of the ring $N_{T_0}(K)$: (1) for each diagonal matrix $d = \sum_{i=1}^{m} d_i e_{ii}$, where $d_i \in K$, $I_d(A) = [d, A]$ is an inner derivation determined by $d$ of the ring of lower triangular matrices over $K$ and it is called a diagonal derivation, and (2) a derivation $s$ of $N_{T_0}(K)$ is called a strongly nilpotent derivation if for all $x \in (N_{T_0}(K))^K$, it follows that $s(x) \in (N_{T_0}(K))^{K^{x-1}}$. The main result of the study by Chun and Park [30] is

**Theorem 2.3.** An arbitrary derivation of $N_{T_0}(K)$ is a sum of a diagonal derivation, a hereditary derivation, and a strongly nilpotent derivation.

In 2010, Levchuk and Radchenko [31] generalized the theorem of Chun and Park replacing the strongly nilpotent derivation with a central derivation. (A derivation (or automorphism) of a ring is called central if it acts like the zero (resp. identity) map modulo the center.)

Derivations of a matrix ring containing a subring of triangular matrices was described in 2011 by Kolesnikov and Mal’tsev [32] using the results of the study by Levchuk and Radchenko [31].

Derivation of matrix rings consisting of sums of a niltriangular matrix and a matrix over an ideal were studies by Kuzucuoğlu and Sayin [33] in 2017.

As infinite matrices, in 2015, Slowik [34] considered the ring $M_{CF}(R)$ consisting of all infinite matrices over an associative ring $R$ with a finite number of nonzero entries in each column. He also denoted by $d_A$ the inner derivation determined by the matrix $A$ and by $\delta$ the hereditary derivation of $M_{CF}(R)$ generated by the derivation $\delta$ of $R$. The first main result is the following:

**Theorem 2.4.** Let $R$ be an associative unital ring. If $d$ is a derivation of $M_{CF}(R)$, then there exists a matrix $A \in M_{CF}(R)$ and a derivation $\delta$ of $R$ such that $d = d_A + \delta$.

The second main result is a similar equality for the ring of infinite matrices with finite number of nonzero entries in every row.

In 2017, Hołubowski et al. [35] showed that any derivation of the Lie algebra of infinite strictly upper triangular matrices over a commutative ring is the sum of an inner derivation and a diagonal derivation.

In 2022, Brešar [36] obtained the following result, closely related to Amitsur’s idea:

**Theorem 2.5.** Let $A$ be a finite-dimensional algebra over a field $F$ with char($F$) \neq 2. If a linear map $D : A \to A$ satisfies $xD(x)x \in [A, A]$ for every $x \in A$, then $D$ is the sum of an inner derivation of $A$ and a linear map from $A$ to rad$(A)$.

Following the theorem, the author drew two conclusions.

**Corollary 2.6.** Let $A$ be a finite-dimensional semisimple algebra over a field $F$ with char($F$) \neq 2. The following conditions are equivalent for a linear map $D : A \to A$:

(i) $xD(x)x \in [A, A]$ for every $x \in A$.

(ii) $D$ is an inner derivation.

A local derivation of an algebra $A$ is defined as a linear map $D : A \to A$ such that for each $x \in A$ there is a derivation $D_x : A \to A$ and $D(x) = D_x(x)$, see for details Kadison [37]. Local automorphisms are defined similarly. Note that local automorphisms play an important role in functional analysis (see the study by Larson and Sourour [38]). A standard question is whether local derivations (resp., local automorphisms) are derivations (resp., automorphisms).

**Corollary 2.7.** Let $A$ be a finite-dimensional semisimple algebra over a field $F$ with char($F$) \neq 2. Then every local inner derivation $D : A \to A$ is an inner derivation.
2.2 Similar research

Exploring whether there are objects in ring theory that share a research history with well-known objects of the same type is an intriguing aspect worth investigating.

Also, in the matrix ring, there is research to present a matrix as a product, a sum, a difference, or linear combinations of idempotent matrices. We leave it to the reader to draw their own conclusions from these investigations.

Over the last 60 years, substantial efforts have been devoted to the examination of idempotent compositions in matrix rings.

In 1966, Howie [39] proved that every transformation of a finite set which is not permutation can be written as a product of idempotents. One year later, Erdos [40] proved that every singular matrix over a field is a product of idempotent matrices. This result was extended to matrices over division rings and Euclidean rings. In many papers, the connection between product decomposition of singular matrices into idempotents and product decomposition of invertible matrices into elementary matrices is considered. An $n \times n$ matrix over ring $R$ is called elementary if it is of the form $I_n + ce_{ij}$, where $c \in R$ and $i \neq j$.

In 2014, Salce and Zanardo [41] studied relations between these two decompositions in the setting of commutative integral domains.

In 2018, Cossu et al. [42] proved that the property every invertible $n \times n$ matrix over integral domain is a product of elementary matrices holds for important classes of non-Euclidean principal integral domains as coordinate rings of elliptic curves having only one rational point.

In 2019, Cossu and Zanardo [43] proved that an integral domain $R$ such that any singular $2 \times 2$ matrix over $R$ is a product of idempotent matrices must be a Prüfer domain in which every invertible $2 \times 2$ matrix is a product of elementary matrices.

In 2022, Cossu and Zanardo [44] proved that any $2 \times 2$ matrix over the ring of integers of the real quadratic number field $Q[\sqrt{d}]$, where $d > 0$ is a square-free integer, with either null row or a null column is a product of idempotents.

We now come to consider idempotent factorizations of matrices over noncommutative rings.

In 2014, Alahmadi et al. [45] provided constructions of idempotents to represent typical singular matrices over a ring (not necessarily commutative) as products of idempotents. The important results are as follows:

If $A$ is $2 \times 2$ matrix over local ring with $\text{ann}(A) \neq 0$, then $A$ is a product of idempotent matrices.

If every $2 \times 2$ invertible matrix over Bézout domain is a product of elementary matrices and diagonal matrices with invertible diagonal entries, then every $n \times n$ singular matrix is a product of idempotent matrices.

In 2014, Alahmadi et al. [46] considered various conditions for ring $R$ connected with the decomposition of singular matrices over $R$ as a product of idempotent matrices.

In 2016, Facchini and Leroy [47] generalized the results of Salce and Zanardo’s paper [41] in noncommutative setting.

Recently, the idempotent matrices over noncommutative rings have been discussed by Hou [48], Drensky [49], and Wright [50].

Very recently, in 2024, Vladeva presented a new formula of all semicentral idempotents of upper triangular matrix rings [51].

3 Semicentral idempotents

3.1 The studies of Birkenmeier and the mathematical community around him

Most research papers by the group around Birkenmeier explore when the properties of a ring (or a module) are transferred to its various ring extensions (or module extensions). Semicentral idempotents are a key tool
for these studies. Therefore, we will review some of the scientific papers where semicentral idempotents play an important role.

In 1983, Birkenmeier [52] defined new idempotent elements of an associative ring \( R \) as follows: an idempotent \( e \in R \) is called a left (resp., right) semicentral idempotent if \( exe = xe \) (resp., \( exe = ex \)) for all \( x \in R \).

Birkenmeier et al. [53] developed the theory of generalized triangular matrix representations. Let \( R \) be an associative \( K \)-algebra with a unity. The authors defined that \( R \) has a generalized triangular matrix representation if it is ring isomorphic to a generalized triangular matrix ring

\[
\begin{pmatrix}
R_1 & R_{12} & R_{13} & \cdots & R_{1n} \\
0 & R_2 & R_{23} & \cdots & R_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & R_n
\end{pmatrix}
\]

where each diagonal element \( R_i \), \( 1 \leq i \leq n \), is a ring with a unity, \( R_i \) is a left \( R_k \)-right \( R_l \)-bimodule for \( 1 \leq i < j \leq n \), and the matrices obey the usual rules for matrix addition and multiplication. Let \( \mathcal{J}(R) \) and \( \mathcal{F}(R) \) denote the sets of left and right semicentral idempotents of \( R \). For some idempotent, \( e \in R \) it follows that \( \mathcal{J}(eRe) = \{0, e\} \) if and only if \( \mathcal{F}(eRe) = \{0, e\} \). When this property for \( e \) is satisfied, \( e \) is called semicentral reduced. A ring is called semicentral reduced if \( 1 \) is a semicentral reduced idempotent.

If each \( R_i \) in matrix (1) is semicentral reduced, then the ring \( R \) has a complete generalized triangular matrix representation.

An ordered set \( \{b_1, ..., b_n\} \) of nonzero distinct idempotents in \( R \) is called a set of left triangulating idempotents of \( R \) if all of the following holds:

- \( b_1 + \cdots + b_n = 1 \);
- \( b_i \in \mathcal{J}(R); \)
- \( b_{i+1} \in \mathcal{F}(c_kRc_k) \), where \( c_k = 1 - (b_1 + \cdots + b_k) \) for \( 1 \leq k \leq n - 1 \).

A set \( \{b_1, ..., b_n\} \) of left triangulating idempotents \( R \) is called complete if each idempotents \( b_k \) is semicentral reduced.

From the main result of the paper (Theorem 2.9), it follows that the ring \( R \) has a complete generalized triangular matrix representation if and only if \( R \) has a complete set of left triangulating idempotents.


New results have been developed for endomorphism algebras of modules and semicentral reduced algebras. One of them is given as follows:

**Theorem 3.1.** For any positive integer \( n \), \( R \) is semicentral reduced if and only if \( M_n(R) \) is semicentral reduced.
A ring $R$ is called a 1-Peirce ring if 0 and 1 are the only Peirce trivial idempotents of $R$. For a natural number $n > 1$, a ring $R$ is called an $n$-Peirce ring if there is a Peirce trivial idempotent $e \in R$ such that $eRe$ is an $m$-Peirce ring for some $m$, $1 \leq m < n$, and $(1 - e)R(1 - e)$ is an $(n - m)$-Peirce ring. An idempotent $e \in R$ is called an $n$-Peirce idempotent if $eRe$ is an $n$-Peirce ring.

For any natural $n$, it follows that $n$-Peirce rings are generalizations of rings with a complete set of triangulating idempotents [52].

The first main result is as follows:

**Theorem 3.2.** Let $R$ be an $n$-Peirce ring and $e \in R$ an arbitrary Peirce trivial idempotent. Then $eRe$ is a $k$-Peirce ring for some $k \leq n$ and $(1 - e)R(1 - e)$ is an $(n - k)$-Peirce ring.

Another significant outcome reveals that the authors have effectively formulated a structural theory for Peirce rings, mirroring the framework established by Bass for semiperfect rings.

Finally, Anh et al. [57], following Jacobson [58], construct the so-called trivial idempotents relative to certain radicals, like $J$-trivial and $B$-trivial idempotents, where $J$ and $B$ are Jacobson and prime (Baer) radical.

### 3.2 Derivations of matrix rings. Links to Amitsur’s theorem

In 2022, Vladeva [59] offers a description of the $R$-derivations of $UTM_n(R)$, the ring of upper triangular matrices over an associative ring $R$ with an identity.

A derivation $d$ of the ring $UTM_n(R)$ is called $R$-derivation if $d(\lambda A) = \lambda d(A)$, where $\lambda \in R$ and $A \in UTM_n(R)$. The author considered the matrices $\hat{c}_k = e_{11} + \cdots + e_{kk}$, where $1 \leq k \leq n$, and proved that $\hat{c}_k$ are left semicentral idempotents of the ring $UTM_n(R)$ (the matrix $e_0$ has $(i,j)$-entry 1 and rest zero is called a matrix unit). The inner derivations $d_{\hat{c}_k}$ defined by $d_{\hat{c}_k}(A) = [\hat{c}_k, A]$, $1 \leq k \leq n - 1$, for any matrix $A \in UTM_n(R)$ are idempotents and linearly independent in the additive group of $R$-derivations of the ring $UTM_n(R)$. The author preferred to work with the derivations $\delta_i$ such that $\delta_i(A) = [e_{ii}, A]$ for any $A \in UTM_n(R)$, $i = 1, \ldots, n$. Then $d_{\hat{c}_k}(A) = \sum_{i=1}^{k} \delta_i(A)$. Let $D$ be the additive group of derivations generated by $\delta_2, \ldots, \delta_n$. Since $\delta_2, \ldots, \delta_n$ are $R$-derivations, it follows that $D$ is an $R$-module. Since $\delta_1 + \delta_2 + \cdots + \delta_n = 0$ it follows that $\delta_1 \in D$. The derivations $\delta_2, \ldots, \delta_n$ form a basis of the $R$-module $D$. The main result of the paper follows that all derivations of $UTM_n(R)$ belong to $D$.

**Theorem 3.3.** Let $D : UTM_n(R) \to UTM_n(R)$ be an arbitrary $R$-derivation of the ring $UTM_n(R)$ and $A = (a_{ij})_{i,j=1}^n \in UTM_n(R)$. Then there are the matrices $M^0, M^0_j, N^0_j \in UTM_n(R)$, $i, j = 1, \ldots, n$, such that

$$D(A) = \sum_{i=1}^{n} a_{ii} \delta_i(M^0) + \sum_{j=1}^{n-1} \sum_{i<j}^{n-1} a_{ij} \left( \delta_i(M^0_j) + \delta_j(N^0_i) \right),$$

where $\delta_i, i = 1, \ldots, n$, are the basic derivations.

Since $d_{\hat{c}_k} = \sum_{i=1}^{k} \delta_i \in D$, it appears that the result similar to the aforementioned theorem, when on the right side of the equality $d_{\hat{c}_k}$ appears, will be true.

Building upon Vladeva’s proposal [59] to represent an arbitrary derivation using derivations generated by left semicentral idempotents in the case $n = 3$.

**Proposition 3.4.** Let $D : UTM_3(R) \to UTM_3(R)$ be an arbitrary $R$-derivation of the ring $UTM_3(R)$ and $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in UTM_3(R)$. Then there are the matrices $M^0, M^0_j \in UTM_3(R)$, $1 \leq i < j \leq 3$, such that

$$D(A) = (a_{11} - a_{22})d_{c_1}(M^0) + (a_{22} - a_{33})d_{c_2}(M^0) + a_{12}d_{c_2}(M^1) + a_{13}d_{c_3}(M^1) + a_{23}d_{c_3}(M^2).$$
Proof. It easily follows that \( d_{ij}(A) = \delta_i(A) = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) and that \( d_{ij}(A) = \delta_i(A) + \delta_j(A) = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \). Since \\
\[ \delta_j(A) = \begin{pmatrix} 0 & 0 & -a_{13} \\ 0 & 0 & -a_{23} \\ 0 & 0 & 0 \end{pmatrix}, \]
we have \( \delta_j(A) = d_{ij}(A), \delta_j(A) = d_{ei}(A) - d_{ij}(A) \) and \( \delta_j(A) = -d_{ei}(A) \).

By using the proof of Theorem 3.4 (Theorem 3.1 of [59]), we have \( D(A) = \sum_{i,j=1}^{n} d_{ij}(A) D(e_{ij}) \).

Let \( D(e_{pq}) = \sum_{i,j=1}^{n} a_{ij}^{(p,q)} e_{ij} \), where \( 1 \leq p \leq q \leq 3 \). Then (as in the study by Vladeva [59]) we obtain that \[ D(e_{11}) = a_{12}^{(1,1)} e_{12} + a_{13}^{(1,1)} e_{13} = d_{ij}(M^p), \]
\[ D(e_{22}) = -a_{12}^{(1,1)} e_{12} + a_{23}^{(2,1)} e_{23} = d_{ij}(M^p) - d_{ij}(M^p), \]
\[ D(e_{33}) = -a_{13}^{(1,1)} e_{13} - a_{23}^{(2,2)} e_{23} = -d_{ij}(M^p), \]
where \( M^p = \begin{pmatrix} 0 & a_{11}^{(1,1)} & a_{12}^{(1,1)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \). Then \( a_{11} D(e_{11}) + a_{12} D(e_{22}) + a_{33} D(e_{33}) = (a_{11} - a_{22}) d_{ij}(M^p) + (a_{22} - a_{33}) d_{ij}(M^p) \).

From (8) in the proof of Theorem 3.1 we obtain \[ D(e_{12}) = a_{12}^{(1,2)} e_{12} + a_{23}^{(2,2)} e_{23}, \]
\[ D(e_{13}) = a_{13}^{(1,3)} e_{13}, \]
\[ D(e_{23}) = a_{12}^{(2,3)} e_{12} + a_{23}^{(3,2)} e_{23}. \]
Now for \( M_{12}^p = \begin{pmatrix} 0 & a_{12}^{(1,2)} & a_{23}^{(2,3)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \), \( M_{13}^p = \begin{pmatrix} 0 & 0 & a_{13}^{(1,3)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) and \( M_{23}^p = \begin{pmatrix} 0 & 0 & a_{23}^{(2,2)} \\ 0 & 0 & a_{23}^{(3,2)} \\ 0 & 0 & 0 \end{pmatrix} \) we have \[ a_{12} D(e_{12}) + a_{13} D(e_{13}) + a_{23} D(e_{23}) = a_{12} d_{ij}(M_{12}^p) + a_{13} d_{ij}(M_{13}^p) + a_{23} d_{ij}(M_{23}^p). \] Hence the result follows. \( \square \)

In 2023, Vladeva [60] investigated the class of endomorphisms \( \alpha \) of a ring \( \text{UTM}_n(R) \) of upper triangular \( n \times n \) matrices over an associative ring \( R \) with a unity.

Let us compare the two results: Proposition 2.5 in [59] and Proposition 2.5 in [60]. In the first one, if \( \epsilon \) is a left semicentral idempotent and \( r \) is a right semicentral idempotent of an arbitrary ring \( R \) and, moreover, \( \epsilon + r = 1 \), then \( d_{ij}(x) = \epsilon x r \) for any \( x \in R \) define a derivation of \( R \). From the second proposition without the restriction \( \epsilon + r = 1 \), it follows that \( \Phi(x) = r x \epsilon \) for any \( x \in R \) define an endomorphism of \( R \). Thus, we may conclude that the endomorphisms of an arbitrary ring, generated by left and right semicentral idempotents, are way more (in general) than the derivations.

For any left semicentral idempotent \( \epsilon_k \in \text{UTM}_n(R) \) [59] the author proves that \( a_k(A) = A \epsilon_k, 1 \leq k \leq n \), for any matrix \( A \in \text{UTM}_n(R) \) defined an endomorphism of \( \text{UTM}_n(R) \). Similarly, \( r_m = \epsilon_{m-1} n - m + 1 + \cdots + \epsilon_n \), where \( 1 \leq m \leq n \) is a right semicentral idempotent of \( \text{UTM}_n(R) \) and \( \beta_m(A) = r_m A \), where \( A \in \text{UTM}_n(R) \), defines an endomorphism of \( \text{UTM}_n(R) \).

The multiplicative semigroup \( E_n(R) \) generated by all \( a_k \) and \( \beta_m, k = 1, \ldots, n \), is a commutative semigroup with an identity. The first main result of the article is as follows:

**Theorem 3.5.** For \( a, \beta \in E_n(R) \), it follows that \( a + \beta \in E_n(R) \) if and only if \( a \beta = 0 \).

An endomorphism \( a \) is called a \((0,1)\)-endomorphism if \( a(e_{ij}) \) is a \((0,1)\)-matrix. An endomorphism \( a \) is called regular if \( a(e_{ii}) = e_{ii} \) or \( a(e_{ii}) = 0 \) for all \( i = 1, \ldots, n \). All endomorphisms that belong to \( E_n(R) \) are regular \((0,1)\)-endomorphisms. The second important result is presented as follows:

**Theorem 3.6.** The class of regular \((0,1)\)-endomorphisms is \( E_n(R) \).
4 Additively idempotent semirings

A natural entry in the semiring theory is offered by Golan’s classical monographs [61–63].
We will use the following definition by Vladeva [64]

An algebra \( S = (S, +, \cdot) \) with two binary operations \(+\) and \(\cdot\) on \(S\), is called a semiring if:
1. \((S, +)\) is a commutative semigroup,
2. \((S, \cdot)\) is a semigroup,
3. distributive laws hold \(\cdot(y + z) = x \cdot y + x \cdot z\) and \((x + y) \cdot z = x \cdot z + y \cdot z\) for any \(x, y, z \in S\).

The semiring \(S\) is called commutative if \(a \cdot b = b \cdot a\) for any \(a, b \in S\).

In some of the considered semirings \(S\) are assumed to exists a zero element \(0 \in S\) such that \(a + 0 = a\)
and \(a \cdot 0 = 0 \cdot a = 0\) for any \(a \in S\) and an identity element \(1 \in S\) such that \(1 \cdot a = a \cdot 1 = a\) for any \(a \in S\).

An element \(a\) of a semiring \(S\) is called additively idempotent if \(a + a = a\). A semiring \(S\) is named additively idempotent if each of its elements is additively idempotent. Additively idempotent semirings are proper semirings, i.e., they are not rings.

Golan [61] comments on the following feature: “On one hand, semirings are abstract mathematical structures and their study is part of abstract algebra - arising from the work of Dedekind, Macaulay, Krull, and others ... On the other, the modern interest in semirings arises primarily from fields of applied mathematics...”.

In addition, it is worth noting that semirings have significant connections to applied mathematics, linguistics, theoretical physics, cryptography, and various other scientific disciplines.

4.1 Applications of additively idempotent semirings

The majority of applications involving semirings pertain to additively idempotent semirings.

4.1.1 Automata theory and linguistics

Schützenberger [65] first acknowledged the significance of semirings in automata theory in 1961 when he formulated the theory of weighted automata and rational power series. Weighted finite automata hold both theoretical and practical importance in computer science, playing a pivotal role in the structural analysis of recognizable languages. Furthermore, they find practical applications in fields such as speech recognition and image compression, as highlighted in the previous studies [66,67].

4.1.2 Logic and theoretical computer science

In 1969, Hoare [68] introduced a formal system, known as Hoare logic, to investigate specification and verification of computer programs. Recently, modal operators for idempotent semirings have been introduced to model properties of programs and transition systems more conveniently and to link algebraic formalisms with traditional approaches such as dynamic and temporal logics [69]. Interpretations of logical formulas over additively idempotent semirings, excluding the Boolean semiring, find applications in various areas of computer science.

Many valued algebras were introduced by Chang [70] as the algebraic counterpart for the infinite valued logic of Łukasiewicz. Recently, Di Nola and Russo [71,72], using MV-algebras that are additively idempotent semirings (in fact, MV-algebras are inclines), have obtained new results of MV-algebras.
4.1.3 Optimizations and max-plus algebras

Max-algebra has been studied in research papers and books from the early 1960s. In 1960, Cuninghame-Green [73] produced the first paper of the topic followed by numerous other articles that were summarized in a lecture notes volume [74] in 1979. Max-algebra is the analog to linear algebra, developed over an additively idempotent semifield $\mathbb{R}_{\text{max}} = \mathbb{R} \cup \{-\infty\}$ (with operations $x \oplus y = \max(x, y)$ and $x \odot y = x + y$) [8,75–77]. The max-plus-based methods described in the monograph by McEneaney [78] are oriented towards solving a Hamilton-Jacobi partial differential equation. It covers as an important fact that the semigroup associated with the nonlinear Hamilton-Jacobi partial differential equation is a linear max-plus operator.

4.1.4 Tropical geometry

Tropical geometry can be thought of as algebraic geometry over the tropical semiring, a piecewise linear version of algebraic geometry, which replaces a variety by its combinatorial shadow. The foremost workers in this area are Mikhalkin [79,80], Itenberg et al. [81], and Sturmfels et al. [82,83]. Some of the basic concepts for amoebas of algebraic varieties and their geometric properties are discussed in by Mikhalkin and Theobald [84,85]. Very recent investigations on these topics have come from Maclagan and Rincon [86] and Ito [87].

4.1.5 Idempotent analysis

The superposition principle (in quantum mechanics) means that the Schrödinger equation is linear. Similarly, in idempotent analysis, the superposition principle means that some important and basic problems and equations (optimization problems, the Bellman equation and its versions and generalizations, the Hamilton-Jacobi equation, etc.), which are nonlinear in the usual sense, can be treated as linear over appropriate idempotent semirings, see studies by Maslov [88] and Maslov and Sambourskiĭ [89].

4.1.6 Petri nets

Modern technology has created dynamic systems that are not easily described by differential equations. The state of such dynamic systems changes only at discrete instants of time instead of continuously, and they are called discrete event dynamic systems. Timed Petri nets are one of the best studied and most widely known models of discrete event dynamic systems. A Petri net is called an event graph, if all arcs have the weight 1 and each place has exactly one input and one output transition. The fact that Petri nets are connected with additively idempotent semirings has been well known since 1992 [90]. Recently, the previous studies [91,92] have initiated an algebraic study of Petri nets.

4.1.7 Cryptography

Modern cryptography is mostly public-key cryptography. One-way trapdoor functions are essential to the study of this subject. A one-way trapdoor function is a one-way function $f$ from a set $X$ to a set $Y$ with the additional property, the trapdoor, and it becomes feasible to find for any $y \in \text{Im}(f)$, an $x \in X$ such that $f(x) = y$. Recently, we have noted the investigations of Grigoriev and Shpilrain [93,94] and the monograph of Roman’kov [95]. We would also like to note the Bulgarian contribution to these studies [96–100].
4.2 The research of Rowen and the mathematical community around him

It is impossible to review all the results of Rowen and the mathematical community around him, due to the large number of articles and their high quality. Perhaps if we were to write ten (or more) surveys called “Rowen: Tropical Algebra,” “Rowen: Supertropical Semirings,” “Rowen: Hyperfields,” “Rowen: The Negation Map,” and so on, we would make a small step toward doing this review.

We point out, in particular, the book [101] and the recent study [102] by Rowen. All other papers are presented in the previous studies [103–110,112–129].

4.3 Derivations of polynomial semirings. Amitsur’s idea and semicentral idempotents

When it comes to polynomial semirings, we refer to Golan [61]. It is well known that the polynomial algebra over an additively idempotent semiring does not satisfy unique factorization. For example, \((x^2 + 1)(x + 1) = x^3 + x^2 + x + 1 = (x + 1)^3\) are two different factorizations of the polynomial \(x^3 + x^2 + x + 1\).

Very recently, Baily et al. [130], Dong [131], Akián et al. [132] have explored polynomial semirings.

In 2000, Thierrin [133] first considered derivations of semirings. He proved that the semiring of languages over some alphabet forms an additively idempotent semiring.

**Why are derivations so important for semirings?**

Let \(S\) be a semiring and \(M_2(S) = \left[\begin{array}{cc} a & b \\ 0 & a \end{array}\right], a, b \in S\). Then \(\delta\) defined by \(\delta\left[\begin{array}{cc} a & b \\ 0 & a \end{array}\right] = \left[\begin{array}{cc} 0 & b \\ 0 & 0 \end{array}\right]\) is a derivation.

Hence, \(M_2(S)\) is a semiring with derivation \(\delta\). Note that the semiring \(S\) need not be additively idempotent.

Since \(S\) may be identified with subsemiring of \(M_2(S)\) consisting of matrices of the form \(\left[\begin{array}{cc} a & 0 \\ 0 & a \end{array}\right]\), it follows that every semiring can be embedded in a semiring with nontrivial derivation.

In 2020, Vladeva [134] investigated derivations of the polynomial semiring \(S[x]\), where \(S\) is a commutative additively idempotent semiring. By using that the map \(\delta_1 : S[x] \rightarrow S[x]\) such that \(\delta_1(P(x)) = a_m x^{m-1} + \cdots + a_1 x + a_0\), for \(P(x) = a_m x^m + \cdots + a_2 x^2 + a_1 x + a_0 \in S[x]\) is a derivation, the author constructed a new derivation \(\delta_{1(f)} : S[x] \rightarrow S[x]\) defined by \(\delta_{1(f)}(P(x)) = \delta_1(P(x)) f(x)\) for a fixed polynomial \(f(x) \in S[x]\).

Following Jacobson [135, p. 530], we can call a derivation an \(S\)-derivation if it is an \(S\)-linear map. Therefore, the following important conclusion can be drawn:

**Theorem 4.1.** Let \(S\) be a commutative additively idempotent semiring and \(d : S[x] \rightarrow S[x]\) be an \(S\)-derivation. Then there exists a polynomial \(f(x) \in S[x]\), such that \(d = \delta_{1(f)}\).

For an arbitrary derivation \(d\) of \(S[x]\) and polynomial \(P(x) = a_m x^m + \cdots + a_1 x + a_0\), the derivatives \(d(a_i)\) are in general polynomials. Then it follows that the map \(\Delta_d : S[x] \rightarrow S[x]\) such that \(\Delta_d(P(x)) = d(a_m) x^{m-1} + \cdots + d(a_2) x^2 + d(a_1) x + d(a_0)\) is a derivation. This derivation referred to as a generalized inner derivation. The key result, where the author proved, that every derivation is a sum of an \(S\)-derivation and a generalized inner derivation, is actually Amitsur’s idea:

**Theorem 4.2.** Let \(S\) be a commutative additively idempotent semiring. For each derivation \(D : S[x] \rightarrow S[x]\), there exists a generalized inner derivation \(\Delta_D\) and a polynomial \(f(x) \in S[x]\) such that \(D = \Delta_D + \delta_{1(f)}\).

In 2020, Vladeva [136] defined a multiplication in a noncommutative additively idempotent semiring \(S[x]\) by the rule \(a x = a x + \delta(a)\), where \(a, \delta(a) \in S\) and \(\delta\) is a derivation of \(S\). It is worth noting that when considering examples of derivations of \(S\) she examined the multiplications by the so-called left (right) Ore elements of \(S\), which are essentially left (right) semicentral idempotents, as we show in the last section.
For the aforementioned derivation $\delta$ the author constructs a map $\delta_{het} : S[x] \to S[x]$ such that $\delta_{het}(P(x)) = \delta(a_0) + \delta(a_1)x + \cdots + \delta(a_m)x^m$, where $P(x) = a_0 + a_1x + \cdots + a_mx^m \in S[x]$, which is a derivation of $S[x]$. Now if $d_\delta(P(x)) = xP(x)$ for $P(x) \in S[x]$, it follows that $d_\delta(P(x)) = \delta_{het}(P(x)) + P(x)x$. This equality implies that $d_\delta$ is a derivation of $S[x]$.

If $C(S)$ is the center of $S$ and $C_0(S)$ is the subsemiring of $S$, consisting the constants $\delta$, then $\Gamma(S) = C(S) \cap C_0(S)$ is a nontrivial subsemiring of $S$. The author proved that $d_\delta^k(P(x)) = x^kP(x)$ and for $a \in \Gamma(S)$ the map $d_{ax^k}(P(x)) = ax^kP(x)$ are derivations of $S[x]$ and then state the result as follows:

**Theorem 4.3.** Each element of the commutative additively idempotent semiring $\Gamma(S)[d_\delta]$ is a derivation of $S[x]$.

A derivation $d : S[x] \to S[x]$ such that $d(\delta(a)) = \delta_{het}(d(a))$, where $a \in S$, is called a $\delta$ - derivation. For each $\delta$-derivation $d : S[x] \to S[x]$, the map $\Delta_d$ defined by $\Delta_d(P(x)) = d(a_0) + d(a_1)x + \cdots + d(a_m)x^m$, where $P(x) = a_0 + a_1x + \cdots + a_mx^m$, is a derivation of $S[x]$.

Vladeva [134] showed that $\partial$ defined by $\partial(P(x)) = a_1 + a_2x + \cdots + amx^{m-1}$ is a derivation of $S[x]$.

The main result is similar to Amitsur’s theorem:

**Theorem 4.4.** Let $S$ be an additively idempotent semiring. For an arbitrary $\delta$ – derivation $D : S[x] \to S[x]$, and for any polynomial $P(x) \in S[x]$, there exists a derivation $\Delta_D$ such that $D(P(x)) = \Delta_D(P(x)) + \partial(P(x))D(x)$.

### 4.4 Derivations of matrix semirings. Amitsur’s idea and semicentral idempotents

When it comes to the history of matrix semirings, we refer to the survey paper of Gondran and Minoux [137].


In 2017, Vladeva [139] studied derivatives of triangular, Toeplitz, and circulant matrices over an additively idempotent semiring.

In 2020, Tan [140] obtained a condition for an idempotent matrix over a commutative semiring to be diagonalizable.

In 2022, Vladeva [141] explored the semiring $UTM_n(S)$ of upper triangular matrices over an additively idempotent semiring $S$. By using the sums $\bar{D}_k = e_{11} + \cdots + e_{kk}$, the author constructed a derivation $\delta_k$ such that $\delta_k(A) = \bar{D}_kA$, where $A \in UTM_n(S)$. Similarly by $\bar{D}_m = e_{1n}+e_{2,n-1}+\cdots+e_{mn}$ is built a derivation $\delta_m(A) = A\bar{D}_m$, where $A \in UTM_n(S)$. But $\bar{D}_k$ and $\bar{D}_m$ are just the left and right semicentral idempotents, respectively, considered in [59] (see Subsection 3.2). The set $D$ of derivations $\delta_k$ and $\delta_m$ is an additively idempotent semiring such that for the products $\delta_k\delta_m$ that the author proved:

**Theorem 4.5.** Let $\delta_k, \delta_m \in D$, where $1 \leq k \leq n$ and $1 \leq m \leq n$. The map $\delta_k\delta_m = \delta_m\delta_k$ is a derivation if and only if $\delta_k + \delta_m$ is the identity map.

Moreover, $D$ is an $S$-semimodule and the derivations $\delta_k\delta_m, k = 1, \ldots, n-1$, and $\delta_k\delta_{m-k+1}, k = 1, \ldots, n$, are the elements of the basis of $D$.

In the central conclusion of the paper, we obtain the Amitsur’s idea:

**Theorem 4.6.** An arbitrary derivation $D : UTM_n(S) \to UTM_n(S)$, where $S$ is an additively idempotent semiring, is a linear combination of elements of the basis $B$ of the $S$-semimodule $D$ with coefficients from $S$.

In 2021, Vladeva [142] studied the semiring $M_n(S)$ of $n \times n$ matrices over an additively idempotent semiring $S$. In the next lemma, the author represented the derivatives of the matrix units under an arbitrary $S$-derivation.
Lemma 4.7. Let \( S \) be an arbitrary (not necessarily commutative) additively idempotent semiring and \( M_n(S) \) be the semiring of \( n \times n \) matrices over \( S \). For a derivation \( D : M_n(S) \rightarrow M_n(S) \), there exists \( g \in S \) such that \( D(e_{ij}) = ge_{ij} \), where \( e_{ij} \) are the matrix units for \( 1 \leq i \leq n \) and \( 1 \leq j \leq n \).

As a consequence in the commutative case it is obtained that any \( S \)-derivation of the matrix semiring \( M_n(S) \) is a hereditary derivation:

**Theorem 4.8.** Let \( S \) be a commutative additively idempotent semiring and \( M_n(S) \) be the semiring of \( n \times n \) matrices over \( S \). For an arbitrary \( S \)-derivation \( D : M_n(S) \rightarrow M_n(S) \) and \( A = (a_{ij}) \in M_n(S) \), the elements of the matrix \( D(A) \) are \( d^A_{ij}(a_{ij}) \), where \( d^A \) is the inner derivation, generated by \( g \in S \) such that \( D(e_{ij}) = ge_{ij} \), where \( e_{ij} \) are the matrix units for \( 1 \leq i \leq n \) and \( 1 \leq j \leq n \).

The element \( g \in S \) is called a generator of the derivation \( D \).

The set of left (resp., right) semicentral idempotent elements of \( S \) is denoted by \( LO(S) \) (resp., \( RO(S) \)). For \( x \in LO(S) \) (resp., \( x \in RO(S) \)), the map \( d^x : S \rightarrow S \) (resp., \( d^x : S \rightarrow S \)) such that \( d^x(a) = xa \) (resp., \( d^x(a) = ax \)) for \( a \in S \) is a derivation in \( S \).

The derivation \( D : M_n(S) \rightarrow M_n(S) \) is called a left (resp., right) derivation if \( g \in LO(S) \) (resp., \( g \in RO(S) \)), where \( g \) is the generator of \( D \). The generalization of the previous theorem for noncommutative semiring \( S \) is the last theorem in Vladeva [142]:

**Theorem 4.9.** Let \( S \) be an arbitrary additively idempotent semiring, \( LO(S) \), the subsemiring of the left semicentral idempotents of \( S \), and \( RO(S) \) be the subsemiring of the right semicentral idempotents of \( S \). Let \( M_n(S) \) be the semiring of \( n \times n \) matrices over \( S \) and \( D : M_n(S) \rightarrow M_n(S) \) be an \( S \)-derivation.

1. If \( A = (a_{ij}) \in M_n(LO(S)) \), the elements of the matrix \( D(A) \) are \( d^A_{ij}(g) \), where \( d^A_{ij} \) are the derivations, generated by \( a_{ij} \) and \( g \in S \) is a generator of \( D \).
2. If \( A = (a_{ij}) \in M_n(RO(S)) \), the elements of the matrix \( D(A) \) are \( d^A_{ij}(g) \), where \( d^A_{ij} \) are the derivations, generated by \( a_{ij} \) and \( g \in S \) is a generator of \( D \).
3. If \( D \) is a left derivation in \( M_n(S) \), then for an arbitrary \( A = (a_{ij}) \in M_n(S) \), the elements of the matrix \( D(A) \) are \( d^A_{ij}(a_{ij}) \), where \( g \in LO(S) \) is a generator of \( D \), and therefore, \( D \) is a hereditary derivation.
4. If \( D \) is a right derivation in \( M_n(S) \), then for an arbitrary \( A = (a_{ij}) \in M_n(S) \), the elements of the matrix \( D(A) \) are \( d^A_{ij}(a_{ij}) \), where \( g \in RO(S) \) is a generator of \( D \), and hence, \( D \) is a hereditary derivation.

Similar studies on semirings of endomorphisms can be found in the study by Vladeva [143].

4.5 Derivations of endomorphism semirings. Derivations of some classes of additively idempotent semirings. Amitsur’s idea and semicentral idempotents

In a set of \( n \) elements, say \( \{0, \ldots, n-1\} \), the operation \( \lor \) is defined by \( a \lor b = \max\{a, b\} \), where \( a, b \in \{0, \ldots, n-1\} \). Thus, we define a chain with \( n \) elements. For a finite chain \( C_n = \{0, \ldots, n-1\}, \lor \), the endomorphisms form a semiring with regard to the addition and multiplication defined by:

- \( y = a + \beta \), if \( y(x) = a(x) \lor \beta(x) \) for all \( x \in C_n \),
- \( y = a\beta \), if \( y(x) = \beta(a(x)) \) for all \( x \in C_n \).

This semiring is called the endomorphism semiring of \( C_n \) as is denoted by \( E_{C_n} \).

In 2009, Ježek and Kepka [144] studied semirings of 1-preserving and of 0,1-preserving endomorphisms. In the same year, Ježek et al. [145] presented the similar research.

In 2016, Vladeva [146] noted that by virtue of Theorem 2.2 of Kim et al. [147], it follows that any finite additively idempotent semiring can be represented as the endomorphism semiring of a finite chain.
Rather than considering all endomorphisms $a \in E_{(0)}$, she defined $a$ such that $\text{Im}(a) \subseteq A$, where $A = \{a_0, a_1, ..., a_{k-1}\}$ is a fixed subset of $C_n$. The set of these endomorphisms is a simplex denoted by $\sigma^{(n)}(a_0, a_1, ..., a_{k-1})$.

The endomorphisms $a \in \sigma^{(n)}(a_0, a_1, ..., a_{k-1})$ such that $a(0) = \cdots = a(i_0 - 1) = a_0$, $a(i_0) = \cdots = a(i_0 + i_1 - 1) = a_1$, $a(i_0 + \cdots + i_2 - 1) = \cdots = a(i_0 + \cdots + i_{k-2} - 1) = a_{k-1}$ are denoted by $a = (a_0)_{i_0} \cdots (a_{k-1})_{i_{k-1}}$, where $\sigma^{k-1}_{i_0 \cdots i_{k-1}} = n$. The simplex $\sigma^{(n)}(a, b)$, $a < b$, is called a string and is denoted by $ST^{(n)}(a, b)$. The elements of this string are endomorphisms $a_b b_{n-k}$, where $k = 0, ..., n$. An important equality for studying the derivations (in [146] of a special type) is the representation $$(a_0)_{i_0} (a_1)_{i_1} \cdots (a_{k-1})_{i_{k-1}} = (a_0)_{n-i_k} (a_i)_{i_k} + (a_0)_{n-i_k} (a_{i_k})_{i_k} + \cdots + (a_0)_{n-i_k} (a_{k-1})_{i_{k-1}}.$$ Therefore, the elements of subsimplices $ST^{(n)}(a_0, a_m)$, where $1 \leq m \leq k - 1$, form an additive base of the simplex.

In 2018, Vladeva [148], using the aforementioned equality, proved that the projections of the considered simplex $\sigma^{(n)}(a_0, a_1, ..., a_{k-1})$ onto the subsimplices of an arbitrary type are derivations.

In 2020, Vladeva [149] considered the maps $\partial^{k-1}_{m-\ell} : \sigma^{(n)}(a_0, ..., a_{k-1}) \rightarrow \sigma^{(n)}(a_{\ell}, ..., a_m)$, where $0 \leq \ell \leq m - 1$ and $a_{\ell}, ..., a_m$ are consecutive elements of the set $\{a_0, ..., a_{k-1}\}$ such that for $a = (a_0)_{i_0} \cdots (a_{k-1})_{i_{k-1}}$, where $\sigma^{k-1}_{i_0 \cdots i_{k-1}} = n$, $$\partial^{k-1}_{m-\ell}(a) = (a_\ell)_{i_0} \cdots (a_{\ell+1})_{i_{\ell+1}} \cdots (a_{m-1})_{i_{m-1}} (a_m)_{i_{m-1}}^{k-1}_{E_{\ell}}.$$ The author constructed a semiring $D^{k-1}_{m-\ell}$ and proved that the mapping $\partial^{k-1}_{m-\ell} : D^{k-1}_{m-\ell} \rightarrow \sigma^{(n)}(a_{\ell}, ..., a_m)$ is a derivation. Moreover, the maximal subsemiring of $\sigma^{(n)}(a_0, a_1, ..., a_{k-1})$ with this property is $D^{k-1}_{m-\ell}$. Subsequently, we note two combinatorial results.

When $k = n$ and $\ell = 0$, the intersection $O N_0 = \bigcap_{m=0}^{n-1} D_m^{n-1}$ is a subsemiring of $E_{(0)}$ of order the $n$-th Catalan number. Moreover, the set of nilpotent endomorphisms $N_0$ is an ideal of $O N_0$ of order the $(n - 1)$-th Catalan number.

Further, in the article, new derivations $d_{\ell,m}$ are constructed and, for a fixed endomorphism $a_0$, the set of endomorphisms $a$ such that $d_{\ell,m}(a) = a_0$ is denoted by $\int_{a_0} d_{\ell,m}$. This set is a subsemiring of $\sigma^{(n)}(a_0, ..., a_{k-1})$ if and only if $a_0$ is an idempotent endomorphism. In the final result of this article, it is proved that the number of the semirings $\int_{a_0} d_{\ell,m}$, where $1 \leq m \leq n - 1$ is the 2$m$th Fibonacci number.

With regards to the various types of the endomorphism semirings of an infinite chain, the reader is referred to the monograph Vladeva [150].

In 2021, Rachev [151] studied the maps $a : [0, 1] \rightarrow [0, 1]$, which preserve the order and called them endomorphisms of the interval $[0, 1]$. For each pair of endomorphisms $a$ and $b$, the sum $a + b$ and the product $ab$ are defined as follows:

$$(a + b)(x) = \max\{a(x), b(x)\},$$

$$(ab)(x) = b(a(x))$$

for every real number $x \in [0, 1]$. Therefore, the set of endomorphisms is the semiring $E_{[0,1]}$. Some derivations of $E_{[0,1]}$ are then constructed.

In 2023, Vladeva [152] researched a class of matrix semirings over an arbitrary (not necessarily commutative) additively idempotent semiring. For an arbitrary additively idempotent semiring $S$, the set $L(S)$ (resp., $R(S)$) of left (resp., right) semicentral idempotents $\ell$ (resp., $r$) such that $\ell \leq 1$ (resp., $r \leq 1$) is a commutative subsemiring of $S$.

The subsemiring of $S$, generated by $L(S) \cup R(S)$, is denoted by $L R(S)$.
The following three definitions are basic for much of what follows:

1. A semiring $S$ is called $\mathcal{L}R$-semiring if $\mathcal{L}R(S) = S$.
2. Let $S$ and $S_0$ be additively idempotent semirings with a zero and an identity. Let $S$ be a noncommutative semiring and an $S_0$-semimodule, with $S_0$ a commutative semiring and $as = sa$ for any $a \in S_0$ and $s \in S$. Then $S$ is called an $S_0$-semialgebra.
3. Let $S$ be an $S_0$-semialgebra and $\mathcal{B}(S) = \{e_j\}$, $i, j = 1, ..., n$, a finite basis of $S$ with the following properties:
   
   (a) $e_i e_j = \begin{cases} e_i, & \text{if } j \leq k, \\ 0, & \text{if } j > k \end{cases}$ where $i, j, k, l = 1, ..., n$.
   
   (b) $\sum_{i=1}^{n} e_{ii} = 1_0$, where $1_0$ is an identity of $S$.
   
   (c) $e_i > 0$ for all $k > j$ and $e_j > 0$ for all $h < i$.

Then $\mathcal{B}(S)$ is called an $e$-basis.

The important results are as follows.

**Theorem 4.10.** Let $S$ be an $S_0$-semialgebra and $\mathcal{B}(S) = \{e_j\}$, $i, j = 1, ..., n$ an $e$-basis of $S$. Then $S$ is isomorphic to a matrix subsemiring of $M_n(S_0)$.

**Theorem 4.11.** Let $S$ be an $S_0$-semialgebra with an $e$-basis $\mathcal{B}(S) = \{e_j\}$, $i, j = 1, ..., n$, where $i \leq j$. Then $S$ is an $\mathcal{L}R$-semiring.

The semiring $S$ from the aforementioned theorems is called an $\mathcal{L}R$-matrix semiring over the semiring $S_0$.

The core discovery by Vladeva [152] refers to the derivations of an $\mathcal{L}R$-matrix semiring $S$. If $\ell_a$ is a left semicentral idempotent of $S$, then the map $\delta_a : S \to S$ such that $\delta_a(a) = \ell_a a$, where $a \in S$ is a derivation of $S$. Similarly, if $\ell_m$ is a right semicentral idempotent, then the map $d_m : S \to S$ such that $d_m(a) = a \ell_m$, where $a \in S$, is a derivation of $S$. Let $\mathcal{D}$ be the semiring generated by the set of all derivations $\delta_a$ and $d_m$. For the products $\delta_a d_m$, we have the following result:

**Theorem 4.12.** Let $S$ be an $\mathcal{L}R$-matrix semiring. Let $\delta_k, d_m \in \mathcal{D}$, where $1 \leq k \leq n$ and $1 \leq m \leq n$. The map $\delta_k d_m$ is a derivation if and only if $k + m \geq n$.

The first result following the Amitsur’s idea is the next theorem.

**Theorem 4.13.** Let $S$ be an $\mathcal{L}R$-matrix semiring which is an $S_0$-semialgebra with an $e$-basis $\mathcal{B}(e_i)$, where $i, j = 1, ..., n$ and $i \leq j$. Let $D$ be an arbitrary derivation of $S$. Then there is a derivation $\Delta_i \in \mathcal{D}$ such that $D(e_j) = \Delta_i(e_j)$, where $i, j = 1, ..., n$ and $i \leq j$.

By replacing the arbitrary commutative semiring $S_0$ with the Boolean semiring $\mathcal{B}$ the author obtained a stronger result:

**Theorem 4.14.** Let $S$ be an $\mathcal{L}R$-matrix semiring which is a $\mathcal{B}$-semialgebra with an $e$-basis $\mathcal{B}(e_i)$, $i, j = 1, ..., n$, $i \leq j$. All arbitrary derivations of the semiring $S$ are elements of the additively idempotent semiring $\mathcal{D}$. The set of nilpotent elements of $S$ is an ideal, which is closed under all of these derivations.

In conclusion, our objective is to scrutinize our paper within the broader context of the development of additively idempotent semirings theory.

The interest in the applications of additively idempotent semirings is of great importance, since we do not know what the foundations of the theory of these semirings are. Therefore, you can refer to the books of Glazek [5] and Golan [61] to gain information on their applications from the last century. You can find details on more recent papers in Subsection 4.1; however, it is important to acknowledge that over the last two decades, more than a thousand papers with similar themes have been published. In Subsection 4.2, we
have specifically highlighted only a selection of papers by Rowen and the mathematical community associated with him, focusing on those that contribute to the future trajectory of additively idempotent semirings theory. In the next three subsections, we present the derivations as an important tool for studying the various additively idempotent semirings.

Perhaps (as in the beginning of the development of differential algebra), the appearance of a book, as Kaplansky’s little book [153], will be a key moment for the future of the theory of additively idempotent semirings. Following the ideas of Kaplansky [153] and Kolchin’s monograph [154], many articles on differential algebra appeared [2].

The perspectives of the theory of additively idempotent semirings will be the same for two reasons:

- a myriad of examples of applications of additively idempotent semirings;
- the quality of the books: Golan [61–63], Vladeva [64,150], Perrin and Pin [66], Bistarelli [69], Butkovič [76], McEneaney [78], and Itenberg et al. [81].

That is why we expect bright horizons for the theory of additively idempotent semirings.

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