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On the Charney Conjecture of Data Assimilation Employing Temperature Measurements Alone: The Paradigm of 3D Planetary Geostrophic Model

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Abstract: Analyzing the validity and success of a data assimilation algorithm when some state variable observations are not available is an important problem in meteorology and engineering. We present an improved data assimilation algorithm for recovering the exact full reference solution (i.e. the velocity and temperature) of the 3D Planetary Geostrophic model, at an exponential rate in time, by employing coarse spatial mesh observations of the temperature alone. This provides, in the case of this paradigm, a rigorous justification to an earlier conjecture of Charney which states that temperature history of the atmosphere, for certain simple atmospheric models, determines all other state variables.

Keywords: Planetary Geostrophic model, data assimilation, nudging, downscaling, Charney's conjecture

MSC: 35Q30, 93C20, 37C50, 76B75, 34D06

1 Introduction

Numerical models for geophysical processes require establishing accurate initial conditions in order to make accurate predictions. To yield results that match observed data careful consideration must be given to model parameters, the highly nonlinear nature of model equations, sparsity of available data, measurement errors and conservation laws that must be satisfied. Due to this multitude of factors classical interpolation methods are often unfavorable starting points when designing the initial conditions.

Meteorologists have pioneered numerous works formulating diagnostic tests for obtaining accurate initialization procedures that minimizes the loss of information and yield a system close to the collected data. In the context of atmospheric physics, data assimilation algorithms where some state variable observations are not available as an input, have been studied in [7, 8, 16, 17, 19, 24] for simplified numerical forecast models. It was noted and tested in several settings that although nonlinear interactions exist between the scales of motion and the parameters in the system, a good dynamical model and sound data assimilation algorithm can identify the full state of the system knowing only coarse observational measurements of the selected partial state of the system. For example, the numerical experiment of Charney in [7], confirms that wind and surface pressure can be determined from coarse mesh measurements of temperatures alone. The numerical experiments in [16, 17] (see also [2]) were leaning towards similar conclusions. It was noted in [17] that "in practice,

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the complexities in weather forecast models together with measurement errors in real data make it unclear that assimilation of temperature data alone will yield initial states of arbitrary accuracy." The authors in [17] developed and tested different time-continuous data assimilation methods using temperature data. Their results vary depending on the assimilation method used. But they have suggested that temperature data can indeed have an impact on a numerical weather forecasts method provided that it is sensitive to the quality of such observed data. They suggested that "a refined method for data assimilation and accurate numerical models can lead to a major improvement in numerical weather prediction using temperature measurements alone." In [2], the experimental numerical results applied to the 2D Bénard convection problem, i.e. the Boussinessq system, showed that it is not sufficient to use temperature measurements alone in order to recover the corresponding state variables of the system.

Another pioneering study was the numerical experiments of Lorenc and Tibaldi in [24] which showed that frontal humidity fields can be determined from height and wind data. Their numerical experiments reported on the influence of the distance between observations, i.e. the size of the grid points, different combinations of the data being assimilated, and the effect of measurement errors in the accuracy of the initialization process. The experiments of [7] and [19] which use the nudging data assimilation scheme, for example, were also able to approximate the size of the relaxation time scale $\frac{1}{\mu}$ in order for the dynamics to adjust properly to the observations. They noted that if μ is set too small, the errors in the observations can get too large which results in the nudging term being ineffective controlling the instabilities. On the other hand, if μ is set too large, the dynamics will not have enough time to relax to the observation values. Concerning the values of the relaxation (nudging) parameter μ in our algorithm it is best to explain it by considering the difference between the approximate algorithm solution and the exact reference solution. The nonlinearity is the cause of instabilities at all spatial scales. Therefore, we require μ to be large enough in order to stabilize the large spatial scales of the difference. However, while the nudging term is stabilizing the large scales it might cause disturbances of the small scales that are magnified by the parameter μ. This, on the other hand, forces us not to choose μ to be too large in order to enable the fixed diffusion/viscous dissipative term to stabilize the overall instabilities in the small spatial scales caused by the above mechanism and the nonlinearity.

In this work, we give estimates using rigorous analysis that properly balances the effect of the relaxation parameter and other physical parameters in order to get an accurate initial condition using a data assimilation algorithm for the 3D planetary geostrophic model. A valuable next step would be to show that these parameter ranges can in fact be narrowed down in the actual numerical implementation, a subject of future investigation. Our claim is based on previous results in [15] (see also [18]) which have shown that in the absence of measurements errors, the continuous data assimilation algorithm for the 2D Navier-Stokes equations performs much better than suggested by the analytical theory, i.e. under more relaxed conditions than those assumed in the theoretical estimates established in [3] would suggest. It is possible that the data assimilation algorithm that we propose here, in practice, will also perform much better than suggested by the analytical results we have established in this work. On a similar note, it is worthwhile to mention recent results in [23], where the authors have derived rigorous conditions for an ODE system that partial observations must have in order to control the inherent uncertainty due to chaos. The authors in [23] presented a summary of 3DVAR for continuous and discrete time observations that highlights some important connection between 3DVAR, nudging and direct insertion (which they call synchronization filters) data assimilation schemes arising from various limiting conditions.

In [13] we proposed a data assimilation algorithm for a two-dimensional Bénard convection problem: two-dimensional Boussinesq system of a layer of incompressible fluid between two solid horizontal walls, with no-normal-flow and stress free boundary conditions on the walls, where fluid is heated from the bottom and cooled from the top. We incorporate the observables as a feedback (nudging) term in the evolution equation of the *horizontal* velocity alone. We show that under an appropriate choice of the nudging parameter and the size of the spatial coarse mesh observables, and under the assumption that the observed data is error free, the solution of the proposed algorithm converges at an exponential rate to the unique exact unknown reference solution of the original system, associated with the observed data on the horizontal component of the velocity. For this system we conjecture that we may not able to show that incomplete historical data on the temperature alone can determine the full state of the system. Recent numerical studies in [2] lends support to this conjecture. On the other hand in [12] we show that for the Bénard convection in porous media one only needs to use discrete spatial-mesh measurements of the temperature to show that the solution of the proposed algorithm converges at an exponential rate in time, to the unique exact unknown reference solution of the original system, associated with the observed finite dimensional projection of temperature data.

Charney's question in [7] of whether temperature observations are enough to determine all the dynamical state of the system results, in many ways, motivated a series of our recent studies, and many others, for example, see [2, 11] and references therein. Starting from many of the pioneering studies that can be traced back to the early 70's through more recent studies mentioned above, one may conclude that the answer to Charney's question would depend on the model. Indeed, as we have mentioned earlier the numerical studies reported in [2] show that the Charney conjecture does not hold for the Bénard convection model, i.e. Bousinessq system. This is not inconsistent with Charney's conjecture where he asserts that his claim is valid for certain simple models. Our study provides a paradigm of such a model for planetary scale dynamics and gives a rigorous support that temperature observations alone can determine the dynamical variables in the system for certain atmospheric models.

Here we consider the following planetary geostrophic viscous model for oceanic and atmosphere dynamics (see, e.g. [5], [6],[27], [28], [29], [32])

$$\nabla p + f\vec{k} \times u + L_1 u = 0, \tag{1.1a}$$
$$\partial_z p + T = 0, \tag{1.1b}$$
$$\nabla \cdot u + \partial_z w = 0, \tag{1.1c}$$
$$\partial_t T + u \cdot \nabla T + w \partial_z T + L_2 T = Q. \tag{1.1d}$$

In the case of ocean dynamics, the system above is studied in the domain

$$\Omega = M \times (-H, 0) \subset \mathbb{R}^3,$$

where M is a bounded smooth domain in $\mathbb{R}^2$, or the square $M = (0,1) \times (0,1)$. Here $\vec{k}$ is the unit vector of vertical direction, $u = (u_1, u_2)$, and (u_1, u_2, w) denote the horizontal and the three-dimensional velocity fields, respectively, T is the temperature, and p is the pressure. Here, $f = f_0 + \beta y$ is the Coriolis Parameter (the β-plane approximation), and Q is a heat source.

Next, we review the set-up and adopt the same notations in [5]. We will take L_1 and L_2 to be the dissipation operators:

$$L_1 = -A_h \Delta - A_v \partial_z^2,$$
$$L_2 = -K_h \Delta - K_v \partial_z^2,$$

where A_h and A_v are positive molecular viscosities, and K_h and K_v are positive conductivity constants. We set $\nabla p = (\partial_x p, \partial_y p)$, $\nabla \cdot u = \partial_x u_1 + \partial_y u_2$ and $\Delta = \partial_x^2 + \partial_y^2$.

We will denote the different parts of the (physical) boundary of Ω by:

$$\Gamma_u = \{(x, y, z) \in \Omega : z = 0\},$$
$$\Gamma_b = \{(x, y, z) \in \Omega : z = -H\},$$
$$\Gamma_s = \{(x, y, z) \in \Omega : (x, y) \in \partial M\},$$

and equip system (1.1a)–(1.1d) with the following (physical) boundary conditions – with wind-driven stress on the top surface and stress-free and non-flux on the side walls and bottom (see, e.g., [25], [26], [28], [29], [30]):

$$\text{on } \Gamma_u : A_v \frac{\partial u}{\partial z} = \tau, \ w = 0, \ -K_v \frac{\partial T}{\partial z} = \alpha(T - T^*); \tag{1.1e}$$
$$\text{on } \Gamma_b : \frac{\partial u}{\partial z} = 0, \ w = 0, \ \frac{\partial T}{\partial z} = 0; \tag{1.1f}$$
$$\text{on } \Gamma_s : u \cdot \vec{n} = 0, \ \frac{\partial v}{\partial \vec{n}} \times \vec{n} = 0, \ \frac{\partial T}{\partial \vec{n}} = 0, \tag{1.1g}$$

where $\tau(x, y)$ is the given wind stress, $\vec{n}$ is the normal vector of Γ_s, $T^\star(x, y)$ is typical temperature of the top (upper) surface, and $\alpha > 0$ is a positive constant. Due to the boundary conditions (1.1e)–(1.1g), one can assume that $T^\star$ satisfies the compatibility boundary condition:

$$\frac{\partial T^\star}{\partial \vec{n}} = 0 \qquad \text{on } \partial M. \tag{1.1h}$$

Finally we supply system (1.1a)-(1.1h) with the initial condition:

$$T(x, y, z, 0) = T_0(x, y, z). \tag{1.1i}$$

Alternate Formulation

Following [27] (see also [5]), we will do our analysis on an equivalent formulation for the system (1.1a)–(1.1i) which we review here. Integrating equation (1.1c) in the z direction, yields

$$w(x, y, z, t) = w(x, y, -H, t) - \int_{-H}^{z} \nabla \cdot u(x, y, \xi, t) d\xi. \tag{1.2}$$

Since $w(x, y, z, t) = 0$ at $z = -H, 0$ (see (1.1e) and (1.1f)), we have

$$w(x, y, z, t) = -\int_{-H}^{z} \nabla \cdot u(x, y, \xi, t) d\xi, \tag{1.3}$$

and

$$\int_{-H}^{0} \nabla \cdot u(x, y, \xi, t) d\xi = \nabla \cdot \int_{-H}^{0} u(x, y, \xi, t) d\xi = 0. \tag{1.4}$$

By integrating equation (1.1b) with respect to z, we obtain

$$p(x, y, z, t) = -\int_{-H}^{z} T(x, y, \xi, t) d\xi + p_s(x, y, t), \tag{1.5}$$

where $p_s(x, y, t)$ is a free function that represents the bottom pressure and needs to be determined. We set

$$T = T^\star + \widetilde{T}. \tag{1.6}$$

Now, the boundary conditions (1.1e)–(1.1g) imply that the fluctuation temperature $\widetilde{T}$ satisfies the following homogeneous boundary conditions:

$$\left.\frac{\partial \widetilde{T}}{\partial z}\right|_{z=-H} = 0; \quad \left.\left(\frac{\partial \widetilde{T}}{\partial z} + \frac{\alpha}{K_v}\widetilde{T}\right)\right|_{z=0} = 0; \quad \left.\frac{\partial \widetilde{T}}{\partial \vec{n}}\right|_{\Gamma_s} = 0, \tag{1.7}$$

where we have also used the compatibility condition (1.1h).

We summarize the new formulation for system (1.1a)–(1.1i) below:

$$\nabla\left[p_s(x,y,t)-\int_{-H}^{z}\widetilde{T}(x,y,\xi,t)d\xi-(z+H)T^{\star}(x,y,t)\right]+f\vec{k}\times u+L_1u=0, \tag{1.8a}$$

$$\nabla\cdot\int_{-H}^{0}u(x,y,z,t)\,dz=0, \tag{1.8b}$$

$$\partial_t\widetilde{T}+L_2\widetilde{T}+u\cdot\nabla\widetilde{T}-\left(\nabla\cdot\int_{-H}^{z}u(x,y,\xi,t)\,d\xi\right)\partial_z\widetilde{T}+u\cdot\nabla T^{\star}=Q^{\star}, \tag{1.8c}$$

$$\left.\frac{\partial u}{\partial z}\right|_{z=0}=\tau,\ \left.\frac{\partial u}{\partial z}\right|_{z=-H}=0,\ u\cdot\vec{n}\big|_{\Gamma_s}=0,\ \left.\frac{\partial u}{\partial\vec{n}}\times\vec{n}\right|_{\Gamma_s}=0, \tag{1.8d}$$

$$\left.\left(\partial_z\widetilde{T}+\frac{\alpha}{K_v}\widetilde{T}\right)\right|_{z=0}=0;\ \partial_z\widetilde{T}\Big|_{z=-H}=0;\ \left.\frac{\partial}{\partial\vec{n}}\widetilde{T}\right|_{\Gamma_s}=0, \tag{1.8e}$$

$$\widetilde{T}(x,y,z,0)=\widetilde{T}_0=T_0(x,y,z)-T^{\star}(x,y), \tag{1.8f}$$

where

$$Q^{\star}=Q+K_h\Delta T^{\star}. \tag{1.8g}$$

We will assume here that the functions $T^{\star}$, τ, $Q^{\star}$ and $\widetilde{T}_0$ are given. This makes the unknown functions in the above system: the vector field $u(x,y,z,t)$, and the scalar functions $p_s(x,y,t)$ and $\widetilde{T}(x,y,z,t)$. The original unknowns of system (1.1a)–(1.1i), i.e., (u,w), T and p, can be recovered from the solution u, $\widetilde{T}$, and p_s of the above system using (1.2), (1.5) and (1.6). The global regularity and global well-posedness results of this model is found in [5, 27].

We will propose and analyze a data assimilation (downscaling) algorithm for recovering the solution u and T of system (1.1) from coarse spatial measurements of temperature T alone, in the absence of initial condition T_0. We add a nudging term that pushes the large spatial scales of the approximating solution towards the reference solution matching the coarse temperature observations. Our algorithm for the approximate velocity v, temperature η and pressure q_s, of the unknown reference solution u, T and p_s, is given by the system:

$$\nabla\left[q_s(x,y,t)-\int_{-H}^{z}\eta(x,y,\xi,t)d\xi-(z+H)T^{\star}(x,y,t)\right]+f\vec{k}\times v+L_1v=0, \tag{1.9a}$$

$$\nabla\cdot\int_{-H}^{0}v(x,y,z,t)\,dz=0, \tag{1.9b}$$

$$\partial_t\eta+L_2\eta+v\cdot\nabla\eta-\left(\nabla\cdot\int_{-H}^{z}v(x,y,\xi,t)\,d\xi\right)\partial_z\eta+v\cdot\nabla T^{\star}=Q^{\star}-\mu\left(I_h(\eta)-I_h(\widetilde{T})\right), \tag{1.9c}$$

$$\left.\frac{\partial v}{\partial z}\right|_{z=0}=\tau,\ \left.\frac{\partial v}{\partial z}\right|_{z=-H}=0,\ v\cdot\vec{n}\big|_{\Gamma_s}=0,\ \left.\frac{\partial v}{\partial\vec{n}}\times\vec{n}\right|_{\Gamma_s}=0, \tag{1.9d}$$

$$\left.\left(\partial_z\eta+\frac{\alpha}{K_v}\eta\right)\right|_{z=0}=0;\ \partial_z\eta\big|_{z=-H}=0;\ \left.\frac{\partial}{\partial\vec{n}}\eta\right|_{\Gamma_s}=0, \tag{1.9e}$$

$$\eta(x,y,z,0)=\eta_0, \tag{1.9f}$$

where

$$Q^{\star}=Q+K_h\Delta T^{\star},$$

and we keep the assumption that $T^{\star}$, τ, and $Q^{\star}$ are given functions. Here, η_0 can be taken arbitrary and $I_h(\cdot)$, which we will define in more detail below, is a linear interpolant operator based on the observational measurements on a coarse spatial resolution of size h, for $t\in[0,T]$. Let us denote by $L^2(\Omega)$ and $H^1(\Omega), H^2(\Omega),\cdots$, the usual L^2–Lebesgue and Sobolev spaces, respectively (see, e.g. [9] and [34]). Two types of interpolants can be considered. One is given by a linear interpolant operator $I_h : H^1(\Omega) \to L^2(\Omega)$ satisfying the approximation property

$$\|\varphi - I_h(\varphi)\|_{L^2(\Omega)}^2 \le c_0 h^2 \|\varphi\|_{H^1(\Omega)}^2, \tag{1.10}$$

for every $\varphi \in H^1(\Omega)$, where $c_0 > 0$ is a dimensionless constant. The other type is given by $I_h : H^2(\Omega) \to L^2(\Omega)$, together with

$$\|\varphi - I_h(\varphi)\|_{L^2(\Omega)}^2 \le c_0 h^2 \|\varphi\|_{H^1(\Omega)} + c_0^2 h^4 \|\varphi\|_{H^2(\Omega)}^2, \tag{1.11}$$

for every $\varphi \in H^2(\Omega)$, where $c_0 > 0$ is a dimensionless constant.

To give an example of an interpolant operator that satisfies (1.10), we consider the positive definite, self-adjoint Laplace operator $(-\Delta)$ for the temperature with the corresponding boundary condition (1.9e). This linear operator has a compact inverse $(-\Delta)^{-1} : L^2(\Omega) \to L^2(\Omega)$, thus there exist a complete orthonormal set of eigenfunctions $\{w_j\}_{j=1}^{\infty} \subset L^2(\Omega)$ such that $-\Delta w_j = \lambda_j w_j$, where $0 < \lambda_j \le \lambda_{j+1}$ for $j \in \mathbb{N}$. Since we can order the eigenvalues we can let I_h to be the orthogonal projection of $L^2(\Omega)$ onto the linear subspace spanned by the first m_h eigenfunctions $\{w_1, w_2, \ldots w_{m_h}\}$, where m_h is chosen large enough so that the corresponding eigenvalue $(\lambda_{m_h})^{-1} \le h^{-2}$. In the case of periodic boundary conditions, an example of an interpolant observable that satisfies (1.10), is the orthogonal projection onto the linear subspace spanned by the low Fourier modes with wave numbers k such that $|k| \le m_h = 1/h$. Physically relevant example is based on volume elements measurements that was studied in [3, 21]. Examples of an interpolant observable that satisfies (1.11) are given by the low Fourier modes and the measurements at a discrete set of nodal points in Ω (see Appendix A in [3]). We are not treating the second type of interpolants in this paper only for the simplicity of presentation. For full details and examples of models on the analysis for the second type of interpolants we refer to [3, 10, 11, 13].

In section 3, we will show that the solution v, η, and q_s of system (1.9) converge, at an exponential rate in time, to the unknown reference solution u, $\widetilde{T}$, and p_s of (1.8) corresponding to the continuous (in time) temperature measurements $I_h(\widetilde{T})$. It is worth mentioning that by combining the tools developed in this paper with those in [4], we can treat the case when the measurements are contaminated with a stochastic error. Furthermore, employing the ideas in [14] with the tools developed here, one can also treat the case of fully discrete measurements in space and time, with deterministic errors, and obtain similar results including statics of the reference solution. That is, one can treat the case where the coarse spatial mesh measurements are collected at discrete times, $\{t_j\}_{j=1}^{\infty}$, provided $|t_{j+1-t_j}| \le \kappa$, for κ small enough depending on physical parameters. To simplify the presentation, we will not treat these cases and focus only on the case of continuous (in time) measurements.

2 Preliminaries and Functional Setting

We will review some analytical results related to system (1.8a)–(1.8f) that we will rely on to prove our convergence result in Section 3. We will use the same notations as in [5] to be consistent with the literature.

2.1 Functional spaces and relevant inequalities

We denote by

$$|T| = \left(\int_{\Omega} |T(x,y,z)|^2 \, dxdydz \right)^{\frac{1}{2}}, \tag{2.1}$$

for every $T \in L^2(\Omega)$, and by

$$\|T\| = \left(\alpha \int_{\Gamma_u} |T(x,y,0)|^2 dxdy + \int_{\Omega} \left[K_h |\nabla T(x,y,z)|^2 + +K_v \left|\partial_z T(x,y,z)\right|^2 \right] \, dxdydz \right)^{\frac{1}{2}}, \tag{2.2}$$

for every $T \in H^1(\Omega)$. Let

$$\widetilde{\mathcal{V}} = \left\{ \widetilde{T} \in C^\infty(\overline{\Omega}) : \left.\frac{\partial \widetilde{T}}{\partial z}\right|_{z=-H} = 0; \left.\left(\frac{\partial \widetilde{T}}{\partial z} + \frac{\alpha}{K_v}\widetilde{T} \right)\right|_{z=0} = 0; \left.\frac{\partial \widetilde{T}}{\partial \vec{n}}\right|_{\Gamma_s} = 0 \right\}.$$

We also denote by $H^{'}$ the dual space of $H^1(\Omega)$, with the dual action $\langle \cdot, \cdot \rangle$.

We will use of the following Poincaré-type inequalities (cf., e.g., [1], [9] [34])

Proposition 2.1. *The norm defined as in* (2.2) *is equivalent to the* $H^1(\Omega)$ *norm. That is, there is a positive constant* K_1 *such that*

$$\frac{1}{K_1}\|T\|^2 \le \|T\|^2_{H^1(\Omega)} \le K_1\|T\|^2 \tag{2.3}$$

for every $T \in H^1(\Omega)$. *Moreover, we have*

$$|T|^2 \le \widetilde{K}\|T\|^2, \qquad \textit{for all} \quad T \in H^1(\Omega), \tag{2.4}$$

where

$$\widetilde{K} = \max\left\{ \frac{2H}{\alpha}, \frac{2H^2}{K_v} \right\}. \tag{2.5}$$

We will also use the Ladyzhenskaya's interpolation inequalities (cf., e.g., [1, 22]):

$$\|\phi(x,y)\|_{L^4(M)} \le C_4\|\phi(x,y)\|^{1/2}_{L^2(M)}\|\phi(x,y)\|^{1/2}_{H^1(M)}, \tag{2.6a}$$

$$\|\phi(x,y)\|_{L^6(M)} \le C_4\|\phi(x,y)\|^{1/3}_{L^2(M)}\|\phi(x,y)\|^{2/3}_{H^1(M)}, \tag{2.6b}$$

and

$$\|g(x,y,z)\|_{L^3(\Omega)} \le C_5|g(x,y,z)|^{1/2}\|g(x,y,z)\|^{1/2}_{H^1(\Omega)}, \tag{2.7a}$$

$$\|g(x,y,z)\|_{L^6(\Omega)} \le C_5\|g(x,y,z)\|_{H^1(\Omega)}, \tag{2.7b}$$

for all $\phi \in H^1(M)$ and $g \in H^1(\Omega)$, respectively. Also, we recall the integral version of Minkowsky inequality for the L^p spaces, $p \ge 1$. Let $\Omega_1 \subset \mathbb{R}^{m_1}$ and $\Omega_2 \subset \mathbb{R}^{m_2}$ be two measurable sets, where m_1 and m_2 are two positive integers. Suppose that $f(\xi, \eta)$ is measurable over $\Omega_1 \times \Omega_2$. Then,

$$\left[\int_{\Omega_1} \left(\int_{\Omega_2} |f(\xi,\eta)| d\eta \right)^p d\xi \right]^{1/p} \le \int_{\Omega_2} \left(\int_{\Omega_1} |f(\xi,\eta)|^p d\xi \right)^{1/p} d\eta. \tag{2.8}$$

Hereafter, C, which may depend on the domain Ω and the constant parameters $f_0, \beta, \alpha, A_h, A_v, K_h, K_v$ in the system (1.1a)–(1.1i), will denote a constant that may change from line to line.

We will apply the following inequality which is a particular case of a more general inequality proved in [21].

Lemma 2.2. *[21] Let* $\tau > 0$ *be fixed. Suppose that* $Y(t)$ *is an absolutely continuous nonnegative function which is locally integrable and that it satisfies the following:*

$$\frac{dY}{dt} + \alpha(t)Y \le \beta(t), \qquad \textit{a.e. on } (0,\infty),$$

such that

$$\liminf_{t\to\infty} \int_t^{t+\tau} \alpha(s)\,ds \ge \gamma, \qquad \limsup_{t\to\infty} \int_t^{t+\tau} \alpha^-(s)\,ds < \infty, \tag{2.9}$$

and

$$\lim_{t\to\infty}\int_t^{t+\tau}\beta^+(s)\,ds = 0, \tag{2.10}$$

for some $\gamma > 0$, *where* $\alpha^- = \max\{-\alpha, 0\}$ *and* $\beta^+ = \max\{\beta, 0\}$. *Then,* $Y(t) \to 0$ *at an exponential rate, as* $t \to \infty$.

Finally, we state the following proposition that was proved in [5].

Proposition 2.3. *[5] Let* $u = (u_1, u_2) \in H^2(\Omega)$, $f \in L^2(\Omega)$ *and* $g \in H^1(\Omega)$. *Then*

$$\left|\int_\Omega \left(\nabla\cdot\int_{-H}^{z} u(x,y,\xi,t)\,d\xi\right) f(x,y,z)\,g(x,y,z)\,dxdydz\right|$$
$$\le C\,|f|\,\|u\|_{H^1(\Omega)}^{1/2}\|u\|_{H^2(\Omega)}^{1/2}\,\|g\|_{H^1(\Omega)}^{1/2}\,|g|^{1/2}.$$

2.2 Regularity Results

We state the definition of weak solutions.

Definition 2.4. [5] Let $\widetilde{T}_0 \in L^2(\Omega)$ and let S be any fixed positive time. The vector field $v(x,y,z,t)$, and the scalar functions $p_s(x,y,t)$ and $\widetilde{T}(x,y,z,t)$ are called a weak solution of *(1.8a)–(1.8f)* on the time interval $[0,S]$ if

$$\begin{aligned}
&p_s(x,y,t) \in C([0,S], L^2(M)) \cap L^2([0,S], H^1(M)),\\
&u(x,y,z,t) \in C([0,S], H^1(\Omega)) \cap L^2([0,S], H^2(\Omega)),\\
&\widetilde{T}(x,y,z,t) \in C([0,S], L^2(\Omega)) \cap L^2([0,S], H^1(\Omega)),\\
&\partial_t\widetilde{T}(x,y,z,t) \in L^1([0,S], H'),
\end{aligned}$$

(recall that H' is the dual space of $H^1(\Omega)$), and if they satisfy

$$\int_\Omega \nabla\left[p_s(x,y,t) - \int_{-H}^{z}(\widetilde{T}(x,y,\xi,t) + T^\star)d\xi\right]\phi\,dxdydz +$$
$$+\int_\Omega \left(f\vec{k}\times u\right)\phi\,dxdydz + \int_\Omega (A_h\nabla u\cdot\nabla\phi + A_v\partial_z u\partial_z\phi)\,dxdydz = \int_{\Gamma_u} A_v\tau\phi\,dxdydz,$$

and

$$\begin{aligned}
&\int_\Omega \widetilde{T}(t)\psi\,dxdydz + \int_{t_0}^{t}\int_\Omega \left(K_h\nabla\widetilde{T}\cdot\nabla\psi + K_v\partial_z\widetilde{T}\partial_z\psi\right)dxdydz\\
&+\alpha\int_{t_0}^{t}\int_{\Gamma_u}\widetilde{T}\psi\,dxdy + \int_{t_0}^{t}\int_\Omega \left(u\cdot\nabla T^\star\right)\psi\,dxdydz +\\
&+\int_{t_0}^{t}\int_\Omega \left[\left(u\cdot\nabla\widetilde{T}\right)\psi - \left(\nabla\cdot\int_{-H}^{z}u(x,y,\xi,t)\,d\xi\right)\partial_z\widetilde{T}\,\psi\right]dxdydz\\
&= \int_\Omega \widetilde{T}(t_0)\psi\,dxdydz + \int_{t_0}^{t}\int_\Omega Q^\star\psi\,dxdydz,
\end{aligned}$$

for every $\phi \in (C^\infty(\overline{\Omega}))^2$ and $\psi \in C^\infty(\overline{\Omega})$, and for almost every $t, t_0 \in [0, S]$.

Moreover, if $\widetilde{T}_0 \in H^1(\Omega)$ a weak solution is called strong solution of *(1.8a)–(1.8f)* on $[0, S]$ if, in addition, it satisfies

$$\begin{aligned} &p_s(x, y, t) \in C([0, S], H^1(M)) \cap L^2([0, S], H^2(M)),\\ &u(x, y, z, t) \in C([0, S], H^1(\Omega)) \cap L^2([0, S], H^2(\Omega)),\\ &\widetilde{T}(x, y, z, t) \in C([0, S], H^1(\Omega)) \cap L^2([0, S], H^2(\Omega)). \end{aligned}$$

Now, we recall the global existence and uniqueness results proved in [5].

Theorem 2.5 (Weak solutions). *[5] Suppose that $\tau \in H_0^1(M)$, $T^\star \in H^2(M)$ and $Q \in L^2(\Omega)$. Then for every $\widetilde{T}_0 = T_0 - T^\star \in L^2(\Omega)$ and $S > 0$, there is a unique weak solution $(p_s, v, \widetilde{T})$ (p_s is unique up to a constant) of the system* (1.8a)–(1.8f) *on the interval* $[0, S]$.

Furthermore, the weak solution of the system (1.8a)–(1.8f) depends continuously on the initial data. That is, the problem is globally well–posed.

Theorem 2.6 (Strong solutions). *[5] Suppose that $\tau \in H_0^1(M)$, $Q \in H^1(\Omega)$ and $T^\star \in H^2(M)$. Then for every $\widetilde{T}_0 = T_0 - T^\star \in H^1(\Omega)$, and $S > 0$, there is a unique strong solution $\widetilde{T}$ of system* (1.8a)–(1.8f).

Theorem 2.7 (Global attractor). *[5] Suppose that $\tau \in H_0^1(M)$, $Q \in L^2(\Omega)$ and $T^\star \in H^2(M)$. Then, there is a finite-dimensional global compact attractor $\mathcal{A} \subset L^2(\Omega)$ for the system* (1.8a)–(1.8f). *Moreover, when t is large enough we have*

$$|\widetilde{T}(t)|^2 \le R_a(T^\star, Q) := 2\widetilde{R}_a(T^\star, Q) + 2\|T^\star\|^2_{L^2(M)}, \tag{2.11a}$$

$$\int_t^{t+r} \|T(s)\|^2 \, ds \le K_r(r, Q, T^\star), \tag{2.11b}$$

$$\|\widetilde{T}(t)\| \le R_v(r, T^\star, Q, \tau), \tag{2.11c}$$

where

$$\widetilde{R}_a(T^\star, Q) := 4\alpha\widetilde{K}\|T^\star\|^2_{L^2(M)} + 8\widetilde{K}^2|Q|^2, \tag{2.12}$$

$$K_r(r, Q, T^\star) := 2R_a(T^\star, Q) + \left[4\alpha\widetilde{K}\|T^\star\|^2_{L^2(M)} + 8\widetilde{K}^2|Q|^2\right] r, \tag{2.13}$$

$$\begin{aligned} R_v(r, T^\star, Q, \tau) := C \Bigg[&\frac{R_a(T^\star, Q)}{r^{1/2}} + \|T^\star\|_{H^1(M)} + |Q| \\ &+ \frac{C}{\lambda_1^{1/2}} \left(1 + \|T^\star\|^2_{H^2(\Omega)} + |Q| + \|\tau\|^2_{H^1(M)} + R_a^2(T^\star, Q)\right) \Bigg] \times \\ &\times e^{C\left[(R_a(T^\star, Q))^4 + \left(\|T^\star\|^4_{H^2(M)} + \|\tau\|^4_{H^1(M)} + (R_a(T^\star, Q))^4\right) r\right]}. \end{aligned} \tag{2.14}$$

3 Analysis and Convergence of the Data Assimilation Algorithm

In this section, we derive conditions under which the solution (q_s, v, η), of the data assimilation algorithm system (1.9a)–(1.9f), converges to the corresponding unique reference solution $(p_s, u, \widetilde{T})$ of the planetary geostrophic system (1.8a)–(1.8f), at an exponential rate, as $t \to \infty$. The steps of the proof we present here are formal in the sense that they can be made rigorous by proving their corresponding counterpart estimates first for the Galerkin approximation system. Then the estimates for the exact solution can be established by passing to the limit in the Galerkin procedure by using the appropriate “Compactness Theorems”.

Theorem 3.1. *Suppose that I_h satisfies the approximation property* (1.10). *Let* $(p_s(t), u(t), \widetilde{T}(t))$, *for* $t \geq 0$, *be a strong solution in the global attractor of* (1.8a)–(1.8f). *Let* $\eta_0 \in L^2(\Omega)$ *and suppose that* $\mu > 0$ *is large enough such that*

$$\mu \geq 2C\left(1 + 5\widetilde{R}_a(T^*, Q) + 4\left\|T^*\right\|^2_{L^2(M)} + \left\|T^*\right\|^{4/3}_{H^1(M)}\right), \tag{3.1}$$

where $\widetilde{R}_a(T^*, Q)$ *is a constant defined in* (2.12), *and* $h > 0$ *is small enough such that* $\mu c_0^2 h^2 \leq 1$. *Then, for any* $S > 0$, *system* (1.9a)–(1.9f) *has a unique weak solution* (q_s, v, η) *on the time interval* $[0, S]$ *(*q_s *is unique up to a constant, i.e.,* ∇q_s *is unique) in the sense of Definition 2.4.*

Moreover, the solution (v, η) *depends continuously on the initial data, and it satisfies* $\left\|\eta(t) - \widetilde{T}(t)\right\|^2_{L^2(\Omega)} \to 0$, *and* $\left\|v(t) - u(t)\right\|^2_{H^1(\Omega)} \to 0$, *at an exponential rate, as* $t \to \infty$.

Proof. We consider the difference between the reference solution and the approximate solution, (1.8a)–(1.8f) and (1.9a)–(1.9f), respectively. We provide here the relevant *a priori* estimates to show simultaneously the global well-posedness and the convergence results. Denote by $U = v - u$, $\chi = \eta - \widetilde{T}$, and $P_s = q_s - p_s$. Then, P_s, U and χ satisfy:

$$\nabla\left[P_s(x, y, t) - \int_{-H}^{z} \chi(x, y, \xi, t) d\xi\right] + f\vec{k} \times U + L_1 U = 0, \tag{3.2a}$$

$$\nabla \cdot \int_{-H}^{0} U(x, y, z, t)\, dz = 0, \tag{3.2b}$$

$$\begin{aligned}\partial_t\chi + L_2\chi + U \cdot \nabla\widetilde{T} + v \cdot \nabla\chi + U \cdot \nabla T^* - \left(\nabla \cdot \int_{-H}^{z} U(x, y, \xi, t)\, d\xi\right)\frac{\partial\widetilde{T}}{\partial z} - \\ - \left(\nabla \cdot \int_{-H}^{z} v(x, y, \xi, t)\, d\xi\right)\frac{\partial\chi}{\partial z} - \mu I_h(\chi) = 0,\end{aligned} \tag{3.2c}$$

$$\left.\frac{\partial U}{\partial z}\right|_{z=0} = 0, \quad \left.\frac{\partial U}{\partial z}\right|_{z=-H} = 0, \quad U \cdot \vec{n}\big|_{\Gamma_s} = 0, \quad \left.\frac{\partial U}{\partial \vec{n}} \times \vec{n}\right|_{\Gamma_s} = 0, \tag{3.2d}$$

$$\left.\left(\frac{\partial\chi}{\partial z} + \frac{\alpha}{K_v}\chi\right)\right|_{z=0} = 0; \quad \left.\frac{\partial\chi}{\partial z}\right|_{z=-H} = 0; \quad \left.\frac{\partial\chi}{\partial\vec{n}}\right|_{\partial M} = 0, \tag{3.2e}$$

$$\chi(x, y, z, 0) = \eta_0(x, y, z) - \widetilde{T}_0(x, y, z). \tag{3.2f}$$

Next, we follow the ideas and arguments in [5]. For an integrable function ϕ defined on Ω we denote by

$$\overline{\phi}(x, y, t) = \frac{1}{H}\int_{-H}^{0} \phi(x, y, z, t)\, dz.$$

By averaging (3.2a) and (3.2c) with respect to z and using (3.2b), we get

$$\nabla\left[P_s(x, y, t) + \frac{1}{H}\int_{-H}^{0} \xi\chi(x, y, \xi, t)\, d\xi\right] + f\vec{k} \times \overline{U} - A_h\Delta\overline{U} = 0, \tag{3.3a}$$

$$\nabla \cdot \overline{U} = 0, \tag{3.3b}$$

$$\overline{U} \cdot \vec{n} = 0, \quad \frac{\partial\overline{U}}{\partial\vec{n}} \times \vec{n} = 0, \qquad \text{on } \partial M. \tag{3.3c}$$

By taking the $L^2(\Omega)$ inner product of equation (3.3a) with $\overline{U}$, we obtain

$$\int_\Omega \left[\nabla\left(P_s(x, y, t) + \frac{1}{H}\int_{-H}^{0} \xi\chi(x, y, \xi, t)\, d\xi\right) - A_h\Delta\overline{U}\right]\overline{U}\, dxdydz = 0.$$

By using integration by parts and applying (3.3b) and (3.3c), we get

$$\int_\Omega |\nabla \overline{U}|^2 \, dxdydz = 0.$$

Thus, $\overline{U}$ is a function of t alone. By (3.3c), we reach $\overline{U} = 0$. As a result, we have

$$P_s(x,y,t) = -\frac{1}{H}\int_{-H}^{0} \xi \chi(x,y,\xi,t)d\xi. \tag{3.4}$$

(P_s is unique up to a constant that depends on time, thus ∇P_s is unique). Therefore, (3.2a) can be written as

$$-\nabla\left[\frac{1}{H}\int_{-H}^{0} \xi\chi(x,y,\xi,t)d\xi + \int_{-H}^{z}\chi(x,y,\xi,t)d\xi\right] + f\vec{k}\times U + L_1 U = 0. \tag{3.5}$$

Notice that U satisfies the boundary condition (3.2d). For the second order elliptic boundary–value problem (3.5) we have the following regularity results (by following similar techniques to those developed in [20] and [33]. For the case of smooth domains see, for example, [22] p. 89, and [31])

$$\|U\|_{H^1(\Omega)} \le \frac{C_2}{\widetilde{A}}|\chi|, \qquad \text{and} \qquad \|U\|_{H^2(\Omega)} \le \frac{C_2}{\widetilde{A}}\|\chi\|, \tag{3.6}$$

where $\widetilde{A} = \min\{A_h, A_v\}$. By taking the H' dual action of equation (3.2c) with χ, we obtain

$$\langle \partial_t\chi + L_2\chi, \chi\rangle + \left\langle U\cdot\nabla\widetilde{T} + v\cdot\nabla\chi + U\cdot\nabla T^*, \chi\right\rangle -$$
$$-\left\langle (\nabla\cdot\int_{-H}^{z} U(x,y,\xi,t)\,d\xi)\partial_z\widetilde{T} - (\nabla\cdot\int_{-H}^{z} v(x,y,\xi,t)\,d\xi)\partial_z\chi, \chi\right\rangle - \mu\left\langle I_h(\chi),\chi\right\rangle = 0.$$

Notice that by integrating by parts and using the boundary conditions (3.2e), we have

$$\begin{aligned}
\int_\Omega \chi L_2\chi \, dxdydz &= -\int_\Omega \chi\left(K_h\Delta\chi + K_v\partial_z^2\chi\right)\, dxdydz \\
&= \int_\Omega \left[K_h|\nabla\chi|^2 + K_v|\partial_z\chi|^2\right]\, dxdydz - \int_{\Gamma_u} K_v\chi\partial_z\chi dxdy \\
&= \int_\Omega \left[K_h|\nabla\chi|^2 + K_v|\partial_z\chi|^2\right]\, dxdydz + \alpha\int_{\Gamma_u}|\chi|^2 dxdy \\
&= \|\chi\|^2.
\end{aligned} \tag{3.7}$$

We use the facts that

$$\langle\partial_t\chi,\chi\rangle = \frac{1}{2}\frac{d|\chi|^2}{dt} \qquad \text{and} \qquad \langle L_2\chi,\chi\rangle = \|\chi\|^2.$$

Moreover,

$$\left\langle U\cdot\nabla\widetilde{T} + v\cdot\nabla\chi + U\cdot\nabla T^*, \chi\right\rangle = \int_\Omega \left[U\cdot\nabla\widetilde{T} + v\cdot\nabla\chi + U\cdot\nabla T^*\right]\chi\, dxdydz,$$

$$\left\langle\left(\nabla\cdot\int_{-h}^{z} U(x,y,\xi,t)\,d\xi\right)\partial_z\widetilde{T} + \left(\nabla\cdot\int_{-H}^{z} v(x,y,\xi,t)\,d\xi\right)\partial_z\chi, \chi\right\rangle\, dxdydz$$
$$= \int_\Omega \left[\left(\nabla\cdot\int_{-H}^{z} U(x,y,\xi,t)\,d\xi\right)\partial_z\widetilde{T} + \left(\nabla\cdot\int_{-H}^{z} v(x,y,\xi,t)\,d\xi\right)\partial_z\chi\right]\chi\, dxdydz.$$

Therefore, we have

$$\frac{1}{2}\frac{d|\chi|^2}{dt} + \|\chi\|^2 = \int\limits_{\Omega} \Big[-U\cdot\nabla\widetilde{T} - v\cdot\nabla\chi - U\cdot\nabla T^{\star} + \\ + \left(\nabla\cdot\int\limits_{-H}^{z} U(x,y,\xi,t)\,d\xi\right)\partial_z\widetilde{T} + \left(\nabla\cdot\int\limits_{-H}^{z} v(x,y,\xi,t)\,d\xi\right)\partial_z\chi \Big]\chi - \mu\left\langle I_h(\chi),\chi\right\rangle .$$

Next, we estimate in the above equation term by term. By integrating by parts and (3.2d), we reach

$$\int\limits_{\Omega}\left[v\cdot\nabla\chi - \left(\nabla\cdot\int\limits_{-H}^{z} v(x,y,\xi,t)\,d\xi\right)\partial_z\chi\right]\chi\,dxdydz = 0. \tag{3.8}$$

The Cauchy-Schwarz inequality yields

$$\left|\int\limits_{\Omega} U\cdot\nabla\left(\widetilde{T}+T^{\star}\right)\chi\,dxdydz\right| \le \|\widetilde{T}+T^{\star}\|_{H^1(\Omega)}\,\|U\|_{L^6(\Omega)}\|\chi\|_{L^3(\Omega)}.$$

Using (3.6) and (2.7), we have

$$\|U\|_{L^6(\Omega)} \le \frac{C}{\widetilde{A}}|\chi|, \quad \text{and} \quad \|\chi\|_{L^3(\Omega)} \le C|\chi|^{1/2}\|\chi\|^{1/2}.$$

Thus,

$$\left|\int\limits_{\Omega} U\cdot\nabla\left(\widetilde{T}+T^{\star}\right)\chi\right| \le C\left[\|\widetilde{T}\| + \|T^{\star}\|_{H^1(M)}\right]|\chi|^{3/2}\|\chi\|^{1/2}. \tag{3.9}$$

Setting $u = U, f = \partial_z\widetilde{T}$ and $g = \chi$, respectively, in Proposition 2.3 implies that

$$\left|\int\limits_{\Omega}\left(\nabla\cdot\int\limits_{-H}^{z} U(x,y,\xi,t)\,d\xi\right)\partial_z\widetilde{T}\chi\right| \\ \le C\left\|\widetilde{T}\right\|_{H^1(\Omega)}\|U\|^{1/2}_{H^1(\Omega)}\|U\|^{1/2}_{H^2(\Omega)}\|\chi\|^{1/2}|\chi|^{1/2}$$

Applying (3.6) to the above estimate, we get

$$\left|\int\limits_{\Omega}\left(\nabla\cdot\int\limits_{-H}^{z} U(x,y,\xi,t)\,d\xi\right)\partial_z\widetilde{T}\,\chi dxdydz\right| \le C\|\widetilde{T}\||\chi|\|\chi\|. \tag{3.10}$$

Finally, thanks to the assumptions $\mu c_0^2h^2 \le 1$, (1.10), and Young inequality, we have

$$\begin{aligned} -\mu(I_h(\chi),\chi) &= -\mu(I_h(\chi)-\chi,\chi) - \mu|\chi|^2 \\ &\le \mu c_0 h\,\|\chi\|\,|\chi| - \mu|\chi|^2 \\ &\le \frac{\mu c_0^2h^2}{2}\|\chi\|^2 - \frac{\mu}{2}|\chi|^2 \\ &\le \frac{1}{2}\|\chi\|^2 - \frac{\mu}{2}|\chi|^2. \end{aligned} \tag{3.11}$$

Therefore, thanks to (3.8)–(3.11), we have the estimate:

$$\frac{1}{2}\frac{d|\chi|^2}{dt} + \frac{1}{2}\|\chi\|^2 \le C(\|\widetilde{T}\| + \|T^{\star}\|_{H^1(M)})|\chi|^{3/2}\|\chi\|^{1/2} + C\|\widetilde{T}\||\chi|\|\chi\| - \frac{\mu}{2}|\chi|^2.$$

By Young's inequality, we obtain

$$\frac{d|\chi|^2}{dt} + \|\chi\|^2 \leq \left[C\left(1 + \|\widetilde{T}\|^2 + \|T^\star\|^{4/3}_{H^1(M)}\right) - \mu\right] |\chi|^2.$$

Recall that from Theorem 2.7 we have

$$\int_t^{t+1} \left\|\widetilde{T}(s)\right\|^2 ds \leq 2R_a(T^\star, Q) + \widetilde{R}_a(T^\star, Q) = 5\widetilde{R}_a(T^\star, Q) + 4\left\|T^\star\right\|^2_{L^2(M)},$$

for any $t \geq 0$, $\widetilde{R}_a(T^\star, Q)$ is a constant defined in (2.12). Thus, applying Lemma 2.2 with $\tau = 1$, and using condition (3.1), we can conclude that condition (2.9) is satisfied, therefore we have

$$\left\|\eta(t) - \widetilde{T}(t)\right\|^2_{L^2(\Omega)} = |\chi(t)|^2 \to 0, \tag{3.12}$$

at at exponential rate, as $t \to \infty$. The regularity result (3.6) yields

$$\|v(t) - u(t)\|^2_{H^1(\Omega)} = \|U(t)\|^2_{H^1(\Omega)} \leq \frac{C_2^2}{\widetilde{A}^2}|\chi(t)|^2 \to 0,$$

at an exponential rate, as $t \to \infty$.

□

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References

[1] Adams R.A., *Sobolev Spaces*. Academic Press, New York, 1975.
[2] Altaf M. U., Titi E.S., Gebrael T., Knio O., Zhao L., McCabe M.F., Hoteit I., *Downscaling the 2D Bénard convection equations using continuous data assimilation*. arXiv:1512.04671, (2015).
[3] Azouani A., Olson E., Titi E.S., Continuous data assimilation using general interpolant observables, J. Nonlinear Sci., 2014, 24(2), 277-304.
[4] Bessaih H., Olson E., Titi E.S., Continuous assimilation of data with stochastic noise, Nonlinearity, 2015, 28, 729-753.
[5] Cao C., Titi E.S., Global well-posedness and finite-dimensional global attractor for a 3-D planetary geostrophic viscous model, Communications on Pure and Applied Mathematics, 2003, 56, 198-233.
[6] Cao C., Titi E.S., Ziane M., Global well-posedness and finite-dimensional global attractor for a 3-D planetary geostrophic viscous model, Nonlinearity, 2004, 17, 1749-1776.
[7] Charney J., Halem J., Jastrow M., Use of incomplete historical data to infer the present state of the atmosphere, Journal of Atmospheric Science, 1969, 26, 1160-1163.
[8] Errico R., Baumhefner D., Predictability experiments using a high-resolution limited-area model, Monthly Weather Review, 1986, 115, 488-505.
[9] Evans L.C., Partial Differential Equations, Graduate Studies in Mathematics, Second Edition, 2010, 19.
[10] Farhat A., Jolly M.S., Titi E.S., Continuous data assimilation for the 2D Bénard convection through velocity measurements alone, 2015, 303, Physica D, 59-66.
[11] Farhat A., Lunasin E., Titi E.S., Abridged continuous data assimilation for the 2D Navier-Stokes equations utilizing measurements of only one component of the velocity field, J. Math. Fluid Mech., 2015, 18(1), 1-23.
[12] Farhat A., Lunasin E., Titi E.S., Data assimilation algorithm for 3D Bénard convection in porous media employing only temperature measurements, Jour. Math. Anal. Appl., 2016, 438(1), 492-506.
[13] Farhat A., Lunasin E., Titi E.S., Continuous data assimilation for a 2D Bénard convection system through horizontal velocity measurements alone. arXiv: 1602.00042 [math.AP], (2015).
[14] Foias C., Mondaini C., Titi E.S., A discrete data assimilation scheme for the solutions of the 2D Navier-Stokes equations and their statistics, SIAM J. on Appl. Dyn. Syst., 2016, 15(4), 2109-2142.

[15] Gesho M., Olson E., Titi E.S., A computational study of a data assimilation algorithm for the two-dimensional Navier–Stokes equations, Communications in Computational Physics, 2016, 19,1094-1110.

[16] Ghil M., Shkoller B., Yangarber V., A balanced diagnostic system compatible with a barotropic prognostic model., Mon. Wea. Rev, 1977, 105, 1223-1238.

[17] Ghil M., Halem M., Atlas R., Time-continuous assimilation of remote-sounding data and its effect on weather forecasting, Mon. Wea. Rev, 1978, 107, 140-171.

[18] Hayden K., Olson E., Titi E.S., Discrete data assimilation in the Lorenz and $2D$ Navier-Stokes equations, Physica D, 2011, 240, 1416-1425.

[19] Hoke J., Anthes R., The initialization of numerical models by a dynamic relaxation technique, Monthy Weather Review, 1976, 104, 1551-1556.

[20] Hu C., Temam R., Ziane M., Regularity results for linear elliptic problems related to the primitive equations, Chin. Annal Math. 2002, 23(2), 277-292.

[21] Jones D.A., Titi E.S., Determining finite volume elements for the 2D Navier-Stokes equations, Physica D, 1992, 60, 165-174.

[22] Ladyzhenskaya O.A., The Boundary Value Problems of Mathematical Physics, Springer-Verlag, 1985.

[23] Law K.J.H., Sanz-Alonso D., Shukla A., Stuart A.M., Filter accuracy for the Lorenz 96 model: Fixed versus adaptive observation operators, Physica D, 2016, 235, 1-13.

[24] Lorenc A., Adams W., Eyre J., The treatment of humidity in ECMWF's data assimilation scheme, Atmospheric Water Vapor, Academic Press New York, 497-512.

[25] Pedlosky J., Geophysical Fluid Dynamics, Springer-Verlag, New York, 1987.

[26] Samelson R., *Coastal boundary conditions and the baroclinic structure of wind–driven continental shelf currents*, Journal of Physical Oceanography, 1997, 27, 2645-2662.

[27] Samelson R., Temam R., Wang S., Some mathematical properties of the planetary geostrophic equations for large scale ocean circulation, Applicable Analysis, 1998, 70, 147-173.

[28] Samelson R., Vallis G., A simple friction and diffusion scheme for planetary geostrophic basin models, J. Phys. Oceanogr., 1997, 27, 186-194

[29] Samelson R., Vallis G., Large–scale circulation with small diapycnal diffusion: The two–thermocline limit, J. Marine Res., 1997, 55, 223-275.

[30] Seidov D., *An intermediate model for large–scale ocean circulation studies*, Dynamics of Atmospheres and Oceans, 1996, 25, 25-55.

[31] Uss A.T., Sevcenko V.I., On the simultaneous regularizability of multidimensional boundary-value problems, Soviet Math. Dokl., 1975, 16, 1151-1154.

[32] Vallis G., Atmospheric and Oceanic Fluid Dynamics: Fundamentals and Large Scale Circulation. Cambridge University Press, 2006.

[33] Ziane M., Regularity results of Stokes type systems, Applicable Analysis, 1995, 58, 263-292.

[34] Ziemer W.P., Weakly Differentiable Functions: Sobolev Spaces and Functions of Bounded Variation, Springer-Verlag, New York, 1989.