Research Article

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Positive periodic solutions for discrete time-delay hematopoiesis model with impulses

Abstract: The present article is devoted to the positive periodic solution for impulsive discrete hematopoiesis model. This model is described by a first-order nonlinear difference equation with multiple delays and impulses. The existence result is obtained by applying the Krasnosel’skiǐ fixed point theorem. A sufficient condition that guarantees that there is at least one positive periodic solution is established. Moreover, illustrative example and numerical simulation analysis are provided to show the existence of positive periodic solutions and the effect of impulses on positive periodic solutions.

Keywords: discrete hematopoiesis model, impulses, positive periodic solutions, Krasnosel’skiǐ fixed point theorem

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1 Introduction

Consider the following discrete nonlinear hematopoiesis model subject to periodic impulses:

\[
\begin{align*}
\Delta x(k) &= -a(k)x(k) + \sum_{i=1}^{m} b_i(k)f(x(k - \tau_i(k))), \quad k \in \mathbb{Z}^+ \text{ and } k \neq k_j, \\
\Delta x(k) &= \gamma x(k), \quad j \in \mathbb{N}
\end{align*}
\]  

with \( m \in \mathbb{N} \). In this model, \( \Delta \) is a forward-difference operator defined by \( \Delta x(k) = x(k + 1) - x(k) \); \( f(u) = u/(1 + u^\alpha) \) \((u \geq 0, n > 1)\) is the production function, which has a unimodal shape and reaches the peak value at point \( u_a = 1/(n - 1) \). Here,

\( A_1 \) \( a : \mathbb{Z} \to (0, 1), b_i : \mathbb{Z} \to (0, \infty), \text{ and } \tau_i : \mathbb{Z} \to \mathbb{Z}^+ \) \((1 \leq i \leq m)\) are \( \omega \)-periodic discrete functions satisfying \( a(k) = a(k + \omega), b_i(k) = b_i(k + \omega), \text{ and } \tau_i(k) = \tau_i(k + \omega) \) for \( k \in \mathbb{Z} \) and the common period \( \omega \in \mathbb{N} \);

\( A_2 \) \( \{\gamma_j\} \) with \( \gamma_j > -1 \) is a \( \omega \)-periodic sequence of real numbers, and \( \{k_j\} \) is a \( \omega \)-periodic sequence of natural numbers, i.e., there exists a natural number \( q \in \mathbb{N} \) such that \( k_{j+q} = k_j + \omega \) and \( \gamma_{j+q} = \gamma_j \). Also,

0 \leq k_1 < k_2 < \cdots < k_j < k_{j+1} \cdots \text{ for } j \in \mathbb{N}.

\( A_3 \) \( \prod_{\delta \leq \delta < a} (1 + \gamma_j)(1 - \alpha(j)) < 1 \).

Let \( \mathbb{Z}[-\tau, 0] = \{-\tau, -\tau + 1, \ldots, 0\} \). We consider the impulsive system (1.1) with the initial value condition

\[ x(k) = \phi(k) > 0 \quad \text{for } k \in \mathbb{Z}[-\tau, 0], \]  

in which \( \tau = \max_{1 \leq i \leq m} \max_{\delta \leq \delta < a} \tau_i(k) \) \( \in \mathbb{Z}^+ \). Denote the solution of equation (1.1) satisfying the initial value condition (1.2) by \( x(\cdot; \phi) \), since \( 0 < a(k) < 1 \) for \( k \in \mathbb{Z} \) and \( \gamma_j > -1 \) for \( j \in \mathbb{N} \), we see that \( x(\cdot; \phi) \) is positive.

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Remark 1.1. The periodic model (1.1) can be reformulated as an impulsive difference equation of form
\[
\begin{align*}
\{ x(k + 1) &= G(k, x(k), x(k - \tau_1(k)), \ldots, x(k - \tau_m(k))), \quad k \in \mathbb{Z}^+ \quad \text{and} \quad k \neq k_j, \\
x(k_j + 1) &= (1 + y_j)x(k_j), \quad j \in \mathbb{N},
\end{align*}
\]
where \( G(k, u_0, \ldots, u_m) = (1 - a(k))u_0 + \sum_{i=1}^{m} b(k)f(u_i) \) is \( \omega \)-periodic with respect to the first variable \( k \) and is continuous with respect to variable \( u_i (1 \leq i \leq m) \).

The nonlinear continuous model with single delay
\[
x'(t) = -a x(t) + b f(x(t - \tau))
\] (1.3)
was first introduced by Mackey and Glass [1] to understand the dynamics of production of blood cells. Here, \( x \) is the density of mature blood cells in the blood circulation; \( a \) is the lost rate of blood cells in the circulation; the term \( b f(x(t - \tau)) \) represents the influx of blood cells into the bloodstream from multipotential hematopoietic stem cells; \( b \) is the maximal production rate; \( \tau \) is the time delay required for immature cells produced in bone marrow to be released into the bloodstream as mature cells. The production function \( f \) in this model reflects a mixed feedback control in the hematopoietic system. Specifically, as the blood cell level gradually increases from zero, it increases monotonically at the beginning, saturates at an appropriate blood cell level, and then decreases monotonically. The positive equilibrium of equation (1.3) has been studied in several works (see, e.g., [2–4]).

Clinical experiments have shown that periodic seasonal changes greatly affect the population density and internal composition of organisms [5,6]. However, the autonomous model with constant coefficients and constant time delays cannot reveal such external periodic effects to the blood cell dynamics. Taking the internal periodicity of organisms into account, modifications of equation (1.3) have been developed by assuming that the formation, maturation, and death of blood cells are periodic. The existence and stability of positive periodic solutions of such periodic hematopoiesis models are main research themes. We refer to [7–12].

It has been recognized that when considering a population with nonoverlapping generations or with small population number, the discrete models are more suitable than continuous ones [13–15]. In addition, discrete models can also provide more efficient methods for numerical computations and simulations. Therefore, it is very meaningful to study the discrete hematopoiesis models. For example, we refer to [16–19] for the discrete hematopoietic dynamics.

In the normal hematopoiesis process of organisms, the number of various blood cells is in a dynamic stable state. However, impulsive phenomena that the states are subject to external perturbations at certain moments of time are observed in many evolutionary processes [20]. In the hematopoietic system, such instantaneous disturbance results in sudden changes in blood cell number [21,22]. There are some research works that investigated the dynamics of blood cells in the disturbed environment [23–25]. These studies are well conducted from continuous perspective. By contrast, discrete dynamical models considering the impulse effects are much less explored.

The purpose of this article is to elucidate a sufficient condition that guarantees the existence of positive \( \omega \)-periodic solutions of discrete impulsive system (1.1). The rest of this article is organized as follows. In Section 2, the solution transformation and some preparations about completely continuous operator in a cone are considered. Section 3 presents the existence of positive \( \omega \)-periodic solutions. In Section 4, a numerical example and some simulations are given. Finally, a short conclusion is provided in Section 5.

## 2 Solution transformation and auxiliary preparations

We discuss the nonlinear difference equation with delays
\[
\Delta y(k) = -\tilde{a}(k)y(k) + \prod_{0 \leq k < k_j} \frac{1}{1 + y_j} \sum_{i=1}^{m} \tilde{b}(k)f \left( y(k - \tau_i(k)) \prod_{0 \leq k < k - \tau_i(k)} (1 + y_j) \right),
\] (2.1)
where
\[ a(k), \quad k \neq k_j, \quad 0, \quad k = k_j, \quad j = 1, 2, \ldots, \] and \[ b(k), \quad k \neq k_j, \quad 0, \quad k = k_j, \quad j = 1, 2, \ldots. \]

The initial value condition is \( y(k) = \phi(k) > 0 \) for \( k \in \mathbb{Z}[-\tau, 0] \). We give the following lemma, which is essential to obtain our main result.

**Lemma 2.1.** Suppose that \((A_1)\) and \((A_2)\) hold. Then,

(i) if \( y(k) \) is a solution of equation (2.1), then \( x(k) = y(k) \prod_{0 \leq k < k_j} (1 + y_j) \) is a solution of equation (1.1);

(ii) if \( x(k) \) is a solution of equation (1.1), then \( y(k) = x(k) \prod_{0 \leq k < k_j} \frac{1}{1 + y_j} \) is a solution of equation (2.1).

**Proof of part (i).** From equation (2.1), we see that
\[
x(k_j + 1) = y(k_j + 1) \prod_{0 \leq k < k_j + 1} (1 + y_j) = (1 + y_j) y(k_j) \prod_{0 \leq k < k_j} (1 + y_j) = (1 + y_j) x(k_j).
\]

For any \( k \neq k_j (j = 1, 2, \ldots) \), it follows that
\[
x(k + 1) - (1 - a(k)) x(k) - \sum_{i=1}^{m} b_i(k) f(x(k - \tau_i(k))) = y(k + 1) \prod_{0 \leq k < k_j + 1} (1 + y_j) - (1 - \tilde{a}(k)) y(k) \prod_{0 \leq k < k_j} (1 + y_j) - \sum_{i=1}^{m} \tilde{b}_i(k) y(k - \tau_i(k)) \prod_{0 \leq k < k_j} (1 + y_j).
\]

Therefore, \( x(k) = y(k) \prod_{0 \leq k < k_j} (1 + y_j) \) is a solution of equation (1.1).

**Proof of part (ii).** It is easy to see that
\[
y(k_j + 1) = y(k) \prod_{0 \leq k < k_j} \frac{1}{1 + y_j} = y(k_j) \prod_{0 \leq k < k_j + 1} \frac{1}{1 + y_j} = x(k_j) \prod_{0 \leq k < k_j + 1} \frac{1}{1 + y_j} = x(k_j).
\]

for \( j = 1, 2, \ldots \), namely, equation (2.1) is satisfied by \( k = k_j \). For any \( k \neq k_j (j = 1, 2, \ldots) \), from equation (1.1), we have
\[
\Delta y(k) = y(k + 1) - y(k) = x(k + 1) \prod_{0 \leq k < k_j + 1} \frac{1}{1 + y_j} - x(k) \prod_{0 \leq k < k_j} \frac{1}{1 + y_j} = \left[ x(k + 1) - x(k) \right] \prod_{0 \leq k < k_j} \frac{1}{1 + y_j} =\]
\[
= \left[ -a(k) x(k) + \sum_{i=1}^{m} b_i(k) f(x(k - \tau_i(k))) \right] \prod_{0 \leq k < k_j} \frac{1}{1 + y_j} = -\tilde{a}(k) y(k) + \sum_{i=1}^{m} \tilde{b}_i(k) f(y(k - \tau_i(k))) \prod_{0 \leq k < k_j} (1 + y_j).
\]

Thus, we can conclude that \( y(k) = x(k) \prod_{0 \leq k < k_j} \frac{1}{1 + y_j} \) is a solution of equation (2.1).

Define
F(k, s) = \frac{\prod_{s+1r-k<s+1}(1 - \bar{a}(r))}{1 - \prod_{s+k<s+1}(1 + y_j)\prod_{s-r<s+1}(1 - \bar{a}(r))} \quad \text{for } k \leq s \leq k + \omega - 1, \ k \in \mathbb{Z}. \quad (2.2)

The assumptions \((A_1)\) and \((A_3)\) ensure the positivity of \(F\), and basic computation shows that
\[
F(k + \omega, s + \omega) = \frac{\prod_{s+k+\omega<s+1}(1 - \bar{a}(r))}{1 - \prod_{s+k<s+1}(1 + y_j)\prod_{s-r<s+1}(1 - \bar{a}(r))}
= \frac{\prod_{s+k<s+1}(1 + y_j)\prod_{s-r<s+1}(1 - \bar{a}(r))}{1 - \prod_{s+k<s+1}(1 + y_j)\prod_{s-r<s+1}(1 - \bar{a}(r))} = F(k, s).
\quad (2.3)

Hence, \(F\) is \(\omega\)-periodic with respect to both variables \(k\) and \(s\) for \(k \leq s \leq k + \omega - 1\) and \(k \in \mathbb{Z}\).

Let \(X\) be the space of all \(\omega\)-periodic discrete functions defined by \(X = \{x : x(k) = x(k + \omega) \in \mathbb{R} \text{ for } k \in \mathbb{Z}\}\) with the norm \(\|x\| = \max_{k \in \mathbb{Z}}|x(k)|\). Then, it follows that \(X\) is a finite-dimensional Banach space. For an element \(x\) of \(X\) satisfying \(x(k) > 0\) for \(k \in \mathbb{Z}\), we make \(\hat{x}(k) = x(k)\) for \(k \in \mathbb{Z}[-\tau, \infty) = [-\tau, \infty) \cap \mathbb{Z}\) and \(\hat{\phi}(k) = x(k)\) for \(k \in \mathbb{Z}[-\tau, 0]\).

**Lemma 2.2.** A positive \(\omega\)-periodic discrete function \(\hat{x}\) is the solution \(\hat{x}(\cdot; \hat{\phi})\) of equation (1.1) if and only if the original function \(x \in X\) satisfies that \(x(k) > 0\) and
\[
x(k) = \sum_{s=k}^{k+\omega-1} \prod_{s+k<s+1}(1 + y_j)\sum_{i=1}^{m} \hat{b}_i(s)f(x(s - \tau_i(s))).
\quad (2.4)

where \(F\) is defined by equation (2.2).

**Proof.** If \(\hat{x}\) is the solution \(x(\cdot; \hat{\phi})\) of equation (1.1), then it is clear that \(x(k) > 0\) for \(k \in \mathbb{Z}\). Equation (2.1) can be rewritten into the following form:
\[
y(k + 1) - (1 - \bar{a}(k))y(k) = \prod_{s+k<s+1} \frac{1}{1 + y_j} \sum_{i=1}^{m} \hat{b}_i(k)f(x(k - \tau_i(k))) \prod_{0s+s<r<s+1}(1 + y_j).
\]

By Lemma 2.1, multiplying both sides of the above equality by \(\prod_{s+r<s+1}(1 - \bar{a}(r))\) leads to
\[
x(k + 1) \prod_{0s+k<s+1} \frac{1}{1 + y_j} \prod_{0s+r<s+1}(1 - \bar{a}(r)) - x(k) \prod_{0s+k<s+1} \frac{1}{1 + y_j} \prod_{0s+r<s+1}(1 - \bar{a}(r))
= \prod_{0s+k<s+1} \frac{1}{1 + y_j} \sum_{i=1}^{m} \hat{b}_i(k)f(x(k - \tau_i(k))) \prod_{0s+r<s+1}(1 - \bar{a}(r)).
\]

Note that \(x(k + \omega) = x(k)\). Sum up both sides from \(k\) to \(k + \omega - 1\), which produces
\[
x(k) \left( \prod_{0s<k<s+1} \frac{1}{1 + y_j} \prod_{0s+r<s+1}(1 - \bar{a}(r)) \right) - \prod_{0s<k<s+1} \frac{1}{1 + y_j} \prod_{0s+r<s+1}(1 - \bar{a}(r))
= \sum_{s=k}^{k+\omega-1} \prod_{0s+k<s+1} \frac{1}{1 + y_j} \sum_{i=1}^{m} \hat{b}_i(s)f(x(s - \tau_i(s))) \prod_{0s+r<s+1}(1 - \bar{a}(r)).
\]

Then, it turns out that
and hence, the periodicity of \( \tilde{a} \) and \( \gamma_j \) results in

\[
x(k) = \frac{1}{1 - \prod_{0 \leq s < k - \omega} \left( 1 + \gamma_j \prod_{s + 1 \leq r < k + \omega} (1 - \tilde{a}(r)) \right)} \sum_{s = k}^{k + \omega - 1} \left( 1 + \gamma_j \prod_{s \leq k < k + \omega} (1 + \gamma_j) \sum_{i = 1}^{m} \tilde{b}(s)f(x(s - \tau(s))) \prod_{s + 1 \leq r < k + \omega} (1 - \tilde{a}(r)) \right)
\]


for \( k \in \mathbb{Z} \). Thus, \( x \) has the expression (2.4).

Conversely, from equations (2.3) and (2.4), one has

\[
x(k + \omega) = \sum_{s = k}^{k + \omega - 1} \left[ F(k + \omega, s) \prod_{s \leq k < k + 2\omega} (1 + \gamma_j) \sum_{i = 1}^{m} \tilde{b}(s)f(x(s - \tau(s))) \right]
\]

\[
= \sum_{s = k}^{k + \omega - 1} \left[ F(k + \omega, s + \omega) \prod_{s + 1 \leq k < k + \omega} (1 + \gamma_j) \sum_{i = 1}^{m} \tilde{b}(s + \omega)f(x(s + \omega - \tau(s + \omega))) \right]
\]

\[
= \sum_{s = k}^{k + \omega - 1} \left[ F(k, s) \prod_{s \leq k < k + \omega} (1 + \gamma_j) \sum_{i = 1}^{m} \tilde{b}(s)f(x(s - \tau(s))) \right] = x(k).
\]

Hence, an element \( x \in X \) verifies equation (2.4). We now make \( \hat{x} \) and \( \hat{\phi} \) from such an \( x \). To prove that \( \hat{x} \) is a positive \( \omega \)-periodic solution of equation (1.1), we only need to show that \( x \) satisfies equation (1.1). To this end, we should first prove that \( x(k + 1) - (1 - a(k))x(k) = \sum_{i = 1}^{m} b_i(x(k - \tau(k))) \) for \( k \neq k_j \). The first term \( x(k + 1) \) of the left side equals

\[
\sum_{s = k + 1}^{k + \omega - 1} \left[ F(k + 1, s) \prod_{s \leq k < k + 1} (1 + \gamma_j) \sum_{i = 1}^{m} \tilde{b}(s)f(x(s - \tau(s))) \right]
\]

\[
= \sum_{s = k + 1}^{k + \omega - 1} \left[ F(k + 1, s + \omega) \prod_{s + 1 \leq k < k + \omega} (1 + \gamma_j) \sum_{i = 1}^{m} \tilde{b}(s + \omega)f(x(s + \omega - \tau(s))) \right]
\]

\[
= \sum_{s = k + 1}^{k + \omega - 1} \left[ F(k + 1, k + \omega) \prod_{s \leq k < k + \omega} (1 + \gamma_j) \sum_{i = 1}^{m} \tilde{b}(k + \omega)f(x(k + \omega - \tau(k + \omega))) \right]
\]

\[
+ \sum_{s = k + 1}^{k + \omega - 1} \left[ F(k + 1, s) \prod_{s \leq k < k + \omega} (1 + \gamma_j) \sum_{i = 1}^{m} \tilde{b}(s)f(x(s - \tau(s))) \right]
\]

Moreover, the second term \( (1 - a(k))x(k) \) of the left side equals
\[
\sum_{s=k}^{k+\omega-1} \left( (1-a(k))F(k, s) \prod_{s \leq k < k + \omega} (1 + \gamma) \sum_{i=1}^{m} b(s)f(x(s - \tau(s))) \right)
\]

\[
= \sum_{s=k+1}^{k+\omega-1} \left( (1-a(k))F(k, s) \prod_{s \leq k < k + \omega} (1 + \gamma) \sum_{i=1}^{m} b(s)f(x(s - \tau(s))) \right)
\]

\[
+ (1-a(k))F(k, k) \prod_{k \leq k < k + \omega} (1 + \gamma) \sum_{i=1}^{m} b(k)f(x(k - \tau(k)))
\]

\[
= \sum_{s=k}^{k+\omega-1} \left( (1-a(k+\omega)) \prod_{s \leq k < k + \omega} (1 + \gamma) \prod_{i=1}^{m} b(s)f(x(s - \tau(s))) \right)
\]

\[
+ (1-a(k)) \prod_{s \leq k < k + \omega} (1 + \gamma) \prod_{i=1}^{m} b(s)f(x(s - \tau(s)))
\]

\[
= \sum_{s=k}^{k+\omega-1} \left( \prod_{s \leq k < k + \omega} (1 + \gamma) \prod_{i=1}^{m} b(s)f(x(s - \tau(s))) \right)
\]

Hence, one obtains

\[
x(k+1) - (1-a(k))x(k) = \frac{1}{1 - \prod_{s=k}^{k+\omega-1} (1 + \gamma) \prod_{i=1}^{m} b(k)f(x(k - \tau(k)))}
\]

Second, we prove that \(x(k+1) = (1 + \gamma)x(k)\). Since \(k + \omega\) is impulse point, we see that \(\tilde{b}(k + \omega) = 0\) for \(1 \leq i \leq m\). Hence, representation (2.4) derives that \(x(k+1)\) equals

\[
\sum_{s=k+1}^{k+\omega-1} \left( F(k+1, s) \prod_{s \leq k < k + \omega+1} (1 + \gamma) \sum_{i=1}^{m} \tilde{b}(s)f(x(s - \tau(s))) \right)
\]

\[
= \sum_{s=k+1}^{k+\omega-1} \left( F(k+1, s) \prod_{s \leq k < k + \omega+1} (1 + \gamma) \sum_{i=1}^{m} \tilde{b}(s)f(x(s - \tau(s))) \right)
\]

\[
+ F(k+1, k + \omega) \prod_{k + \omega}^{m} \tilde{b}(k + \omega)f(x(k + \omega - \tau(k + \omega)))
\]

\[
= \sum_{s=k+1}^{k+\omega-1} \left( \prod_{s \leq k < k + \omega+1} (1 + \gamma) \prod_{i=1}^{m} \tilde{b}(s)f(x(s - \tau(s))) \right)
\]

Meanwhile, we have
\[ x(k_j) = \sum_{s=k_j}^{k_j+\omega-1} \left( F(k_j, s) \prod_{s \leq k < k_j + \omega} (1 + y_i) \sum_{i=1}^{m} \tilde{b}(s)f(x(s - \tau(s))) \right) \]

\[ = \sum_{s=k_j+1}^{k_j+\omega-1} \left( F(k_j, s) \prod_{s \leq k < k_j + \omega} (1 + y_i) \sum_{i=1}^{m} \tilde{b}(s)f(x(s - \tau(s))) \right) + F(k_j, k_j)(1 + y_j) \prod_{k_j \leq r < k_j + \omega} (1 - \tilde{a}(r)) \sum_{i=1}^{m} \tilde{b}(s)f(x(k_j - \tau(k_j))) \]

\[ = \sum_{s=k_j+1}^{k_j+\omega-1} \left( \prod_{s \leq k < k_j + \omega} (1 + y_i) \prod_{k_j \leq r < k_j + \omega} (1 - \tilde{a}(r)) \sum_{i=1}^{m} \tilde{b}(s)f(x(s - \tau(s))) \right). \]

Since \( y_j \) is \( \omega \)-periodic, we see that \( x(k_j + 1) = (1 + y_j)x(k_j) \). Hence, \( x \) satisfies system (1.1). The proof is complete. \( \Box \)

Denote a constant \( 0 \leq \lambda = \prod_{k_j \leq r < k_j + \omega} (1 - \tilde{a}(r)) < 1 \). Let \( P = \{ x \in X : x(k) \geq \lambda||x|| \text{ for } k \in \mathbb{Z} \} \). Then, \( P \) is a cone in \( X \). Define an operator \( \Phi \) by

\[ \Phi x(k) = \sum_{s=k}^{k+\omega-1} \left( F(k, s) \prod_{s \leq k < k + \omega} (1 + y_i) \sum_{i=1}^{m} \tilde{b}(s)f(x(s - \tau(s))) \right) \quad (2.5) \]

for \( x \in P \). It is obvious that \( \Phi \) is an operator from cone \( P \) in the finite dimensional space \( X \) into itself. Due to the expression (2.4) of \( x \in X \) and its periodicity, we obtain \( \Phi \) is \( \omega \)-periodic, i.e., \( \Phi x(k + \omega) = \Phi x(k) \) for \( k \in \mathbb{Z} \). Therefore, Lemma 2.2 reveals that \( \hat{x} \), which is made from a fixed point \( x \) of \( \Phi \), is the positive \( \omega \)-periodic solution \( x(\cdot; \hat{\phi}) \) of (1.1).

From the above discussion, it is obvious that \( \Phi x \in X \). In addition, we can estimate that

\[ F(k, s) \geq \prod_{k+1 \leq r < k+\omega} (1 - \tilde{a}(r)) \]

\[ \geq \prod_{k \leq r < k+\omega} (1 - \tilde{a}(r)) \]

\[ = \prod_{k \leq r < k+\omega} (1 - \tilde{a}(r)) \]

\[ = \prod_{k \leq r < k+\omega} (1 - \tilde{a}(r)) \]

\[ = \mu, \]

and

\[ F(k, s) \leq \prod_{k+1 \leq r < k+\omega} (1 - \tilde{a}(r)) \]

\[ = \prod_{k \leq r < k+\omega} (1 - \tilde{a}(r)) \]

\[ = \prod_{k \leq r < k+\omega} (1 - \tilde{a}(r)) \]

\[ = \nu. \]

Then, we have

\[ \Phi x(k) \geq \mu \sum_{s=k}^{k+\omega-1} \left( \prod_{s \leq k < k + \omega} (1 + y_i) \sum_{i=1}^{m} \tilde{b}(s)f(x(s - \tau(s))) \right) \quad (2.6) \]

and
The leads to

\[
\Phi x(k) \leq v \sum_{s=k}^{k+\omega-1} \left( 1 + \gamma_j \sum_{i=1}^{m} \bar{b}_i(s)f(x(s - \tau_i(s))) \right) \leq \frac{v}{\mu} \Phi x(k),
\]

and therefore,

\[
\Phi x(k) \geq \prod_{r \neq k, 0 < r < \omega} (1 - a(r)) \|\Phi\| = \lambda \|\Phi\|.
\]

Thus, we see that \( \Phi x \in P \). We can verify that \( \Phi \) is a completely continuous operator by straightforward calculations, namely, \( \Phi \) is continuous and maps any bounded subset of \( P \) into a relatively compact subset of \( P \) (see, e.g., [26], Lemma 4). The Krasnosel'skii fixed point theorem [27] utilized for the above cone \( P \) is as follows:

**Lemma 2.3.** Let \((X, \|\|)\) be a Banach space, and let \( P \subset X \) be a cone in \( X \). Suppose that \( \Omega_1 \) and \( \Omega_2 \) are open bounded subsets of \( X \) with \( \emptyset \subset \subset \Omega_1 \subset \Omega_2 \). Let \( \Phi : P \rightarrow P \) be a completely continuous operator on \( P \) such that, either

(i) \( \|\Phi x\| \leq \|x\| \) for \( x \in P \cap \partial \Omega_1 \) and \( \|\Phi x\| \geq \|x\| \) for \( x \in P \cap \partial \Omega_2 \), or

(ii) \( \|\Phi x\| \geq \|x\| \) for \( x \in P \cap \partial \Omega_1 \) and \( \|\Phi x\| \leq \|x\| \) for \( x \in P \cap \partial \Omega_2 \).

Then, \( \Phi \) has a fixed point in \( P \cap \left( \Omega_2 \setminus \Omega_1 \right) \).

### 3 Existence result

In this section, we state the result concerning the existence of positive \( \omega \)-periodic solution of system (1.1). The main result is assured by the following theorem.

**Theorem 3.1.** Let \( b_i = \min_{k \neq k_0, 0 \leq k < \omega} b_i(k) \) for each \( i = 1, 2, \ldots, m \) and \( \gamma_j^* = \min\{\gamma_j, 0\} \) for \( j = 1, 2, \ldots \). Suppose that

\[
\prod_{r \neq k, 0 \leq r < \omega} (1 - a(r)) \sum_{i=1}^{m} b_i \prod_{0 \leq k < \omega} (1 + \gamma_j^*) \lambda u \geq \prod_{0 \leq k < \omega} (1 + \gamma_j^*) \prod_{r \neq k, 0 \leq r < \omega} (1 - a(r)) u.
\]

Then, system (1.1) has at least one positive \( \omega \)-periodic solution.

**Proof.** As it follows from the facts exposed in Section 2, the positive \( \omega \)-periodic solution of system (1.1) is a fixed point of the completely continuous operator \( \Phi \) defined by equation (2.5). Owing to this solution transformation, we have only to ensure the existence of the fixed point of \( \Phi \). Let \( b_l = \max_{k \neq k_0, 0 \leq k < \omega} b_l(k) \) for each \( l = 1, 2, \ldots, m \) and \( \gamma_j^* = \max\{\gamma_j, 0\} \) for \( j = 1, 2, \ldots \). Recall that \( f(u) = u/(1 + u^n) \) and \( n > 1 \). Since \( \lim_{u \to 0} f(u) = 0 \), we can choose a constant \( \varepsilon_0 \) satisfying \( 0 < \varepsilon_0 < 1/(\nu_0 \sum_{l=1}^{m} b_l(1 + \gamma_l^*)) \) and a number \( M \) with \( 0 < u < M \) such that

\[
f(u) < \varepsilon_0 \lambda \quad \text{for} \quad u \geq M,
\]

in which

\[
u = \frac{1}{1 - \sum_{0 \leq k < \omega} (1 + \gamma_j^*) \prod_{r \neq k, 0 \leq r < \omega} (1 - a(r))}.
\]

Recall that \( \lambda = \prod_{r \neq k, 0 \leq r < \omega} (1 - a(r)) \). Let \( u^* = M/\lambda > u \). Then, we define two subsets as follows:
\[ \Omega_1 = \{ x \in X : ||x|| < u_+ \} \quad \text{and} \quad \Omega_2 = \{ x \in X : ||x|| < u^* \}. \]

It is obvious that \( \Omega_i \) (1 \( \leq \) i \( \leq \) 2) are open bounded subsets with the inclusion relation \( \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2 \subset \bar{\Omega}_2 \).

Case (i): If an element \( x \) belongs to \( P \cap \partial \Omega_1 \subset X \), then \( x \) is \( \omega \)-periodic discrete function satisfying \( u_+ \geq x(k) \geq \lambda \ ||x|| = \lambda u_+ \) for \( k \in \mathbb{Z} \). By the monotonically increasing property of function \( f \) over the interval \([0, u_+]\), we see that

\[
\min_{\lambda u_+ \leq ||x||} f(u) = f(\lambda u_+). \tag{3.3}
\]

From equations (2.6) and (3.3), we obtain

\[
(\Phi x)(k) \geq \mu \sum_{s=k}^{k+\omega-1} \prod_{j \leq k < k + \omega} (1 + y_j') \sum_{i=1}^{m} \tilde{b}_i(s) f(x(s - \tau(s)))
\]

\[
\geq \mu \sum_{s=k}^{k+\omega-1} \prod_{j \leq k < k + \omega} (1 + y_j') \sum_{i=1}^{m} \tilde{b}_i(s) f(x(s - \tau(s)))
\]

\[
= \sum_{i=1}^{m} \tilde{b}_i(0) \prod_{s \leq k < \omega} (1 + y_j') \left \{ \prod_{r = k \omega, 0 \leq r < \omega} (1 - a(r)) \right \} f(\lambda u_+)
\]

for \( k \in \mathbb{Z} \). Hence, the condition (3.1) leads to the relationship \((\Phi x)(k) > u_+\). Therefore, \(||x|| > u_+ = ||x||\) for \( x \in P \cap \partial \Omega_1 \).

Case (ii): If an element \( x \) belongs to \( P \cap \partial \Omega_2 \subset X \), then \( x \) is \( \omega \)-periodic discrete function satisfying \( \lambda u^* \leq x(k) \leq u^* \) for \( k \in \mathbb{Z} \). From equation (2.7), we have

\[
(\Phi x)(k) \leq v \sum_{s=k}^{k+\omega-1} \prod_{j \leq k < k + \omega} (1 + y_j') \sum_{i=1}^{m} \tilde{b}_i(s) f(x(s - \tau(s)))
\]

\[
\leq v \sum_{i=1}^{m} \tilde{b}_i(0) \prod_{s \leq k < \omega} (1 + y_j') \left \{ \prod_{r = k \omega, 0 \leq r < \omega} (1 - a(r)) \right \} f(\lambda u^*)
\]

for \( k \in \mathbb{Z} \). Using equation (3.2), we obtain \( \sum_{s=k}^{k+\omega-1} f(x(s - \tau(s))) < \varepsilon \delta \omega M \) for \( i = 1, 2, …, m \). Hence, we see that

\[
(\Phi x)(k) < \varepsilon \delta \omega \sum_{i=1}^{m} \tilde{b}_i(0) \prod_{s \leq k < \omega} (1 + y_j') M < M \quad \text{for} \quad k \in \mathbb{Z}.
\]

Note that \( u_+ < M = \lambda u^* < u^* \). It turns out that \((\Phi x)(k) < u^*\), which implies that \(||\Phi x|| < u^* = ||x||\) for \( x \in P \cap \partial \Omega_2 \).

Thus, Lemma 2.3 ensures that operator \( \Phi \) has a fixed point \( x \) in \( P \cap (\bar{\Omega}_2 \setminus \Omega_2) \). The fixed point \( x \) is a positive \( \omega \)-periodic discrete function satisfying \( x(k) > \lambda ||x|| \) for \( k \in \mathbb{Z} \) and \( 0 < u_+ \leq ||x|| \leq u^* \). Let \( \hat{x}(k) = x(k) \) for \( k \in \mathbb{Z}[\tau, \infty) \) and \( \hat{\phi}(k) = x(k) \) for \( k \in \mathbb{Z}[-\tau, 0] \). Now, we are in a position to conclude that \( \hat{x} \) is a positive \( \omega \)-periodic solution of equation (1.1) with the initial discrete function \( \phi \).
4 Example and numerical simulations

To illustrate Theorem 3.1, we give an example in this section. The existence of a positive $\omega$-periodic solution is obtained, and the region where a positive $\omega$-periodic solution is located can be clarified.

Consider the impulsive delay difference system

\[
\begin{align*}
\Delta x(k) &= -a(k)x(k) + \frac{b_1(k)x(k-7) + b_2(k)x(k-2)}{1 + x^2(k-2)}, \quad k \in \mathbb{Z}^+ \quad \text{and} \quad k \neq k_j, \\
\Delta x(k_j) &= y_j x(k_j), \quad j = 1, 2, \ldots
\end{align*}
\]

(4.1)

with $k_j = 2j - 1$. Let

\[
a(k) = \begin{cases} 1/5 & \text{if } k = 0, \\ 1/2 & \text{if } k = 2, \end{cases}
\]

(4.2)

\[
b_1(k) = \begin{cases} 4 & \text{if } k = 0, \\ 2 & \text{if } k = 2, \end{cases}
\quad b_2(k) = \begin{cases} 6 & \text{if } k = 0, \\ 7 & \text{if } k = 2. \end{cases}
\]

(4.3)

Moreover, the common period $\omega$ is 4. It is obvious that $m = n = 2$. Fix the impulse values $y_1 = 5/6 > 0$ and $y_2 = -1/11 < 0$. We can verify in detail that there exists at least one positive 4-periodic solution of system (4.1) with the coefficients defined by equations (4.2) and (4.3).

Note that $f(u) = u/(1 + u^n) = u/(1 + u^2)$ for $u \geq 0$. The unimodal function $f$ reaches its maximum at $u^* = \sqrt{1/(2 - 1)} = 1$. By a basic calculation, it follows that

\[
\prod_{r \neq k_j, \ 0 \leq r < 4} (1 - a(r)) = \left(1 - \frac{1}{5}\right)\left(1 - \frac{1}{2}\right) = \frac{2}{5},
\]

\[
\prod_{0 \leq j < 4} (1 + y_j^n) = 1 \times \frac{10}{11} = \frac{10}{11},
\]

and

\[
\prod_{0 \leq j < 4} (1 + y_j) = \left(1 + \frac{5}{6}\right) \times \left(1 - \frac{1}{11}\right) = \frac{5}{3}.
\]

Hence, we obtain

\[
\prod_{0 \leq j < 4} (1 + y_j) \prod_{r \neq k_j, \ 0 \leq r < 4} (1 - a(r)) = \frac{5}{3} \times \frac{2}{5} = \frac{2}{3} < 1,
\]

Figure 1: A graph of four arbitrary positive solutions of system (4.1). The numerical simulations show that there is a positive 4-periodic solution of system (4.1) located in [6.8, 12.5].
namely, the assumption \(A_3\) holds. Moreover, the minimum values of \(b_1\) and \(b_2\) are \(b_1 = 2\) and \(b_2 = 6\). Then, we can check that
\[
\frac{\prod_{r^* k_j < a < b} (1 - a(r))^2 \prod_{0 < a < b} (1 + y) \sum_{i=1}^{2} b_i}{(1 + \prod_{r^* k_j < a < b} (1 - a(r))^2)} = \frac{(2 + 6) \times \frac{4}{5} \times \frac{10}{11}}{1 + \frac{4}{5}} = \frac{320}{319} > 1,
\]
and
\[
\left\{ 1 - \prod_{0 < a < b} (1 + y) \prod_{r^* k_j < a < b} (1 - a(r)) \right\} u_{\ast} = 1 - \frac{2}{3} = \frac{1}{3}.
\]
Therefore, condition (3.1) is satisfied. Theorem 3.1 shows that system (4.1) has at least one positive 4-periodic solution (see Figure 1).

If we consider system (4.1) with constant impulsive force \(\gamma_1 = \gamma_2 = \gamma\), then there are two important cases to be discussed: (a) \(-1 < \gamma < 0\); (b) \(\gamma > 0\).

**Case (a).** For the negative impulse value \(-1 < \gamma < 0\), we see that condition (3.1) becomes to
\[
(1 + y)^2 \prod_{r^* k_j < a < b} (1 - a(r))^2 \sum_{i=1}^{2} b_i > \left\{ 1 - (1 + y)^2 \prod_{r^* k_j < a < b} (1 - a(r)) \right\}.
\]
(4.4)

Note that assumption \(A_3\) holds for any \(-1 < \gamma < 0\). From equation (4.4), we have
\[
(1 + y)^2 > \frac{1 + \prod_{r^* k_j < a < b} (1 - a(r))^2}{\prod_{r^* k_j < a < b} (1 - a(r))^2 + \sum_{i=1}^{2} b_i - \prod_{r^* k_j < a < b} (1 - a(r))^2 + \prod_{r^* k_j < a < b} (1 - a(r))}
\]
\[
= \frac{1 + 0.4^2}{0.4^2 + 8 \times 0.4^2 + 0.4} = \frac{145}{218},
\]
which implies that \(y > -0.184 \cdots = y^\ast\). Hence, system (4.1) has a positive 4-periodic solution for \(y^\ast < y < 0\) (see Figure 2(a)). As shown in Figure 2(b), there also exists a positive 4-periodic solution of system (4.1) when condition (3.1) is not satisfied by \(y = -0.3 < y^\ast\). This suggests that equation (3.1) is a sufficient but not necessary condition for the existence of positive 4-periodic solution.

**Case (b).** For the positive impulse value \(\gamma > 0\), we see that condition (3.1) becomes
\[
\frac{\prod_{r^* k_j < a < b} (1 - a(r))^2 \sum_{i=1}^{2} b_i}{1 + \prod_{r^* k_j < a < b} (1 - a(r))^2} > \left\{ 1 + (1 + y)^2 \prod_{r^* k_j < a < b} (1 - a(r)) \right\}.
\]
(4.5)

\[\text{Figure 2:}\] The numerical simulations of four arbitrary positive solutions of system (4.1). (a) There is a positive 4-periodic solution of system (4.1) located in \([3.65, 5.1]\) with \(y = 0.1 > y^\ast\) and (b) there is a positive 4-periodic solution of system (4.1) located in \([2.45, 4.45]\) with \(y = 0.3 < y^\ast\).
In this case, assumption \((A_3)\) holds if 

\[
\gamma < \frac{1}{\prod_{r \neq k, \delta r < \epsilon}(1 - a(r))} - 1 = \frac{1}{\sqrt{0.4}} - 1 = 0.581 \ldots.
\]

Consequently, we obtain from equation (4.5) that 

\[
(1 + \gamma)^2 > \frac{1 + \left(1 - \sum_{i=1}^{2} b_i\right)\prod_{r \neq k, \delta r < \epsilon}(1 - a(r))^2 + \prod_{r \neq k, \delta r < \epsilon}(1 - a(r))}{0.4^3 + 0.4} = -0.258 \ldots,
\]

which is always true. Hence, system (4.1) has a positive 4-periodic solution for \(0 < \gamma < \sqrt{5/2} - 1\). Then, we can obtain an approximate threshold \(\gamma_\ast = \sqrt{5/2} - 1\), which determines the existence or nonexistence of positive 4-periodic solution. Specifically, we state as follows: 

(i) When \(0 < \gamma < \gamma_\ast\), there is a positive 4-periodic solution of system (4.1).

(ii) When \(\gamma > \gamma_\ast\), there is no positive 4-periodic solution of system (4.1). In fact, assumption \((A_3)\) does not hold in this case, and the periodic solution of system (4.1) loses its positivity.

Note that for such \(\gamma_\ast\), the periodic solution of system (4.1) is undefined if \(\gamma = \gamma_\ast\). Hence, there is no need to consider the case that \(\gamma = \gamma_\ast = \sqrt{5/2} - 1\). These analytical results are also verified by simulations (Figures 3 and 4).

Figure 3: The numerical simulations of four arbitrary positive solutions of system (4.1). There is a positive 4-periodic solution of system (4.1) located in \([32.2, 64.5]\) with \(\gamma = 0.57 < \sqrt{5/2} - 1 = \gamma_\ast\).

Figure 4: The numerical simulations of four arbitrary positive solutions of system (4.1). The four positive solutions increase continuously, and there is no positive 4-periodic solution of system (4.1) with \(\gamma = 0.59 > \sqrt{5/2} - 1 = \gamma_\ast\).
In Figure 3(a), one observes that the arbitrary four positive solutions of system (4.1) have the same behavior (periodic behavior in fact) for large values of $k$. This means the existence of a positive 4-periodic solution with $y = 0.57 < \sqrt{5}/2 - 1 = y_\gamma$. The details of this positive 4-periodic solution are revealed in Figure 3(b). Figure 4 presents the nonexistence of positive 4-periodic solution of system (4.1). Although the four arbitrary positive solutions increase slowly at the beginning, they grow rapidly after a certain value of $k$.

5 Conclusion

The discrete hematopoiesis model that incorporates the impulses effect is studied in this article. This model provides a more realistic illustration of the sudden change of blood cell number. By applying the well-known Krasnosel’skii fixed point theorem, we explored the sufficient condition under which there is a positive periodic solution. In the case where the impulse is constant, we obtained a threshold for the existence of positive periodic solution. Through the numerical simulations, the fact that the impulse affects the maximum and minimum of positive periodic solutions is also revealed.

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