Enhanced Young-type inequalities utilizing Kantorovich approach for semidefinite matrices

Abstract: This article introduces new Young-type inequalities, leveraging the Kantorovich constant, by refining the original inequality. In addition, we present a range of norm-based inequalities applicable to positive semidefinite matrices, such as the Hilbert-Schmidt norm and the trace norm. The importance of these results lies in their dual significance: they hold inherent value on their own, and they also extend and build upon numerous established results within the existing literature.

Keywords: Kantorovich constant, unitarily invariant norms, Young-type inequalities, matrix inequalities

MSC 2020: 26D07, 26D15, 15A18, 15A60, 47A63

1 Introduction

Consider two positive numbers, denoted as $a$ and $b$. Following the widely recognized Young inequality, as formulated by

$$a^{1-\nu}b^{\nu} \leq va + (1 - \nu)b,$$

where $\nu$ ranges from 0 to 1 inclusively. To facilitate the discussion, we introduce the concepts of weighted arithmetic and geometric means as follows:

$$a\varpi, b = va + (1 - \nu)b \quad \text{and} \quad a\gamma, b = a^{\nu}b^{1-\nu},$$

and by employing a weighted arithmetic-geometric means inequality, we can express the Young inequality as follows:

$$a\gamma, b \leq a\varpi, b.$$

In scholarly discussions, this inequality has been garnering increasingly more attention. An advancement in the Young inequality was proposed by Kittaneh and Manasrah in their work [1], which can be expressed as follows:

$$\rho(\sqrt{a} - \sqrt{b})^2 + a\gamma, b \leq a\varpi, b,$$

where the parameter $\rho$ is defined as the minimum of $\nu$ and $1 - \nu$. The Young inequality underwent further enhancement through the subsequent extension by Hirzallah and Kittaneh, as detailed in their work [2], yielding the ensuing inequality.
\[ r_0^2(a - b)^2 + (a \nabla b)^2 \leq (a \nabla b)^2, \]

where \( r_0 \) is determined as the minimum of \( v \) and \( 1 - v \).

The constant function \( K(m, M) = \frac{(m + M)^2}{4mM} \), \( 0 < m < M \), is called the Kantorovich constant. This constant can be expressed as \( K(t) = \frac{(t + 1)^2}{4t} \), where \( t = \frac{M}{m} \). It is also denoted as \( K(t, 2) \). The Kantorovich constant has properties \( K(t, 2) = K \left( \frac{1}{2}, 2 \right) \geq 1 \) \((t > 0)\), and \( K(t, 2) \) is increasing on \([1, \infty]\) and is decreasing on \((0, 1)\). To learn more about the Kantorovich constant, interested readers are recommended to consult the following sources: [3–7].

The subsequent multiplicative refinement and reverse of the Young inequality, formulated with respect to Kantorovich constant, can be expressed as follows:

\[ K(h, 2) a \nabla b \leq a \nabla b \leq K(h, 2) r a \nabla b; \]

here, it is essential to note that \( a \) and \( b \) are both positive, \( v \) lies within the interval \([0, 1]\), and we define \( r = \min(v, 1 - v), R = \max(v, 1 - v), \) and \( h = \frac{b}{a} \). The second inequality within equation (1.6) is credited to Liao et al. [8], while the first inequality is attributed to Zou et al. [7].

In reference to [9], the authors have achieved an enhanced version of the Young inequality and its converse, which are described as follows:

First, the multiplicative refinement and reverse of the Young inequality, incorporating Kantorovich constant, are stated as follows:

\[ r(\sqrt{a} - \sqrt{b})^2 + K(\sqrt{h}, 2) r a \nabla b \leq a \nabla b, \]

and

\[ a \nabla b \leq K(\sqrt{h}, 2) r a \nabla b + R(\sqrt{a} - \sqrt{b})^2, \]

where the parameters \( h, r, R, \) and \( r' \) are defined as follows: \( h \) is the ratio \( \frac{b}{a} \), \( r \) is the minimum of \( v \) and \( 1 - v \), \( R \) is the maximum of \( v \) and \( 1 - v \), and \( r' \) is the minimum of \( 2r \) and \( 1 - 2r \). In addition, another variation of the reverse Young inequality, utilizing Kantorovich constant, is elucidated in [8], employing the same notation as mentioned above:

\[ a \nabla b - R(\sqrt{a} - \sqrt{b})^2 \leq K(\sqrt{h}, 2) r' a \nabla b, \]

where \( R' \) is calculated as the maximum of \( 2r \) and \( 1 - 2r \). Furthermore, Rashid and Bani-Ahmad [5] have recently refined the inequalities (1.7) and (1.8) with the following results:

(i) When \( v \) falls within the range \( \left[ 0, \frac{1}{2} \right] \), the inequalities are given as follows:

\[
\begin{align*}
K(\sqrt{\frac{h}{1 - v}}, 2)^v(a - b)^2 + v^2(a - b)^2 + v b(\sqrt{\frac{h}{1 - v}} - \sqrt{b})^2 + r_0 b(\sqrt{\frac{h}{1 - v}}ab - \sqrt{b})^2 \\
\leq v^2 a^2 + (1 - v)^2 b^2 \leq K \left( \frac{h}{1 - v}, 2 \right) \frac{r_1}{r} a^2 [(1 - v)b]^2 - v^2(a - b)^2 \\
+ (1 - v) a(\sqrt{\frac{h}{1 - v}} - \sqrt{1 - v}b)^2 - v b(\sqrt{\frac{h}{1 - v}}a - \sqrt{a})^2,
\end{align*}
\]

where \( h = \frac{a}{b}, r \) is the minimum of \( 2v \) and \( 1 - 2v \), \( r_0 \) is the minimum of \( 2r \) and \( 1 - 2r \), and \( r_1 \) is the minimum of \( 2r_0 \) and \( 1 - 2r_0 \).

(ii) If \( v \) lies in the range \( \left[ \frac{1}{2}, 1 \right] \), the inequalities are expressed as follows:

\[
\begin{align*}
K \left( \sqrt{\frac{h}{1 - v}}, 2 \right)^v a^2 [(1 - v)b]^2 - v^2(a - b)^2 \\
+ (1 - v) a(\sqrt{\frac{h}{1 - v}} - \sqrt{1 - v}b)^2 + v b(\sqrt{\frac{h}{1 - v}}ab - \sqrt{a})^2 \\
\leq v^2 a^2 + (1 - v)^2 b^2 \leq K(\sqrt{\frac{h}{1 - v}}, 2)^v a^2 [(1 - v)b]^2 - v^2(a - b)^2 \\
+ v b(\sqrt{\frac{h}{1 - v}} - \sqrt{b})^2 - r_0 b(\sqrt{\frac{h}{1 - v}}ab - \sqrt{b})^2.
\end{align*}
\]

where \( h = \frac{a}{b}, r = \min[2v - 1, 2 - 2v], r_0 = \min[2r, 1 - 2r], \) and \( r_1 \) is the minimum of \( 2r_0 \) and \( 1 - 2r_0 \).
The Young inequality for two real numbers, serving as a weighted extension of the generalized arithmetic-geometric mean inequality, constitutes a fundamental connection between two nonnegative real values, and we acknowledge its significance. For further elaboration on this type of inequality, the author suggests referring to [1,4,10–14] and the related references.

This study focuses on enhancing and reversing the inequalities presented in equations (1.10) and (1.11). In addition, we offer several matrix inequalities employing the Hilbert-Schmidt norm and trace norm as practical applications of our methodology.

This article comprehensively addresses all of these topics, and its structure is organized as follows: in Section 2, we derive several supplementary results, which serve as refinements and reversals of the inequalities (1.4)–(1.9), (1.10), and (1.11); in Section 3, we establish the matrix counterparts of the inequalities (2.1)–(2.7) for both the Hilbert-Schmidt norm and the trace norm. We accomplish this task by relying on Lemmas 3.1, 3.2, 3.3, and the Cauchy-Schwarz inequality.

2 Scalar-type inequalities using mean

Lemma 2.1. Let \( a, b > 0 \) and \( 0 \leq \nu < \kappa \leq 1 \). Then,

\[
r(\sqrt[\kappa]{a \nu b} - \sqrt{b})^2 + K(\frac{\nu}{\kappa}) a \nu b \leq (1 - \nu)b - \left(\frac{\nu}{\kappa}\right)(a \nu b - a \nu b),
\]

where \( r = \min\left\{\frac{\nu}{\kappa}, 1 - \frac{\nu}{\kappa}\right\}, h = \frac{a}{b}, \) and \( r' = \min\{2r, 1 - 2r\}. \)

Proof. A simple argument shows that

\[
va + (1 - \nu)b - \frac{\nu}{\kappa}(a \nu b - a \nu b) = va + (1 - \nu)b - \frac{\nu}{\kappa}(\kappa a + (1 - \kappa)b - a^\nu b) = \frac{\nu}{\kappa}a^\nu b + \left(1 - \frac{\nu}{\kappa}\right)b = (a \nu b) \nu b.
\]

By applying the inequality (1.7) for the relation (2.2), it follows that

\[
r(\sqrt[\kappa]{a \nu b} - \sqrt{b})^2 + K(\frac{\nu}{\kappa}) a \nu b \leq (a \nu b) \nu b.
\]

Hence, the inequality (2.1) follows. \( \square \)

Lemma 2.2. Let \( a, b > 0 \) and \( 0 \leq \nu < \kappa \leq 1 \). Then,

\[
va + (1 - \nu)b - \frac{\nu}{\kappa}(a \nu b - a \nu b) \leq K(\frac{\nu}{\kappa}) a \nu b + R(\sqrt[\kappa]{a \nu b} - \sqrt{b})^2
\]

where \( R = \max\\{\frac{\nu}{\kappa}, 1 - \frac{\nu}{\kappa}\}, r = \min\left\{\frac{\nu}{\kappa}, 1 - \frac{\nu}{\kappa}\right\}, h = \frac{a}{b}, \) and \( r' = \min\{2r, 1 - 2r\}. \)

Proof. By applying the inequality (1.8) for the relation (2.2), it follows that

\[
va + (1 - \nu)b - \frac{\nu}{\kappa}(a \nu b - a \nu b) \leq K(\frac{\nu}{\kappa}) a \nu b + R(\sqrt[\kappa]{a \nu b} - \sqrt{b})^2.
\]

So, we obtain the inequality (2.3). \( \square \)

Theorem 2.3. Let \( a, b > 0 \) and \( 0 \leq \nu < \kappa \leq 1 \).

(i) If \( 0 \leq \nu < \kappa \leq \frac{1}{2} \), then
\[
\begin{align*}
&v^2a^2 + (1 - v)^2b^2 \geq v^2(a - b)^2 + K(\sqrt{vh}, 2)^r(\nu a)^{2v}b^{2-2v} \\
&\quad + \left(\frac{v}{\kappa}\right)^2((\nu a)^{2v}b - (\nu a)^{2v}b) \\
&\quad + \frac{br}{\kappa}(\nu a)^{2v}b - (\nu a)^{2v}b],
\end{align*}
\] (2.4)

where \( h = \frac{a}{b}, r = \min\left[\frac{v}{\kappa}, 1 - \frac{v}{\kappa}\right], \) and \( r' = \min\{2r, 1 - 2r\} \).

(ii) If \( \frac{1}{2} \leq v < \kappa \leq 1 \), then

\[
\begin{align*}
&v^2a^2 + (1 - v)^2b^2 \geq (1 - v)^2(a - b)^2 + K\left(\frac{h}{\sqrt{1 - v}}\right)^r \left(\nu a\right)^{2v}b^{2-2v} \\
&\quad + ar((\nu a)^{2v}b - (\nu a)^{2v}b)^2 \\
&\quad + \frac{(2v - 1)a}{2\kappa - 1}(\nu a)^{2v}b - (\nu a)^{2v}b],
\end{align*}
\] (2.5)

where \( h = \frac{a}{b}, r = \min\left[\frac{2v - 1}{2\kappa - 1}, 1 - \frac{2v - 1}{2\kappa - 1}\right], \) and \( r' = \min\{2r, 1 - 2r\} \).

**Proof.** The proof of the second inequality follows by applying again Lemma 2.1, but to \((1 - v)b\) and parameter \(2v - 1\). We only need to investigate the first one. Applying Lemma 2.1 to \(va, b\) and parameter \(2v\), we have

\[
v^2a^2 + (1 - v)^2b^2 - v^2(a - b)^2 = [2v(va) + (1 - 2v)b] \\
\]

\[
\geq b \left[r((\nu a)^{2v}b - \sqrt{b})^2 + K(\sqrt{vh}, 2)^r(\nu a)^{2v}b^{1-2k} \\
\quad + \left(\frac{v}{\kappa}\right)((\nu a)^{2v}b - (\nu a)^{2v}b) \\
\quad + \frac{br}{\kappa}(\nu a)^{2v}b - (\nu a)^{2v}b] \\
\quad + \left(\frac{2v - 1}{2\kappa - 1}\right)(\nu a)^{2v}b - (\nu a)^{2v}b].
\]

So, the result. \(\square\)

**Theorem 2.4.** Let \(a, b > 0\) and \(0 \leq v < \kappa \leq 1\).

(i) If \(0 \leq v < \kappa \leq \frac{1}{2}\), then

\[
v^2a^2 + (1 - v)^2b^2 \geq (1 - v)^2(a - b)^2 + K\left(\frac{h}{\sqrt{1 - v}}\right)^r \left(\nu a\right)^{2v}b^{2-2v} \\
\quad + ar((\nu a)^{2v}b - (\nu a)^{2v}b)^2 \\
\quad + \frac{(2v - 1)a}{2\kappa - 1}(\nu a)^{2v}b - (\nu a)^{2v}b],
\] (2.6)

where \( h = \frac{a}{b}, r = \min\left[\frac{v}{\kappa}, 1 - \frac{v}{\kappa}\right] \) and \( r' = \min\{2r, 1 - 2r\} \).

(ii) If \(\frac{1}{2} \leq v < \kappa \leq 1\), then

\[
v^2a^2 + (1 - v)^2b^2 \geq v^2(a - b)^2 + K(\sqrt{vh}, 2)^r(\nu a)^{2v}b^{2-2v} \\
\quad + \left(\frac{v}{\kappa}\right)^2((\nu a)^{2v}b - (\nu a)^{2v}b) \\
\quad + \frac{br}{\kappa}(\nu a)^{2v}b - (\nu a)^{2v}b],
\] (2.7)

where \( h = \frac{a}{b}, r = \min\left[\frac{2v - 1}{2\kappa - 1}, 1 - \frac{2v - 1}{2\kappa - 1}\right], \) and \( r' = \min\{2r, 1 - 2r\} \).

**Proof.** The proof of the inequality (2.6) follows by applying Lemma 2.2 to \((1 - v)b\) and parameter \(2v - 1\). While the proof of the inequality (2.7) follows by applying Lemma 2.2 to \(va, b\) and parameter \(2v\). \(\square\)
Remark 2.5. Thanks to the Kantorovich constant, the inequalities in Theorems 2.3 and 2.4 are improved versions of findings in earlier works. These include the results [6, Theorem 2.5], [15, Theorem 2.1, Theorem 2.4], [5, Theorem 2.3], as well as the inequalities (1.10) and (1.11), and [4, Theorem 2.1].

3 Norm and trace-type inequalities using matrix means

In the subsequent text, we denote the space of all complex matrices of size $n \times n$ as $M_n(\mathbb{C})$. The notation $X \geq Y$ for $X, Y \in M_n(\mathbb{C})$ signifies that both $X$ and $Y$ are Hermitian matrices, and their difference $X - Y$ belongs to the set $M_n^+(\mathbb{C})$, which represents all positive semidefinite matrices in $M_n(\mathbb{C})$. Furthermore, $M_n^{++}(\mathbb{C})$ denotes the collection of strictly positive definite matrices in $M_n(\mathbb{C})$. The symbol $||\cdot||$ applied to matrices in $M_n(\mathbb{C})$ represents a unitarily invariant norm. In other words, for any matrix $A \in M_n(\mathbb{C})$ and all unitary matrices $U$ and $V$ within $M_n(\mathbb{C})$, it holds that $||UAV|| = ||A||$.

For a matrix $A = [a_{ij}] \in M_n(\mathbb{C})$, the Hilbert-Schmidt (or Frobenius) norm and the trace norm of $A$ are defined as follows:

$$\|A\|_2 = \sqrt{\sum_{i,j=1}^{n} a_{ij}^2} = \sqrt{\sum_{j=1}^{n} s_j^2(A)}, \quad \|A\|_1 = \sum_{j=1}^{n} s_j(A),$$

(3.1)

where $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$ represent the singular values of $A$, which are essentially the eigenvalues of the positive matrix $|A| = (A^*A)^{1/2}$. These singular values are arranged in decreasing order and repeated according to their multiplicity. It is a well-known fact that $\|\cdot\|_2$ is invariant under unitary transformations (as elaborated in [16] and [17]).

In the realm of matrices within the space $M_n(\mathbb{C})$, particularly focusing on matrices denoted as $A$ and $B$ that fall under the category of positive semidefinite matrices, Rashid and Bani-Ahmad, as presented in [5, Theorem 4.5], have innovatively introduced a matrix-based counterpart to the inequalities (1.10) and (1.11).

In this section, we will provide a concise overview of the Hilbert-Schmidt norm, unitarily invariant norm, trace norm, and trace as they pertain to the intriguing matrix formulations found in Theorem 2.3. To facilitate this discussion, we will make use of the following Lemmas. It is important to emphasize that the initial lemma, concerning unitarily invariant norms, takes the form of a Heinz-Kato inequality.

Lemma 3.1. [18] Let $A, B, X \in M_n(\mathbb{C})$ be such that $A, B \in M_n^+(\mathbb{C})$. If $\nu \in [0, 1]$, then

$$\|A^\nu XB^{1-\nu}\| \leq \|A\|\|X\|\|B\|^{1-\nu}.$$  

In particular,

$$\text{tr}|A^\nu B^{1-\nu}| \leq (\text{tr}A)^\nu(\text{tr}B)^{1-\nu}.$$  

Lemma 3.2. [16] Let $A, B \in M_n(\mathbb{C})$. Then,

$$\sum_{j=1}^{n} s_j(AB) \leq \sum_{j=1}^{n} s_j(A)s_j(B).$$

Lemma 3.3. [15] Let $A, B \in M_n(\mathbb{C})$ be such that $A$ and $B$ are positive definite. If $\nu \in [0, 1]$, then

$$\|A^\nu B^{1-\nu}\|^2 \leq \sum_{j=1}^{n} [s_j^\nu(A)s_j^{1-\nu}(B)]^2.$$
Cauchy-Schwarz’s inequality. Let \( a_i, b_i \geq 0 \) (\( i = 1, \ldots, n \)). Then,
\[
\sum_{i=1}^{n} a_i b_i \leq \left( \sum_{i=1}^{n} a_i^2 \right)^{1/2} \left( \sum_{i=1}^{n} b_i^2 \right)^{1/2}.
\]

We will now establish a matrix counterpart of Theorem 2.3 specifically designed for the Hilbert-Schmidt norm. The proof of this matrix version relies on the spectrum theorem.

**Theorem 3.4.** Let \( A, B, X \in M_n(\mathbb{C}) \) be such that \( A, B \leq M_\kappa I \), and \( \nu \in [0, 1] \). Then, we have the following:

(i) If \( 0 \leq \nu < \kappa \leq \frac{1}{2} \), then
\[
\|AX + (1-\nu)XB\|_2^2 \geq \nu^2 \|AX - XB\|_2^2 + \nu^{2\kappa}K(\sqrt{\nu h}, 2)^{r'}\|A^\nu XB^{1-\nu}\|_2^2
\]
\[+ 2\nu(1-\nu)\|A^\nu XB^{1-\nu}\|_2^2 + \left(1 - \nu^2\right)\|XB\|_2^2 - 2\nu^{2\kappa}\|A^\nu XB^{1-\nu}\|_2^2
\]
\[+ \frac{\nu}{\kappa} \left(2\kappa - 1\right)\nu \kappa (2 - 2\kappa)\|A^\nu XB^{1-\nu}\|_2^2
\]
where \( h = \frac{M}{m}, r = \min\left\{ \frac{\nu}{\kappa}, 1 - \frac{\nu}{\kappa} \right\}, \) and \( r' = \min\{2r, 1 - 2r\} \).

(ii) If \( \frac{1}{2} \leq \nu < \kappa \leq 1 \), then
\[
\|AX + (1-\nu)XB\|_2^2 \geq (1-\nu)^2 \|AX - XB\|_2^2 + (1-\nu)^{2-2\kappa}K(\sqrt{\frac{1-\nu}{\nu}}, 2)^{r'}\|A^\nu XB^{1-\nu}\|_2^2
\]
\[+ 2\nu(1-\nu)\|A^\nu XB^{1-\nu}\|_2^2 + \left(1 - \nu^2\right)\|XB\|_2^2 - 2\nu^{2\kappa}\|A^\nu XB^{1-\nu}\|_2^2
\]
\[+ \frac{2\nu - 1}{2\kappa - 1}\left(2\kappa - 1\right)\nu \kappa (2 - 2\kappa)\|A^\nu XB^{1-\nu}\|_2^2
\]
where \( h = \frac{M}{m}, r = \min\left\{ \frac{2\nu - 1}{2\kappa - 1}, 1 - \frac{2\nu - 1}{2\kappa - 1} \right\}, \) and \( r' = \min\{2r, 1 - 2r\} \).

**Proof.** Since every positive semidefinite is unitarily diagonalizable, it follows by spectral theorem that there are unitary matrices \( U, V \in M_n(\mathbb{C}) \) such that \( A = UD_1U^* \) and \( B = VD_2V^* \), where \( D_1 = \text{diag}(\lambda_1, \ldots, \lambda_n) \), \( D_2 = \text{diag}(\mu_1, \ldots, \mu_n) \), and \( \lambda_i, \mu_i \geq 0 \) (\( 1 \leq i \leq n \)). Let
\[
Y = U^*XV = [y_{ij}].
\]

The following system of equations describes the relationship between the matrices \( A, X \), and \( B \) in terms of the elements \( y_{ij} \) of the matrix \( Y \):
\[
vAX + (1-\nu)XB = U[(v\lambda_i + (1-\nu)\mu_i)y_{ij}]V^*,
\]
\[
AX - XB = U[(\lambda_i - \mu_i)y_{ij}]V^*,
\]
\[
A^\nu XB^{1-\nu} = U[(\lambda_i^\nu \mu_i^{1-\nu})y_{ij}]V^*
\]
\[
AX = U[\lambda_i y_{ij}]V^* \quad \text{and} \quad XB = U[\mu_i y_{ij}]V^*.
\]

If \( 0 \leq \nu < \kappa \leq \frac{1}{2} \), then by inequality (2.4) and the unitary invariance of the Hilbert-Schmidt norm, we have
\[ \|vAX + (1 - v)XB\|_F^2 = \sum_{i,j=1}^n (\nu_l + (1 - v)\mu_j)^2|y_{ij}|^2 \]
\[ \geq v^2 \sum_{i,j=1}^n (\lambda_i - \mu_j)^2|y_{ij}|^2 \]
\[ + \nu^2 k \sum_{i,j=1}^n \min \{K\left(\sqrt{v}t_{ij}, 2\right), \nu^{2k}\} \left(\lambda_i \mu_j^{1-k}\right) |y_{ij}|^2 + 2v(1 - v) \sum_{i,j=1}^n \left(\lambda_i \mu_j^{1-k}\right) |y_{ij}|^2 \]
\[ + r \left(2 v \sum_{i,j=1}^n (\lambda_i^2 \mu_j^{1-k})|y_{ij}|^2 + \sum_{i,j=1}^n \mu_j^2 |y_{ij}|^2 - 2v^k \sum_{i,j=1}^n (\lambda_i^2 \mu_j^{1-k})^2 |y_{ij}|^2 \right) \]
\[ + \left(\frac{v}{k}\right) \left[2v^k \sum_{i,j=1}^n (\frac{3}{4} \lambda_i \mu_j^{1-k}) |y_{ij}|^2 + (1 - 2k) \sum_{i,j=1}^n \mu_j^2 |y_{ij}|^2 - v^{2k} \sum_{i,j=1}^n (\lambda_i^2 \mu_j^{1-k})^2 |y_{ij}|^2, \right. \]

where \( t_{ij} = \frac{\lambda_i}{\mu_j}. \) Utilizing the condition \( 0 < ml \leq A, B \leq M, \frac{m}{M} = \frac{1}{h} \leq t_{ij} = \frac{\lambda_i}{\mu_j} \leq h = \frac{M}{m} \) and the property of the Kantorovich constant, we have
\[ \|vAX + (1 - v)XB\|_F^2 \geq v^2 \|AX - XB\|_F^2 + v^{2k} K\left(\sqrt{v}h, 2\right) \|A^kXB^{1-k}\|_F^2 \]
\[ + 2v(1 - v) \left\| A^2XB \right\|_F^2 + \left\| A^kXB^{1-k}\right\|_F^2 + \left\| XB\right\|_F^2 - 2v^k \left\| A^kXB^{1-k}\right\|_F^2 \]
\[ + \left(\frac{v}{k}\right) \left[2v^k \left\| A^2XB \right\|_F^2 + (1 - 2k) \left\| XB\right\|_F^2 - v^{2k} \left\| A^kXB^{1-k}\right\|_F^2 \right]. \]

where \( h = \frac{M}{m}, r = \min\left\{\frac{v}{h}, 1 - \frac{v}{h}\right\}, \) and \( r' = \min\{2r, 1 - 2r\}. \)

The following results for unitarily invariant norm and trace are established by Lemma 3.1.

**Theorem 3.5.** Let \( A, B, X \in M_n(C) \) be such that \( A, B \in M_n(C), I \) is the identity matrix, and \( 0 < ml \leq A, B \leq M, \) and \( v \in [0, 1] \). Then for every unitarily invariant norm \( \|\| \| \)\, we have

(i) If \( 0 \leq \nu < \kappa \leq \frac{1}{2} \), then
\[ \|\nu\|A\| \| + (1 - \nu)\|X\|B\| \| \| \| \| \geq \|\nu\|A\| \| + (1 - \nu)\|XB\| \| \| + 2v(1 - v) \left\| A^2XB \right\|_F^2 \]
\[ + v^{2k} \left(\|A\| \| - \|X\|B\| \| \right)^2 + \left(\nu^{2k} \left\| A^kXB^{1-k}\right\|_F^2 + \left\| XB\right\|_F^2 - v^{2k} \left\| A^kXB^{1-k}\right\|_F^2 \right) \]
(3.2)

where \( h = \frac{\|A\|}{\|X\|B\|}, r = \min\left\{\frac{\nu}{h}, 1 - \frac{\nu}{h}\right\}, \) and \( r' = \min\{2r, 1 - 2r\}. \)

(ii) If \( \frac{1}{2} \leq \nu < \kappa \leq 1 \), then
\[ \|\nu\|A\| \| + (1 - \nu)\|X\|B\| \| \| \| \| \| \|
\[ \geq \|\nu\|A\| \| + (1 - \nu)^{2-2k} \left(\frac{\nu}{h - 2v}\right) \left(\|A\| \| - \|X\|B\| \| \right)^2 + 2v(1 - v) \left\| A^2XB \right\|_F^2 \]
\[ + (1 - v)^{2(\|A\| \| - \|X\|B\| \|)} + r^{(\nu^{1-k} \left\| A^kXB^{1-k}\right\|_F^2 + \left\| XB\right\|_F^2 - v^{2k} \left\| A^kXB^{1-k}\right\|_F^2) \]
\[ + \left(\frac{2v - 1}{2k - 1}\right) \left(\|A\| \| - \|X\|B\| \| \right) + (2 - 2k)(1 - v)\|X\|B\| - (1 - v)^{2-2k} \left(\|A\| \| - \|X\|B\| \| \right)^2 \]
(3.3)

where \( h = \frac{\|A\|}{\|X\|B\|}, r = \min\left\{\frac{2v - 1}{2k - 1}, 1 - \frac{2v - 1}{2k - 1}\right\}, \) and \( r' = \min\{2r, 1 - 2r\}. \)
Proof. (i) Suppose that 0 ≤ ν < κ ≤ 1 2. By Lemma 3.1 and inequality (2.4), we have

\[
[v|AX|| + (1 - v)||XB||] ^2 \\
\geq v^2 K(\sqrt{v}h, 2^r)[(v|AX|)^r||XB||^{1 - r})^2 + 2v(1 - v)|AX| ||XB|| \\
+ v^2[(v|AX|| - ||XB||)^2 + r||XB||][\sqrt{v}[|AX||]|\bar{\nu}_k||XB||| - \sqrt{||XB||}^2] \\
+ \left(\frac{v}{K}\right)||XB||[(v|AX|)||\bar{\nu}_k||XB||| - (v||AX||)|\bar{\nu}_k||XB|||] \\
\geq v^2 K(\sqrt{v}h, 2^r)[|AXB|^2] + 2v(1 - v)\left(\frac{1}{2} + \frac{1}{2}\right)^2 \\
+ v^2[(v|AX|) - ||XB||] + r||AX||^2 - ||XB||^2 \\
+ \left(\frac{v}{K}\right)||XB||[2v||AX|| + (1 - 2\kappa)||XB|| - v^2(||AXB||^2)].
\]

In this manner, we have successfully concluded the proof of equation (3.2). The demonstration of Statement (ii) follows a similar approach to that of Statement (i), making use of Lemma 3.1 and inequality (2.5). Therefore, we will not provide the detailed proof for Statement (ii).

\[\square\]

Theorem 3.6. Let A, B ∈ M_n^*(C), I is the identity matrix, and 0 < mI ≤ A, B ≤ MI, and ν ∈ [0, 1]. It holds that

(i) If 0 ≤ ν < κ ≤ 1 2, then

\[
[\text{tr}(A\bar{V}_{1-\nu}B)]^2 \geq v^2 K(\sqrt{v}h, 2^r)[(\text{tr}(AB^{-1}))]^2 + 2v(1 - v)\left(\frac{1}{2} + \frac{1}{2}\right)^2 \\
+ v^2(\text{tr}(A) - \text{tr}(B))^2 + r||v^2\text{tr}(A)B^{-1}||^2 - \text{tr}(B)^2 \\
+ \left(\frac{v}{K}\right)||2v\text{tr}(A)B - (1 - 2\kappa)(\text{tr}(B))^2 - v^2(\text{tr}(A)B^{-1})||^2,
\]

where h = \frac{\text{tr}(A)}{\text{tr}(B)}, r = \min\{\frac{v}{\kappa}, 1 - \frac{v}{\kappa}\}, and r' = \min\{2r, 1 - 2r\}.

(ii) If 1 2 ≤ ν < κ ≤ 1, then

\[
[\text{tr}(A\bar{V}_{1-\nu}B)]^2 \geq (1 - v)^2 K\left(\sqrt{1 - v}h, 2 \right)^2 \left[|\text{tr}(AB^{-1})|^2 \right] \\
+ 2v(1 - v)\left(\frac{1}{2} + \frac{1}{2}\right)^2 + (1 - v)^2(\text{tr}A - \text{tr}B)^2 \\
+ r\text{tr}(A)\left[\sqrt{1 - v}\text{tr}(B)\right] - (1 - v)^2(\text{tr}B)^2 \\
+ \left(\frac{2v - 1}{2 - 1}\right)|\text{tr}(A)|\text{tr}(AB^{-1}) - (1 - v)^2(\text{tr}B)|^2,
\]

where h = \frac{\text{tr}(A)}{\text{tr}(B)}, r = \min\{\frac{2v - 1}{2 - 1}, 1 - \frac{2v - 1}{2 - 1}\}, and r' = \min\{2r, 1 - 2r\}.

Proof. (i) Suppose that 0 ≤ ν < κ ≤ 1 2. By Lemma 3.1 and inequality (2.4), we have

\[
[\text{tr}(A\bar{V}_{1-\nu}B)]^2 = [\text{tr}(A) + (1 - v)\text{tr}(B)]^2 \\
\geq v^2 K(\sqrt{v}h, 2^r)[(\text{tr}(A))^r(\text{tr}(B))^{1 - r})^2 + 2v(1 - v)(\text{tr}(A)\text{tr}(B)) \\
+ v^2(\text{tr}(A) - \text{tr}(B))^2 + r\text{tr}(B)[\sqrt{v}\text{tr}(A)]\bar{\nu}_k|B - \sqrt{\text{tr}(B)}|^2 \\
+ \left(\frac{v}{K}\right)(\text{tr}(A))\bar{\nu}_k|\text{tr}(B) - (\text{tr}(A))\bar{\nu}_k|\text{tr}(B)| \\
\geq v^2 K(\sqrt{v}h, 2^r)[(\text{tr}(AB^{-1})]^2 + 2v(1 - v)(\text{tr}(A)B^{-1})^2 \\
+ v^2(\text{tr}(A) - \text{tr}(B))^2 + r\text{tr}(AB^{-1})^2 - \text{tr}(B)^2 \\
+ \left(\frac{v}{K}\right)||2v\text{tr}(A)B - (1 - 2\kappa)(\text{tr}(B))^2 - v^2(\text{tr}(A)B^{-1})||^2.
\]
In this manner, we have successfully concluded the proof of (3.4). The demonstration of Statement (ii) follows a similar approach to that of Statement (i), making use of Lemma 3.1 and the inequality (2.5). Therefore, we will not provide the detailed proof for Statement (ii).

Furthermore, we derive the subsequent outcome, which offers improvements to the trace norm variation of the Young-type inequality.

**Theorem 3.7.** Let $A, B \in M_n^*(\mathbb{C})$, $I$ is the identity matrix, and $0 < mI \preceq A, B \preceq MI$, and $\nu \in [0, 1]$. It holds that

(i) If $0 \leq \nu < \kappa \leq \frac{1}{2}$, then

$$v^2 \|A\|_2^2 + (1 - v)^2\|B\|_2^2 = \text{tr}(v^2 A^2 + (1 - v)^2 B^2) \geq v^{2\nu} K\left(\sqrt{\nu h}, 2^\nu \|A^{1-x}\|_2^2 + v^{2\nu} (\|A\|_2 - \|B\|_2)^2ight)$$

$$+ r\left(v^{2\nu} \|A^{1-x}\|_2^2 + \|B\|_2^2 - 2v^{\nu} \|A^{2-\nu}\|_2 \|B^{2-\nu}\|_2 \right)$$

where $h = \frac{M}{m}$, $r = \min\{\nu, 1 - \nu\}$, and $r' = \min\{2r, 1 - 2r\}$.

(ii) If $\frac{1}{2} \leq \nu < \kappa \leq 1$, then

$$v^2 \|A\|_2^2 + (1 - v)^2\|B\|_2^2 = \text{tr}(v^2 A^2 + (1 - v)^2 B^2) \geq v^{2\nu} K\left(\sqrt{\nu h}, 2^\nu \|A^{1-x}\|_2^2 + (1 - v)^2(\|A\|_2 - \|B\|_2)^2ight)$$

$$+ r\left(v^{2\nu} \|A^{1-x}\|_2^2 + (1 - v)^2\|AB\|_1 - 2(1 - \nu)^2\|AB\|_1 - 2v^{\nu} \|A^{2-\nu}\|_2 \|B^{2-\nu}\|_2 \right)$$

where $h = \frac{M}{m}$, $r = \min\{\nu, 1 - \nu\}$, and $r' = \min\{2r, 1 - 2r\}$.

**Proof.** (i) Assume that $0 \leq \nu < \kappa \leq \frac{1}{2}$. Employing the inequality (2.4), we can establish the following:

$$v^2 \|A\|_2^2 + (1 - v)^2\|B\|_2^2 = \sum_{j=1}^{n} v^2 s_j^2(A) + (1 - v)^2 s_j^2(B) \geq \sum_{j=1}^{n} \text{min} K\left(\nu t_j, 2^\nu \|s_j(A)\|_2^2 \|s_j(B)\|_2^2 \right) + v^2 \sum_{j=1}^{n} (s_j(A) - s_j(B))^2$$

$$+ r \sum_{j=1}^{n} s_j(B)\left(\sqrt{\nu t_j} s_j(B) - s_j(B)\right)^2$$

where $t_j = \frac{s_j(A)}{s_j(B)}$. Utilizing the condition $0 < mI \preceq A, B \preceq MI$, $\frac{m}{M} = \frac{1}{n} \leq t_j = \frac{s_j(A)}{s_j(B)} \leq h = \frac{M}{m}$ and the property of the Kantorovich constant, we have
\[ v^2 \|A\|_2^2 + (1 - v)^2 \|B\|_2^2 \geq v^{2k}K(\sqrt{vh}, 2)^n \sum_{j=1}^n [s_j(A)]^2 \cdot s_{j^{-v}}(B)^2 \]

\[ + v\left[ \sum_{j=1}^n s_j^2(A) + \sum_{j=1}^n s_j^2(B) - 2 \frac{n}{\sum_{j=1}^n s_j^2(B)} \left( \sum_{j=1}^n s_j^2(A) \right)^2 \right] \]

\[ + r \sum_{j=1}^n [v^{2k}s_j^{2k}(A)s_{j^{-2k}}(B) + s_j(B) - 2v^k s_j^2(A) s_{j^{-2k}}(B)] \]

\[ + \frac{v}{K} \sum_{j=1}^n s_j(B)[2vks_j(A) + (1 - 2k)s_j^{-v}(B) - v^{2k}s_j^{2k}(A) s_{j^{-2k}}(B)] \]

Consequently, by using the Cauchy-Schwarz’s inequality, we obtain

\[ v^2 \|A\|_2^2 + (1 - v)^2 \|B\|_2^2 \geq v^{2k}K(\sqrt{vh}, 2)^n \sum_{j=1}^n [s_j(A)]^2 \cdot s_{j^{-v}}(B)^2 \]

\[ + v\left[ \sum_{j=1}^n s_j^2(A) + \sum_{j=1}^n s_j^2(B) - 2 \frac{n}{\sum_{j=1}^n s_j^2(B)} \left( \sum_{j=1}^n s_j^2(A) \right)^2 \right] \]

\[ + r \sum_{j=1}^n [v^{2k}s_j^{2k}(A)s_{j^{-2k}}(B) + s_j(B) - 2v^k s_j^2(A) s_{j^{-2k}}(B)] \]

\[ + \frac{v}{K} \sum_{j=1}^n s_j(B)[2vks_j(A) + (1 - 2k)s_j^{-v}(B) - v^{2k}s_j^{2k}(A) s_{j^{-2k}}(B)] \]

\[ \times v^{2k}K(\sqrt{vh}, 2)^n \sum_{j=1}^n [s_j(A)]^2 \cdot s_{j^{-v}}(B)^2 \]

\[ + v\left[ \sum_{j=1}^n s_j^2(A) + \sum_{j=1}^n s_j^2(B) - 2 \frac{n}{\sum_{j=1}^n s_j^2(B)} \left( \sum_{j=1}^n s_j^2(A) \right)^2 \right] \]

\[ + r \sum_{j=1}^n [v^{2k}s_j^{2k}(A)s_{j^{-2k}}(B) + s_j(B) - 2v^k s_j^2(A) s_{j^{-2k}}(B)] \]

\[ + \frac{v}{K} \sum_{j=1}^n s_j(B)[2vks_j(A) + (1 - 2k)s_j^{-v}(B) - v^{2k}s_j^{2k}(A) s_{j^{-2k}}(B)] \]

Hence, it follows by Lemma 3.2 that
\[
\begin{align*}
\geq v^{2}K(\sqrt{\nu h},2^{\nu})\|A^\nu B^{-1}v\|_2^2 + v^2(\|A\|_2 - \|B\|_2)^2 \\
+ \nu(\|A^\nu B^{-1}v\|_2^2 - \|B\|_2^2 - 2\nu v \|A^\nu B^{-1}v\|_2^2) \\
+ \left(\frac{v}{\kappa}\right)2\nu v \|AB\|_1 + (1 - 2\kappa)\|B\|_2^2 - v^{2\kappa} \|A^\nu B^{-1}v\|_2^2 - v^2 \|A^\nu B^{-1}v\|_2^2 - 2\nu \|A^\nu B^{-1}v\|_2^2).
\end{align*}
\]

Finally, by using Lemma 3.3, we obtain
\[
\text{tr}(v^2A^2 + (1 - v)^2B^2) = v^2\text{tr}(A^2) + (1 - v)^2\text{tr}(B^2) \\
= v^2 \sum_{j=1}^{n} s_j^2(A) + (1 - v)^2 \sum_{j=1}^{n} s_j^2(B) \\
= v^2 \|A\|_2^2 + (1 - v)^2 \|B\|_2^2.
\]

In this manner, we have successfully concluded the proof of equation (3.6). The demonstration of Statement (ii) follows a similar approach to that of Statement (i), making use of the Cauchy-Schwarz inequality, Lemmas 3.2 and 3.3, and inequality (2.5). Therefore, we will not provide the detailed proof for Statement (ii). \(\square\)

**Remark 3.8.** Clearly, owing to the characteristics of the Kantorovich constant, it is evident that inequalities (3.6) and (3.7) represent enhancements of the established outcomes in [4, Theorem 3.2] and [15, Theorem 4.4].

### 4 Conclusion and future work

This study has introduced novel Young-type inequalities, capitalizing on the Kantorovich constant, by enhancing the original inequality. Furthermore, a collection of norm-based inequalities has been presented, specifically designed for positive semidefinite matrices, encompassing norms like the Hilbert-Schmidt norm and the trace norm. The significance of these findings is twofold: they possess intrinsic value in their own right and serve as extensions and advancements of numerous established results in the existing literature.

As for future work, there is a promising avenue for further exploration in this area. Future research could focus on applying these inequalities in various mathematical and scientific contexts, potentially leading to new insights and applications. In addition, the refinement and extension of these inequalities to more general matrix settings or other mathematical structures could also be an intriguing direction for future investigations.

**Acknowledgement:** The authors wish to extend sincere gratitude to the referees and the editor for providing valuable comments and suggestions that will enhance the quality of this article.

**Funding information:** None declared.

**Author contributions:** The authors declare that they have contributed equally to this article. Both authors have read and approved the final version of the manuscript.

**Conflict of interest:** The authors state that there is no conflicts of interest.

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