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Revisiting the Charney Baroclinic Instability Problem and Point-jet Barotropic Instability Problem, Part II: Matched Asymptotic Expansions & Overreflection Without Delta-Functions

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Abstract: Baroclinic instability generates the cyclones and anticyclones of midlatitude weather. Charney developed the first effective theory for the infancy of this cyclogenesis in 1947. His linear eigenproblem is analytically solvable by confluent hypergeometric functions. It is also, with extension of the domain of the coordinate from $[0, \infty]$ to $[-\infty, \infty]$ by reflection about the origin, the point-jet model of barotropic instability, important for tropical cyclogenesis. (Note that the coordinate is height z in the Charney model, but latitude y for the point-jet barotropic instability. It is a great simplification that the Charney and point-jet instability problems are mathematically identical, but it also is confusing that the mathematical analysis in y also applies to the Charney problem with the substitution of z for y .)

Unfortunately, the theory is full of distributions like the Dirac delta-function and the reflected Charney eigenfunction has a discontinuous first derivative at $y = 0$. Here we regularize the Charney problem by replacing a linear mean current, $U = |y|$, by either $U = \epsilon \log(\cosh(y/\epsilon))$ or $U = \epsilon y \operatorname{erf}(y/\epsilon)$, followed by matched asymptotic perturbation expansions in powers of the small regularization parameter ϵ . The series is carried to third order because the lowest nonzero correction to the phase speed is $O(\epsilon^2)$ and this correction is determined simultaneously with the third order approximation to the eigenfunction. The result is both an explicit, analytic regularization of a problem important in atmospheric and ocean dynamics, but also a good school problem because the series is explicit with nothing worse than polylogarithms and confluent hypergeometric functions. The primary meteorological conclusion is that the delta functions in the Charney problem are harmless as demonstrated both by third order perturbation theory and by spectrally-accurate numerical solutions. The physics of the regularized Charney problem is not significantly changed from that of the original Charney problem.

Keywords: baroclinic instability; singular perturbation; barotropic instability

MSC: 76E20; 86A10; 35B25; 33C15

"The argument which demands no horizontal variation of potential temperature at the boundary but a finite gradient an infinitesimal distance above I find most disquieting. It is, for example, difficult to escape the conclusion that since N^2 is continuous there must be large areas of static instability in this boundary layer. In addition such a velocity profile is itself in a sense "critical" for (as reported in Green (1960), equation 11) an additional baroclinic wave may exist if the horizontal temperature gradient changes sign a finite,

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possibly small, height above the boundary.”

John S. A. Green, in comments on Francis P. Bretherton’s papers “Critical layer instability in baroclinic flows” and “Baroclinic instability and the short wavelength cutoff in terms of potential vorticity”. Published in the discussion section of the Quarterly Journal Of the Royal Meteorological Society, page 552 (1967) [12].

1 Introduction

In recent years, two quite different conceptual models have competed to explain barotropic and baroclinic instability. Wave Overreflection Theory, usually abbreviated as OR, depends upon a configuration of turning points and critical levels that allow waves to grow as normal modes like light bouncing back and forth between the silvered ends of a laser. Richard Lindzen and his students were the most enthusiastic contributors to OR theory, mostly in the 80’s [1, 19, 21–25, 32, 33, 33, 34, 36]. The very different Counter-propagating Rossby Wave (CRW) paradigm depends on a resonant interaction between a *pair* of Rossby waves propagating in different directions [7, 13–17, 26, 27]. The two conceptual perspectives are not mutually exclusive, but are quite dramatically different. The Orr mechanism plays a significant role in both theories of exponentially-growing instability [19], but in its original form [2, 10, 28] is a theory of the transient growth of the continuous spectrum, followed by asymptotic decay for large times. It is important to understand these mechanisms because baroclinic instability generates the continent-sized storms that constitute most of the large-scale weather in the middle latitudes; barotropic instability of the East African jet initiates most Atlantic hurricanes.

The prototype of all baroclinic instability models is the “Charney problem”, described in detail below, which is an eigenproblem for waves linearized about a mean flow which varies linearly with height [9, 11, 18]. Schoeberl and Lindzen showed that a barotropic flow with linear mean shear in latitude is mathematically isomorphic to the Charney problem of baroclinic instability if the intervals of linear shear are joined by an infinitesimal layer in which the relative vorticity is a *Dirac delta function* and the *shear* of the mean current U is discontinuous. Although the underlying physics of baroclinic and barotropic instability is quite different, the Charney problem is a key to both.

This paper is the second in a multipart series, following [6], which hopes to create a better synthesis of the OR, CRW and Orr paradigms by revisiting and extending simple, classic problems with modern analytical and computational technologies.

One flaw in current theories of barotropic and baroclinic instability is that both OR and CRW are inordinately fond of delta functions. Here, we use the method of matched asymptotic expansions to show that the Charney problem can be regularized to delete the delta functions and also, it turns out, show that the infinitesimal layer results of earlier work are correct in the limit that the layer depth goes to zero.

Our tool is the method of matched asymptotic expansions. The Charney eigenrelation falls out at first order. Nevertheless, we have extended the expansion to third order for three reasons. First, some perturbation series have logarithms of the perturbation parameter appear at higher order, or may break down entirely, as common in the method of strained coordinates. The regularity of the third order approximation strongly suggests that nothing unconventional happens at any order. Second, the lowest non trivial correction to the phase speed c is at *second order*, but is determined in the calculation of the *third order* eigenfunction. Third, this is a good pedagogical example because it is a rare problem where one can explicitly go to third order while encountering only polylogarithms and confluent hypergeometric functions.

We explicitly employ two different regularized mean current profiles $U(y)$. Both are smoothed versions of the function $U(y) = |y|$ and are chosen for smoothness and simplicity rather than observational curve-fitting. Both profiles give solutions to the wave equation which are analytic for all real y . The $\log(\cosh)$ function has singularities in the complex plane and asymptotes to $|y|$ for large $|y|$ as $U \sim |y| + O(\exp(-2|y|))$. The linear-erf function is an entire function that asymptotes to $|y|$ for large $|y|$ as $U \sim |y| + O(\exp(-y^2))$.

Regularization has been previously considered by Bretherton [8] and Lindzen and Tung [24]. Without performing either numerical or perturbative calculations these authors noted that a delta function in potential

vorticity can be removed by smoothing the linear variation of the mean flow so that the shear at the ground is zero. Lindzen, Farrell and Tung [21] number-crunched to show that it is possible to completely eliminate the instability by curving the wind profile so that U_z is zero at the ground. They also did not derive any perturbative expansions.

2 The Charney Model of Baroclinic Instability

The vertical structure function $u(z)$ where u is the *streamfunction* and not the zonal velocity, is

$$u_{zz} + \left\{ \frac{r}{z-c} - \frac{1}{4} \right\} u = 0, \quad z \in [0, \infty] \quad (1)$$

$$cu_z + u = 0 \text{ at } z = 0, \quad u(z) \rightarrow 0 \text{ as } z \rightarrow \infty \quad (2)$$

The complete streamfunction is the product of a trigonometric function in latitude multiplied by $\exp(ik(x-ct))$ where c is the phase speed and k is the zonal wavenumber. The parameter r is the nondimensionalized beta parameter (the latitudinal derivative of the Coriolis parameter).

This equation arises through separation-of-variables from a special case of the general equation for a wave, linearized about a mean zonal wind $U(y, z)$, in the quasi-geostrophic approximation, which is given as (6.70) and (13.42) of the textbooks by Vallis [35] and Lindzen [20], respectively. Good reviews of quasi-geostrophic instability can also be found in the text by Pedlosky [29] and the review by Pierrehumbert and Swanson [31]. The history of Charney's own work is described in [30].

Kuo gave an alternative form which is more convenient for analysis [18]:

$$u_{\eta\eta} + \left\{ \frac{r}{\eta} - \frac{1}{4} \right\} u = 0 \quad (3)$$

$$\eta u_{\eta} - u = 0 \text{ at } \eta = \eta_0, \quad u(\eta) \rightarrow 0 \text{ as } \eta \rightarrow \infty \quad (4)$$

where η_0 is the unknown eigenparameter where the solution which is bounded as $\eta \rightarrow \infty$ satisfies the boundary condition. The coordinates are related by

$$\eta \equiv z - c \quad \leftrightarrow \quad z = \eta + c \quad (5)$$

where c is the complex-valued phase speed. Note that at the ground $\eta = -c = \eta_0$, which translates the boundary condition of the first form into the boundary condition of the second form.

Instabilities occur for all $r > 0$ except when r is an integer, but the primary branch, the fastest-growing waves, is on $r \in [0, 1]$, so we shall concentrate on this range. The eigenvalues for this parametric interval is shown in Fig. 1.

The solution to the Kuo form which decays exponentially with increasing z (or η) so as to be bounded as $|z| \rightarrow \infty$ is the one-parameter Whittaker function. Denoting the confluent hypergeometric by $M(a, b; z)$ and $U(a, b; z)$,

$$W_{r,1/2}(\eta) = \exp(-\eta/2) \eta U(1-r, 2, \eta) \quad (6)$$

The second linearly independent solution is

$$M_{r,1/2}(\eta) = \exp(-\eta/2) \eta M(1-r, 2, \eta) \quad (7)$$

The M-function is analytic at $\eta = 0$, which is the location of the pole in the coefficient of the differential equation. This is in contrast to the W-function, which is logarithmically singular at $\eta = 0$. The M-function grows exponentially as $z \rightarrow \infty$ and so must be rejected as a solution in its own right, but ironically is still needed because the coefficient of the logarithm in the Whittaker W-function is none other than a constant times the Whittaker M-function!

Note that η_0 is always the *negative* of the corresponding phase speed c . The power series in the Whittaker functions have *infinite* radii of convergence:

$$M(1-r, 2, \eta) = \sum_{m=0}^{\infty} \frac{(1-r)_m}{(2)_m} \frac{\eta^m}{m!} \quad (8)$$

$$\begin{aligned} \Gamma(1-r) U(1-r, 2, \eta) &= \frac{1}{\eta} - r M(1-r, 2, \eta) \{\log |\eta| - \alpha \pi i\} \\ &\quad - r \sum_{m=0}^{\infty} \frac{(1-r)_m}{(2)_m} \frac{\eta^m}{m!} \{\psi(1-r-m) - \psi(1+m) - \psi(2+m)\} \\ (x)_m &\equiv x(x+1) \dots (x+m-1) \end{aligned} \quad (9)$$

where ψ is the digamma function.

The branch cut is in the *upper* η -plane implying that $\alpha = 1$ in (9). The logarithmic part of the U-function is

$$\begin{aligned} U^{\log}(1-r, 2, \eta) &= -\frac{r}{\Gamma(1-r)} \frac{1}{\eta} M(1-r, 2, \eta) \log(\eta) \\ &= -\frac{r}{\Gamma(1-r)} \frac{1}{\eta} M(1-r, 2, \eta) \{\log |\eta| - \pi i\} \end{aligned} \quad (10)$$

Figure 1: Real and imaginary parts for the phase speed c in the Charney problem.

3 The Point-Jet Model of Barotropic Instability

Suppose that the mean current is

$$U(y) = |y|, \quad y \in [-\infty, \infty] \quad [\text{“Point-Jet”}] \quad (11)$$

where y is latitude. Lindzen, Rosenthal and Farrell showed that the barotropic instability of the “point-jet” in an unbounded flow is *mathematically* identical with Charney’s eigenproblem [23]. (The Charney problem is defined on a semi-infinite domain; strictly speaking, therefore, the solution to the barotropic point jet instability is the Charney eigenfunction $\psi(y)$, $y = [0, \infty]$ plus its reflection about $y = 0$, $\psi(-y) = \psi(y)\forall y$). The Charney problem and the point jet barotropic instability are not *physically* isomorphic because baroclinic instability is a transfer of available potential energy whereas barotropic instability is an exchange of kinetic energy.

Nevertheless, this mathematical equivalence greatly strengthens the case for Lindzen’s sweeping view of overreflection as the primary mechanism of both barotropic and baroclinic instability.

Figure 2: Solid: The point-jet mean current $U(y) = |y|$. Dashed; the regularized mean flow $U(y) = \epsilon[\log(\cosh(y/\epsilon)) - \log(2)]$ for $\epsilon = 1$.

4 Regularizing the Point-Jet and Charney problems

The mean vorticity of the point-jet is a Dirac delta function. It is obviously desirable to remove the singularity. (Students, I have found in forty-one years of teaching, are particularly unhappy with infinitesimally thin, infinitely large quantities.) To regularize the point jet, we replace the piecewise-linear mean wind by a smooth function with the following properties where ϵ is a small positive parameter:

1.
$$U(y) \sim y + \text{exponentially fast in } (y/\epsilon), \quad |y| \rightarrow \infty \quad (12)$$

2.
$$U(-y) = U(y) \forall y \text{ [Symmetry with respect to the origin]} \quad (13)$$

Two specific instances of this class are the “log(cosh)” jet defined by

$$U(y; \epsilon) = \epsilon \{ \log(\cosh(y/\epsilon)) + \log(2) \} \quad (14)$$

$$\sim |y|, \quad |y| \gg \epsilon \quad (15)$$

and the “linear-erf” jet defined by

$$U(y; \epsilon) = \epsilon \{ (y/\epsilon) \operatorname{erf}(y/\epsilon) \} \quad (16)$$

$$\sim |y|, \quad |y| \gg \epsilon \quad (17)$$

Fig. 2 compares the piecewise linear mean flow with the regularized current that will replace it. In the limit that the parameter ϵ tends to zero, the $\log(\cosh)$ profile reverts to the point jet. For all nonzero ϵ , the infinite vorticity is smeared out into a layer of thickness $O(\epsilon)$.

There are many functions that share the same limiting and approximation properties. Our choice of the logarithm of the hyperbolic cosine and the linear-erf were motivated by the fact that the derivatives of both are known explicitly, which means that the coefficients of the regularized differential equation will be explicit elementary functions also; in addition, all the integrals are known analytically as cataloged in Table 1 and Table 2, which is very helpful in applying perturbation theory. A further advantage is that both functions tends *exponentially* fast to the point-jet as $|y| \rightarrow \infty$. When ϵ is small, the differential equation close to Charney’s except in a region of width $O(\epsilon)$ around the ground (in the baroclinic instability interpretation) or the discontinuity at the maximum of the point-jet (in the barotropic instability application).

Our chosen analytical method is to apply matched asymptotic expansions in powers of ϵ . The narrow regularization interval around the origin is the “inner region” in the parlance of perturbation theory; the rest of the domain is the “outer region” where the approximation is a confluent hypergeometric function to all orders, just as in the original Charney problem.

We will treat general $U(y)$ first and then specialize to the “ $\log(\cosh)$ ” and “linear-erf” jets later.

The regularized problem is, introducing the inner coordinate $Y \equiv y/\epsilon$,

$$u_{yy} + \left\{ \frac{r - \epsilon^{-1} \tilde{U}_{YY}(y/\epsilon)}{\epsilon \tilde{U}(y/\epsilon) - c} - 1/4 \right\} u = 0, \quad u_y(0) = 0, \quad u(\pm\infty) = 0 \quad (18)$$

where $U(y/\epsilon; \epsilon) \equiv \epsilon \tilde{U}(y/\epsilon)$. Since the ODE is symmetric about $Y = 0$, it is easy to show by the parity analysis of Chapter 8 of [4] that all the eigenmodes are either symmetric or antisymmetric with respect to the origin. We shall focus on symmetric modes only as consistent with the boundary condition $u_y(0) = 0$.

Antisymmetric instability modes are also possible. However, these will not be analyzed here since the symmetric modes suffice to analyze OR and CRW concepts.

The eigenfunctions need to be normalized; we shall impose

$$u(0) = 1 \quad (19)$$

which is a condition that must be satisfied by the *inner* solution.

5 Matched Asymptotic Expansions

The key assumption is

$$\epsilon \ll 1 \quad (20)$$

The regularized eigenfunction will be well-approximated by the Whittaker function except in a thin boundary layer around $y = 0$, the “inner region”.

5.1 Outer Solution

By assumption, the mean wind $\epsilon U(y/\epsilon)$ tends *exponentially* fast to the point-jet as $|y| \rightarrow \infty$. This implies that there will be no corrections in powers of ϵ to the coefficients of the differential equation in the outer region. Therefore, the outer equation to *all orders* is

$$u_{yy} + \left\{ \frac{r}{y - c} - 1/4 \right\} u = 0 \quad (21)$$

However, note that the phase speed c has $O(\epsilon^2)$ corrections. The outer solution is

$$u^{outer}(y) = \mathfrak{A} \frac{W_{r,1/2}(y-c)}{W_{r,1/2}(-c_0)} \quad (22)$$

where the denominator $W_{r,1/2}(-c_0)$ is convenient because then $\lim_{\epsilon \rightarrow 0} u^{outer}(y=0) = \mathfrak{A}$. Both the amplitude $\mathfrak{A}$ and c are functions of ϵ and therefore must be formally expanded as

$$c \sim c_0 + c_1\epsilon + c_2\epsilon^2 + c_3\epsilon^3 + \dots \quad (23)$$

$$\mathfrak{A} \sim P_0 + P_1\epsilon + P_2\epsilon^2 + P_3\epsilon^3 + \dots \quad (24)$$

It is convenient to define

$$w(y; r) = W_{r,1/2}(y) \quad (25)$$

where $W_{r,k}(z)$ is the standard Whittaker function. We shall usually suppress the parameter r for notational simplicity. Furthermore, we shall adopt the convention that whenever w or one of its y -derivatives is written *without arguments*, the implied argument is $-c_0$, corresponding to $y = 0$. Thus,

$$w \leftrightarrow w(-c_0) = W_{r,1/2}(-c_0) \quad (26)$$

$$w_y \leftrightarrow w_y(-c_0) \quad (27)$$

and so on.

5.2 Inner Approximation

5.2.1 The inner problem

The inner coordinate Y is defined by $y = \epsilon Y$.

The inner equation is

$$u_{YY} + \left\{ \frac{r\epsilon^2 - \epsilon U_{YY}(Y)}{\epsilon U(Y) - c} - (1/4)\epsilon^2 \right\} u = 0 \quad (28)$$

and the boundary conditions are $u_Y(0) = 0$ (to impose symmetry), and $u(0) = 1$ (arbitrary normalization of the eigenfunction).

$$u(Y) = u_0^i(Y) + u_1^i(Y)\epsilon + u_2^i(Y)\epsilon^2 + \dots \quad (29)$$

$$c = c_0 + c_1\epsilon + c_2\epsilon^2 + \dots \quad (30)$$

5.2.2 The inner solution

At lowest order, we have, after imposing the boundary conditions, $u_0^i(0) = 1$,

$$u_{0,YY}^i = 0, \quad \rightarrow u_0^i(Y) = 1 \quad (31)$$

The first-order equation is

$$u_{1,YY}^i = -\frac{1}{c_0} U_{YY}(Y), \quad u_1(0) = 0, u_{1,Y}(0) = 0 \quad (32)$$

which is solved by

$$u_1^i(Y) = -\frac{1}{c_0} (U(Y) - U(0)) \quad (33)$$

The second order equation and second-order solution are

$$u_{2,YY}^i = \frac{1}{4} + \frac{r}{c_0} + \frac{c_1/c_0}{c_0} U_{YY}(Y) \quad (34)$$

$$u_2^i = \left(\frac{1}{8} + \frac{r}{2c_0} \right) Y^2 + \left\{ \frac{c_1 - U(0)}{c_0^2} \right\} \{U(Y) - U(0)\} \quad (35)$$

At third order,

$$u_{3,YY}^i = \frac{r U(0)}{c_0^2} + \left\{ \frac{c_2}{c_0^2} - \frac{U(0)^2}{c_0^3} \right\} U_{YY}(Y) - \frac{1}{4c_0} (U(Y) - U(0)) - \frac{1}{c_0} \left\{ \frac{1}{8} + \frac{r}{2c_0} \right\} Y^2 U_{YY} \quad (36)$$

It is convenient to define $V(Y)$ as the iterated integral of the mean wind, more precisely,

$$V_{YY} = U(Y), \quad V(0) = V_Y(0) = 0 \quad (37)$$

$$\begin{aligned} u_3^i = & \frac{r U(0)}{2c_0^2} Y^2 + \left\{ \frac{c_2}{c_0^2} - \frac{U(0)^2}{c_0^3} \right\} (U(Y) - U(0)) - \frac{1}{4c_0} (V(Y) - (U(0)/2)Y^2) \\ & - \frac{1}{c_0} \left\{ \frac{1}{8} + \frac{r}{2c_0} \right\} \left\{ Y^2 U(Y) - 4YV_Y + 6V(Y) \right\} \end{aligned} \quad (38)$$

We need the following to derive the outer limit of the inner solution.

$$U(Y) \sim Y \quad (39)$$

$$V_Y \sim v_1 + Y^2/2 \quad (40)$$

$$V(Y) \sim v_0 + v_1 Y + Y^3/6 \quad (41)$$

where $V(Y)$ is the iterated integral of the mean wind defined by (37). Note that V_Y cannot include an asymptotic term that is linear in Y because this would imply that $U(Y) = (V_Y)_Y$ would asymptote to a constant plus a linear term, contrary to our assumptions on $U(Y)$. Similarly, V_Y cannot include a constant in its asymptotic form unless $V(Y)$ has a term linear in Y . $V(Y)$ cannot include a quadratic term because this would force a constant in $U(Y)$. The outer limit of the inner approximation is

$$u_0^{i,outer} = 1 \quad (42)$$

$$u_1^{i,outer} = -\frac{1}{c_0} Y + \frac{1}{c_0} U(0)$$

$$u_2^{i,outer} = \left(\frac{1}{8} + \frac{r}{2c_0} \right) Y^2 + \left\{ \frac{c_1 - U(0)}{c_0^2} \right\} Y - \left\{ \frac{c_1 - U(0)}{c_0^2} \right\} U(0)$$

$$u_3^{i,outer} = -\frac{1}{24} \frac{Y^3}{c_0} + \left(\frac{rU(0)}{2c_0^2} + \frac{U(0)}{8c_0} \right) Y^2 \quad (43)$$

$$+ \left(-2 \left(\frac{1}{8} + \frac{r}{2c_0} \right) \frac{v_1}{c_0} + \frac{c_2}{c_0^2} - \frac{(U(0))^2}{c_0^3} - \frac{v_1}{4c_0} \right) Y \quad (44)$$

$$- \left(\frac{c_2}{c_0^2} - \frac{(U(0))^2}{c_0^3} \right) U(0) - 6 \left(\frac{1}{8} + \frac{r}{2c_0} \right) \frac{v_0}{c_0} - \frac{v_0}{4c_0} \quad (45)$$

The inner limit of the outer solution is

$$\begin{aligned} u^{o,inner} \sim & P_0 + \epsilon \left\{ P_1 + P_0 \left(-\frac{w_y}{w} c_1 + \frac{w_y}{w} Y \right) \right\} \\ & + \epsilon^2 \left\{ P_2 + P_1 \left(-\frac{w_y}{w} c_1 + \frac{w_y}{w} Y \right) \right\} \\ & + P_0 \left(\left[\frac{1}{2} \frac{w_{yy}}{w} c_1^2 - \frac{w_y}{w} c_2 + \right] - \frac{w_{yy}}{w} c_1 Y + \frac{1}{2} \frac{w_{yy}}{w} Y^2 \right) + P_1 \Big\} + \dots \end{aligned} \quad (46)$$

6 Term-by-Term Matching

6.1 Overview of Matching the Inner & Outer Solutions

When $Y \gg 1 \Leftrightarrow y \gg 1/\epsilon$, the exponentially decaying terms are negligible and the inner approximation, truncated at any finite order m in ϵ , is a polynomial in Y . If $y \ll 1$, the outer approximation is accurately approximated by its power series in y . There is thus an overlap region, $\epsilon \ll y \ll 1$, where both approximations are simultaneously valid. Note that in the overlap region, the exponentials have decayed so that the inner approximation is just a polynomial in Y and ϵ . Simultaneous validity requires that the coefficients of the power series from the outer approximation and the polynomials that are the inner approximation must match, term by term, so that the approximations agree simultaneously not merely at *points*, but on an *interval*.

The inner limit of the outer solution is just a bivariate power series in ϵ and y with coefficients we shall denote by p_{jk} :

$$\text{Inner limit}(u_{\text{outer}}) \sim \sum_{j=0} \sum_{k=0} p_{jk} \epsilon^j y^k \quad (47)$$

Before conversion to the outer variable y , the inner solution is the bivariate power series with coefficients denoted by q_{jk} :

$$\text{Outer limit}(u_{\text{inner}}) \sim \sum_{j=0} \sum_{k=0} q_{jk} \epsilon^j Y^k \quad (48)$$

Recall that $y = \epsilon Y \Leftrightarrow Y = y/\epsilon$. Then

$$\text{OuterLimit}(u_{\text{inner}}) \sim \sum_{j=0} \sum_{k=0}^j q_{jk} \epsilon^{j-k} y^k \quad (49)$$

It follows that in the overlap region at $O(\epsilon^m)$,

$$\sum_{k=0}^m p_{m-k,k} y^k = \sum_{k=0} q_{mk} y^k \quad (50)$$

which yields the matching condition

$$p_{m-k,k} = q_{mk}, \quad k = 0, 1, \dots, m \quad (51)$$

If $k = m + 1$, matching requires $p_{-1,k}$, a coefficient of $1/\epsilon$ in the outer solution. If this coefficient is zero, then the inner solution must be a polynomial in Y of at most degree m , or in other words q_{mk} is a triangular array in the sense that

$$q_{m,k} = 0, \quad \forall k > m, \quad m = 0, 1, \dots \quad (52)$$

The p_{jk} up to third order are explicitly

$$O(1) : \begin{cases} p_{00} = P_0 \\ p_{01} = \frac{w_y}{w} = -\frac{1}{c_0} \\ p_{02} = \frac{1}{2} \frac{w_{yy}}{w} = \frac{r}{2c_0} + \frac{1}{8} \\ p_{03} = \frac{1}{6} \frac{w_{yyy}}{w_y} \end{cases} \quad (53)$$

$$O(\epsilon) : \begin{cases} p_{10} = P_1 - c_1 \frac{w_y}{w} \\ p_{11} = P_1 \frac{w_y}{w} - c_1 \frac{w_{yy}}{w} \\ p_{12} = P_1 \frac{1}{2} \frac{w_{yy}}{w} - c_1 \frac{1}{2} \frac{w_{yyy}}{w} \\ p_{13} = P_1 \frac{1}{6} \frac{w_{yyy}}{w_y} - c_1 \frac{1}{6} \frac{w_{yyyy}}{w} \end{cases} \quad (54)$$

$$O(\epsilon^2) : \begin{cases} p_{20} = P_2 - P_1 c_1 \frac{w_y}{w} + \frac{1}{2} c_1^2 \frac{w_{yy}}{w} - c_2 \frac{w_y}{w} \\ p_{21} = P_2 \frac{w_y}{w} - c_1 P_1 \frac{w_{yy}}{w} + c_1^2 \frac{1}{2} \frac{w_{yyy}}{w} - c_2 \frac{w_{yy}}{w} \\ p_{22} = P_2 \frac{1}{2} \frac{w_{yy}}{w} - P_1 c_1 \frac{1}{2} \frac{w_{yyy}}{w} + c_1^2 \frac{1}{4} \frac{w_{yyyy}}{w} - c_2 \frac{1}{2} \frac{w_{yyy}}{w} \\ p_{23} = P_2 \frac{1}{6} \frac{w_{yyy}}{w} - P_1 c_1 \frac{1}{6} \frac{w_{yyyy}}{w} + c_1^2 \frac{1}{12} \frac{w_{5y}}{w} - c_2 \frac{1}{6} \frac{w_{yyyy}}{w} \end{cases} \quad (55)$$

$$O(\epsilon^3) : \begin{cases} p_{30} = P_3 - P_2 c_1 \frac{w_y}{w} + P_1 \frac{1}{2} c_1^2 \frac{w_{yy}}{w} - P_1 c_2 \frac{w_y}{w} - \frac{1}{6} c_1^3 \frac{w_{yyy}}{w} + c_1 c_2 \frac{w_{yy}}{w} - c_3 \frac{w_y}{w} \\ p_{31} = P_3 \frac{w_y}{w} - P_2 c_1 \frac{w_{yy}}{w} + P_1 \frac{1}{2} c_1^2 \frac{w_{yyy}}{w} - P_1 c_2 \frac{w_{yy}}{w} - \frac{1}{6} c_1^3 \frac{w_{yyyy}}{w} + c_1 c_2 \frac{w_{yyy}}{w} - c_3 \frac{w_{yy}}{w} \\ p_{32} = P_3 \frac{1}{2} \frac{w_{yy}}{w} - P_2 \frac{1}{2} c_1 \frac{w_{yyy}}{w} + P_1 \frac{1}{4} c_1^2 \frac{w_{4y}}{w} - P_1 c_2 \frac{1}{2} \frac{w_{yyy}}{w} - \frac{1}{12} c_1^3 \frac{w_{5y}}{w} + c_1 c_2 \frac{1}{2} \frac{w_{4y}}{w} - c_3 \frac{1}{2} \frac{w_{yyy}}{w} \\ p_{33} = P_3 \frac{1}{6} \frac{w_{yyy}}{w} - P_2 \frac{1}{6} c_1 \frac{w_{4y}}{w} + P_1 \frac{1}{12} c_1^2 \frac{w_{5y}}{w} - P_1 c_2 \frac{1}{6} \frac{w_{4y}}{w} - \frac{1}{36} c_1^3 \frac{w_{6y}}{w} + c_1 c_2 \frac{1}{6} \frac{w_{5y}}{w} - c_3 \frac{1}{6} \frac{w_{4y}}{w} \end{cases} \quad (56)$$

The pertinent coefficients of the bivariate power series for the *inner* solution are the following:

$$O(1) : \quad q_{00} = 1 \quad (57)$$

$$O(\epsilon) : \begin{cases} q_{10} = -\frac{1}{c_0} U(0) \\ q_{11} = -\frac{1}{c_0} \end{cases} \quad (58)$$

$$O(\epsilon^2) : \begin{cases} q_{20} = -\left\{ \frac{c_1 - U(0)}{c_0^2} \right\} U(0) \\ q_{21} = \frac{c_1 - U(0)}{c_0^2} \\ q_{22} = \frac{1}{8} + \frac{r}{2c_0} \end{cases} \quad (59)$$

$$O(\epsilon^3) : \begin{cases} q_{30} = -\frac{v_0}{c_0} - 3r \frac{v_0}{c_0^2} - \frac{c_2}{c_0^2} U(0) + \frac{U(0)^3}{c_0^3} \\ q_{3,1} = -\frac{v_1}{2c_0} - r \frac{v_1}{c_0^2} + \frac{c_2}{c_0^2} - \frac{(U(0))^2}{c_0^3} \\ q_{3,2} = U(0) \left(\frac{1}{8c_0} + \frac{r}{2c_0^2} \right) \\ q_{3,3} = -\frac{1}{24c_0} \end{cases} \quad (60)$$

6.2 Zeroth Order Matching

The match of lowest order terms $p_{00} = q_{00}$ gives, recalling that $p_{00} = P_0$ and $q_{00} = 1$, requires that

$$P_0 = 1 \quad (61)$$

where P_0 is the leading coefficient in the amplitude constnat that appears in the outer approximation, $\mathfrak{A}(\epsilon) = P_0 + \epsilon P_1 + \epsilon^2 P_2 + \dots$

6.3 First Order Matching

The first order terms in Y gives

$$u_1^{o,inner} = P_1 + \left(-\frac{w_y}{w} c_1 + \frac{w_y}{w} Y \right) \quad (62)$$

$$u_1^{i,outer} = -\frac{1}{c_0} Y + \frac{U(0)}{c_0} \quad (63)$$

The terms proportional to Y yield

$$\frac{w_y}{w} = -\frac{1}{c_0} \quad (64)$$

This is equivalent to, restoring the arguments to w and w_y from (26) and (27),

$$c_0 w_y(-c_0) + w(-c_0) = 0 \quad (65)$$

In the coordinate $\eta = y - c$, we can define η_0 as the value of η corresponding to $y = 0$ when c is approximated by c_0 :

$$\eta_0 = -c_0 \quad (66)$$

Recalling that $d/d\eta = d/dy$, we can rewrite the matching condition as

$$-\eta_0 w_\eta(\eta_0) + w(\eta_0) = 0 \quad (67)$$

This is the usual Charney eigencondition.

We also have the first order match of the parts of the first order solutions which are independent of Y in the overlap region.

$$P_1 - \frac{w_y}{w} c_1 = \frac{U(0)}{c_0} \quad (68)$$

Substituting in (64), $w_y/w = -1/c_0$, gives

$$P_1 = \frac{U(0)}{c_0} - \frac{c_1}{c_0} \quad (69)$$

Matching at *second order* will force $c_1 = 0$, but we will keep c_1 as an unknown for the moment.

6.4 Second Order Matching

Invoking $P_0 = 1$ and

$$P_1 = \frac{U(0)}{c_0} - \frac{c_1}{c_0} \quad (70)$$

we obtain

$$\begin{aligned} u_2^{o,inner} &= P_2 + \left[\frac{U(0)}{c_0} - \frac{c_1}{c_0} \right] \left(-\frac{w_y}{w} c_1 + \frac{w_y}{w} Y \right) \\ &+ \frac{1}{2} \frac{w_{yy}}{w} c_1^2 - \frac{w_y}{w} c_2 - \frac{w_{yy}}{w} c_1 Y + \frac{1}{2} \frac{w_{yy}}{w} Y^2 \end{aligned}$$

which collecting powers of Y is the inner limit of the second order outer term

$$\begin{aligned} u_2^{o,inner} &= P_2 + \left[\frac{U(0)}{c_0} - \frac{c_1}{c_0} \right] \left(-\frac{w_y}{w} c_1 \right) \\ &+ \frac{1}{2} \frac{w_{yy}}{w} c_1^2 - \frac{w_y}{w} c_2 + \left\{ \left[\frac{U(0)}{c_0} - \frac{c_1}{c_0} \right] \frac{w_y}{w} - \frac{w_{yy}}{w} c_1 \right\} Y + \frac{1}{2} \frac{w_{yy}}{w} Y^2 \end{aligned}$$

The outer limit of the inner solution is

$$u_2^{i,outerlimit} = \left(\frac{1}{8} + \frac{r}{2c_0} \right) Y^2 + \left\{ \frac{c_1 - U(0)}{c_0^2} \right\} Y - \left\{ \frac{c_1 - U(0)}{c_0^2} \right\} U(0) \quad (71)$$

Matching the terms quadratic in Y yields

$$\frac{r}{2c_0} + \frac{1}{8} \sim \frac{1}{2} \frac{w_{yy}}{w} \quad (72)$$

However, the confluent hypergeometric equation evaluated at $y = 0$ is

$$w_{yy} = (r/c_0 + 1/4)w \quad \Leftrightarrow \quad \frac{1}{2} \frac{w_{yy}}{w} = \frac{r}{2c_0} + \frac{1}{8} \quad (73)$$

It follows that this matching condition is automatically fulfilled.

Matching the terms in Y requires, with the outer limit of the inner solution on the left,

$$\begin{aligned} \frac{c_1}{c_0^2} - \frac{U(0)}{c_0^2} &\sim \frac{c_1}{c_0^2} - \frac{U(0)}{c_0^2} - c_1 \frac{w_{yy}}{w} \\ 0 &\sim -c_1 \frac{w_{yy}}{w} \end{aligned} \quad (74)$$

where the second line follows by subtracting identical terms $c_1/c_0^2 - U(0)/c_0^2$ from both sides of the first line. Since w_{yy} is zero only at isolated points, the match requires that the first order phase speed correction is zero:

$$c_1 = 0 \quad (75)$$

Setting $c_1 = 0$ and showing just terms independent of Y :

$$u_2^{o, inner; Y\text{-independent}} = P_2 - \frac{w_y}{w} c_2 + \frac{U(0)}{c_0}$$

Substituting in (64), $w_y/w = -1/c_0$, gives

$$u_2^{o, inner; Y\text{-independent}} = P_2 + \frac{c_2}{c_0}$$

Matching the terms independent of Y yields

$$-\left\{ \frac{c_1 - U(0)}{c_0^2} \right\} U(0) \sim P_2 + \frac{1}{c_0} c_2 \quad (76)$$

Applying results of two previous matches, $c_1 = 0$ and $w_{yy} = -\left(\frac{r}{c_0} + \frac{1}{4}\right)w$, this simplifies to

$$P_2 \sim \frac{U(0)^2}{c_0^2} - \frac{1}{c_0} c_2 \quad (77)$$

Note that c_2 is as yet undetermined. Just as the first order phase speed correction was determined by a *second* order matching condition, similarly c_2 will fall out of the third order matching in the next subsection.

6.5 Third Order Matching

The matches at $O(y^3)$ and $O(y^2)$ are trivially satisfied in the sense, after replacing c_1 and P_2 by their values determined at lower orders,

$$O(y^3): \quad p_{0,3} = q_{3,3} = -1/(24c_0) \quad (78)$$

$$O(y^2): \quad p_{1,2} = q_{3,2} = \frac{U(0)(c_0 + 4r)}{8c_0^2} \quad (79)$$

$$O(y): \quad p_{2,1} = -\frac{(U(0))^2}{c_0^2} + \frac{c_2}{c_0^2} - \frac{1}{4} c_2 - r \frac{c_2}{c_0} \quad (80)$$

$$q_{3,1} = -\frac{1}{2} \frac{v_1}{c_0} - r \frac{v_1}{c_0^2} + \frac{c_2}{c_0^2} - \frac{(U(0))^2}{c_0^3} \quad (81)$$

Solving the equation $p_{2,1} - q_{3,1} = 0$ gives the second order phase speed correction as

$$c_2 = v_1 \left(1 + 2 \frac{r}{c_0} \right) \frac{2}{c_0 + 4r} \quad (82)$$

The terms that are asymptotically y -independent are, using an overline to denote $p_{3,0}$ and $p_{3,0}$ multiplied by a denominator-clearing common factor of $c_0^3(c_0 + 4r)$,

$$\begin{aligned} O(y^0): \quad \bar{p}_{3,0} &= P_3 c_0^4 + 4 P_3 c_0^3 r + 2 U(0) v_1 c_0 + 4 U(0) v_1 r + c_3 c_0^3 + 4 c_3 c_0^2 r \\ \bar{q}_{3,0} &= -2 U(0) v_1 c_0 - 4 U(0) v_1 r + (U(0))^3 c_0 + 4 (U(0))^3 r - v_0 c_0^3 \\ &\quad - 7 v_0 c_0^2 r - 12 v_0 c_0 r^2 \end{aligned} \quad (83)$$

Matching gives

$$\begin{aligned} c_0^3(c_0 + 4r)P_3 &= (U(0))^3 c_0 + 4 (U(0))^3 r - 4 U(0) v_1 c_0 - 8 U(0) v_1 r \\ &\quad - v_0 c_0^3 - 7 v_0 c_0^2 r - 12 v_0 c_0 r^2 - c_3 c_0^3 - 4 c_3 c_0^2 r \end{aligned} \quad (84)$$

The value of the *third* order phase speed correction and also P_3 in the amplitude multiplying the outer solution are determined by matching at *fourth* order. Fortunately, these do not explicitly appear in the third order *inner* approximation, which therefore is complete.

7 Example One: Log(Cosh) Mean Wind

The mean wind is

$$U(y; \epsilon) = \epsilon \{ \log(\cosh(y/\epsilon)) + \log(2) \} \quad (85)$$

$$\sim |y|, \quad |y| \gg \epsilon \quad (86)$$

The regularized Charney problem is

$$u_{yy} + \left\{ \frac{r - \epsilon^{-1} \operatorname{sech}^2(y/\epsilon)}{\epsilon \{ \log(\cosh(y/\epsilon)) + \log(2) \} - c} - 1/4 \right\} u = 0, \quad u_y(0) = 0, \quad u(\pm\infty) = 0 \quad (87)$$

The outer approximation is independent-in-form of the mean wind; to all orders in ϵ , the outer solution is the Whittaker function. This depends on $U(y)$ through the phase speed c and the amplitude $\mathfrak{A}$:

$$u^0(y) = \mathfrak{A} \frac{W_{r,1/2}(y - c)}{W_{r,1/2}(-c_0)} \quad (88)$$

$$c \sim c_0 + \epsilon^2 \frac{\pi^2}{12} \left(1 + 2 \frac{r}{c_0} \right) \frac{1}{c_0 + 4r} + \dots \quad (89)$$

$$\mathfrak{A} \sim 1 + \frac{\log(2)}{c_0} \epsilon + P_2 \epsilon^2 + \dots \quad (90)$$

$$u_0^i(Y) = 1 \quad (91)$$

$$u_1^i(Y) = -\frac{1}{c_0} \log(\cosh(Y)) \quad (92)$$

$$u_2^i = \left(\frac{1}{8} + \frac{r}{2c_0} \right) Y^2 + A \left\{ \frac{c_1 - \log(2)}{c_0^2} \right\} \log(\cosh(Y)) \quad (93)$$

$$\begin{aligned}
u_3^i = & \frac{3\zeta(3)}{16c_0} \left\{ 1 + 3\frac{r}{c_0} \right\} - \frac{1}{8c_0} Y^2 \log(1 + \exp(2Y)) \left\{ 4\frac{r}{c_0} + 1 \right\} \\
& + \frac{\log(2)}{8c_0} Y^2 \left\{ 4\frac{r}{c_0} + 1 \right\} + \frac{\pi^2}{96c_0} \left\{ 4\frac{r}{c_0} + 1 \right\} Y \\
& - \frac{1}{4c_0} \left\{ 4\frac{r}{c_0} + 1 \right\} Y \operatorname{Li}_2(-\exp(2Y)) + \frac{1}{c_0^2} \left\{ c_2 - \frac{\log^2(2)}{c_0} \right\} \log(\cosh(Y)) \\
& + \frac{1}{32c_0} \operatorname{Li}_3(-\exp(-2Y)) + \frac{1}{32c_0} \left\{ 7 + 24\frac{r}{c_0} \right\} \operatorname{Li}_3(-\exp(2Y))
\end{aligned} \tag{94}$$

where Li_n is a polylogarithm function, defined by (112) in the first appendix.

Fig. 3 shows the different orders in the inner solution for the log-of-cosh regularization. Only a small range in the inner coordinate Y is shown because the regularized wind $\log(\cosh(Y))$ decays to a linear profile as $O(\exp(-2Y))$, and is thus within a quarter of a percent of its asymptote for $Y = 3$. Although the cubic growth of u_3^i is not obvious on the limited range shown,

$$u_n^i \sim \text{constant} |Y|^n, \quad |Y| \rightarrow \infty \tag{95}$$

The power-law growth of the inner order terms is not a pathology, but rather is essential so that the inner solution matches the inner limit of the outer solution, which is a power series in Y , or at leading order a power series in ϵY .

Figure 3: The lowest four orders of the inner solution (after matching) for $r = 1/2$ and $U(y) = \epsilon \log(2 \cosh(y/\epsilon))$. Solid: $u_0^i \equiv 1$. Long red dash: first order. Short green dash: u_2^i . Medium-length blue dash: third order.

8 Example Two: The Linear-Erf Jet

The “linear-erf” jet is defined by

$$U(y; \epsilon) = \epsilon \{ (y/\epsilon) \operatorname{erf}(y/\epsilon) \} \quad (96)$$

$$\sim |y|, \quad |y| \gg \epsilon \quad (97)$$

$$u_{yy} + \left\{ \frac{r - \epsilon^{-1} [4/\sqrt{\pi}] (1 - (y/\epsilon)^2) \exp(-(y/\epsilon)^2)}{y \operatorname{erf}(y/\epsilon) - c} - 1/4 \right\} = 0, \quad u_y(0) = 0, \quad u(\pm\infty) = 0 \quad (98)$$

$$u_0^i(Y) = 1 \quad (99)$$

$$u_1^i(Y) = -\frac{1}{c_0} Y \operatorname{erf}(Y) \quad (100)$$

$$u_2^i = \left(\frac{1}{8} + \frac{r}{2c_0} \right) Y^2 \quad (101)$$

$$u_3^i = \left\{ \frac{c_2}{c_0^2} \right\} (U(Y)) - \frac{1}{4c_0} (V(Y)) Y^2 - \frac{1}{c_0} \left\{ \frac{1}{8} + \frac{r}{2c_0} \right\} \left\{ Y^2 U(Y) - 4YV_Y + 6V(Y) \right\} \quad (102)$$

$$c = c_0 + \epsilon^2 c_2 + \dots \quad (103)$$

where

$$c_2 = -\frac{1}{2} \left(1 + 2 \frac{r}{c_0} \right) \frac{1}{c_0 + 4r} \quad (104)$$

$$V(Y) = (1/12) (2Y^3 - 3Y) \operatorname{erf}(Y) + (1/[6\sqrt{\pi}]) Y^2 \exp(-Y^2) \quad (105)$$

In the outer limit,

$$V(Y) \sim (1/6) Y^3 - (1/4) Y \quad Y \gg 1 \quad (106)$$

For $\epsilon = 1$, the corrections are as large as the unperturbed solution, so numerical solutions are used to extend regularized/unregularized comparisons to $\epsilon = 1$ in the next section.

9 Numerical Comparison of the Charney Solution with the Regularized Solution with $\epsilon = 1$

The numerical calculations were computed using a rational Chebyshev pseudospectral method as described in [3, 6]. The solution is computed in the form of a Fourier series but with rational Chebyshev functions replacing the sines and the cosines:

$$u(y) = \sum_{n=0}^{\infty} a_n \operatorname{TL}_n(y) \quad (107)$$

where the explicit form of the TL functions is given in [3]. Fig. 4 shows the computed coefficients a_n . The plot is consistent with the theoretical prediction that the coefficients are proportional to $\exp(-qn^{1/2})$ for some constant q [3–5]; the error when the series is truncated after the N -th term is therefore proportional to $\exp(-q\sqrt{N})$ as explained in Chapter 2 of [4].

TL coefficients $L=1$ Charney: black $\epsilon=1$: red

Figure 4: Comparison, for $r = 1/2$ and map parameter $L = 1$, of the TL coefficients for the Charney eigenfunction [black] with the corresponding coefficients of the log(cosh) regularization of the Charney problem for $\epsilon = 1$.

Fig. 5 shows that the reflection of Charney's mode about $y = 0$ has a discontinuous slope because $\psi_y(0) \neq 0$. In contrast, the slope of the regularized solutions is always zero at the origin and the reflection is smooth and singularity-free.

For $r = 1/2$, the complex phase speeds are (from rational Chebyshev numerical solutions)

$$c_{Charney} = 0.901016078298 + 0.544989507567 i \quad (108)$$

$$c(\epsilon = 1/4) = 0.928711457199 + 0.530656566091 i \quad (109)$$

$$c(\epsilon = 1/2) = 0.9945303130 + 0.4819021245 i \quad (110)$$

$$c(\epsilon = 1) = 1.1430477218 + 0.3217382311 i \quad (111)$$

Figure 5: Comparison of Charney eigenfunction with log(cosh) regularization for $\epsilon = 1$.

The absolute value of the complex modes shows little change (Fig. 6). When normalized properly, meaning that the coefficients of TL_0 are the same for both the Charney and regularized modes, the real and imaginary parts are also similar.

10 Over-reflection Theory

Fig 7 is a schematic of the Lindzen-Tung-Farrell-Rosenthal Wave Overreflection Theory (OR). The bottom of the diagram is the ground (for baroclinic instability) or the center of the symmetric “point jet” (for barotropic instability).

In OR theory, waves propagate from the ground upward to a turning point. This turning point is **ESSENTIAL** because it stops the wave propagation and allows the waves to *tunnel* to the critical level. Lindzen *et al.* assert that it is not possible for waves to propagate directly to the critical level (or latitude), there to be overreflected, because the *vertical group velocity* goes to zero at the critical level, so the waves would never make it as oscillations, but only by tunneling through an interval of exponential growth away from the critical level. (“Tunneling” is used in the same meaning as in quantum mechanics.) The turning point above the critical level, which creates the region of exponential decay upwards to infinite height, is **OPTIONAL**. This

Figure 6: Comparison of the absolute value of the Charney eigenfunction with $\log(\cosh)$ regularization for $\epsilon = 1$.

turning point could be replaced by a rigid upper boundary without modifying the overreflection process, and is therefore labeled “irrelevant”. The idea is that waves from the ground are overreflected downward from the critical level, bounce off the ground, and are overreflected again so as to build up an amplifying normal mode in a process reminiscent of a laser signal growing through repeated bounces within a ruby crystal.

Lindzen’s review [19] shows that the theory also applies to gravity waves governed by the Taylor-Goldstein equation and its invariants, but this is outside the scope of the present discussion.

The first obvious difficulty with OR theory is that the region of waves near the ground, the “resonance cavity” of the laser analogy, is compressed into a delta function. The inner solution of the matched asymptotic expansion approximates the solution in this wavy region, number “1” in the schematic, by the inner approximation. However, the leading order inner solution is just the *constant*, which is as far from a wave as one could possibly imagine. The first-order correction is merely the mean wind $U(y)$. This does curve, but speedily and monotonically asymptotes to a linear function. The first order inner approximation is not very wavy either.

11 Implications

The delta functions in the Charney problem and barotropic point-jet, so worrisome to J. S. A. Green in the quote at the beginning of the article, have been removed. The regularized solutions confirm the soundness of the point-jet model of Lindzen, Rosenthal and Farrell [23]. In particular the lowest order eigenvalue is identical with theirs. The eigenfunctions are the same, too, to all orders everywhere outside the inner region of width $O(\epsilon)$ around the jet maximum.

Figure 7: Schematic of Overreflection Theory, similar to Fig. 1 of Lindzen and Tung [24]. The wiggly lines denote regions of wave propagation; straight vertical lines denote regions of exponential growth or decay in space. Far from the ground or the center of the point jet, the wave is confined by either a turning point with exponential decay on the far side, as true of the Charney problem, or a wall boundary, as marked by the horizontal dashed line (top of the figure). The numbers on the far right refer to the corresponding numbered regions in Lindzen and Tung's article. In the Charney problem, the all-important oscillation region "1" — the "resonant cavity" of the "laser" — is squished down into a layer of INFINITESIMAL THICKNESS where the curvature of the mean wind is a DELTA FUNCTION.

Yet larger issues remain. A linear shear that is linear everywhere, and not merely piecewise-linear, is always stable. Therefore the inner region of width ϵ , which may be arbitrarily thin, somehow controls the dynamics everywhere. And yet the Overreflection Theory completely ignores the physics in the inner region, and analyzes instability entirely in terms of properties of the outer region. Even so, the outer region is completely impotent to create instability on its own. The inner region is both all-powerful and irrelevant!

We shall return to these mysteries after analyzing the WKB-thinking that underlies OR theory in the third part of this series.

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A Dilogs, Polylogarithms and the log(cosh) Function

A.1 Definitions

The matched asymptotic expansions require the unfamiliar transcendentals known as the “polylogarithms” defined by

$$\text{Li}_m(z) \equiv \sum_{n=1}^{\infty} \frac{z^n}{n^m} \quad (112)$$

$\text{Li}_2(x)$ is the “dilogarithm”, often written simply as Li . Table 1 shows that the integrals and derivatives of the $\log(\cosh)$ function can be expressed in terms of the polylogarithms and ordinary hyperbolic functions. This allows explicit, analytical expressions for the lowest three orders of the ϵ series for the regularized Charney problem.

The next two subsections solve subproblems essential to the third order solution. An important simplification is that the inner approximation is symmetric about the origin to all orders.

An arbitrary function is said to be “symmetric with respect to the origin” if and only if it has the property that $f(-Y) = f(Y)$ for all Y . The integrands for both $\diamond$ and $\mathfrak{U}$, the functions derived in the rest of this section, have this symmetry. The following helpful lemma is a slight variation on the analysis of Chapter 9 of [4]. It may also be proved by observing that all odd powers of Y are antisymmetric with respect to the origin, all even powers are symmetric, and then expanding the integrands in power series.

Lemma 1. *The integrand of a symmetric function is antisymmetric except for the arbitrary constant of integration. The integral-of-the-integral of a symmetric function is symmetric except for the additive term CY where C is a constant of integration.*

Table 1: Derivatives and Integrals of the $\log(\cosh)$ Function

The p_j are constants of integration.

U_{YY}	$\text{sech}^2(Y)$
U_Y	$\tanh(Y)$
U	$\log(\cosh(Y)) + \log(2)$
$\int U(Y') dY'$	$p_0 + (1/2)Y^2 + (1/2)\text{Li}_2(-\exp(-2Y))$
$\int dY'' \int^{Y''} U(Y') dY'$	$p_0 + p_1 Y + (1/6)Y^3 - (1/4)\text{Li}_3(-\exp(-2Y))$
$m\text{-th iterated integral of } U$	$P_{m-1}(Y) + (1/(m+1)!)Y^{m+1} - (-1/2)^m \text{Li}_{m+1}(-\exp(-2Y))$
	P_{m-1} with coefficients p_j is an arbitrary polynomial of degree $(m-1)$. $\text{Li}_m(z) \equiv \sum_{n=1}^{\infty} \frac{z^n}{n^m}$.

A.2 Analytical evaluation of $\diamond(Y)$, the second integral of $\log(\cosh(Y))$ [part of the third order solution]

Theorem 1. 1.

$$\diamond(Y; p_0, p_1) \equiv \int^Y dY'' \int^{Y''} \log(\cosh(Y')) dY' \quad (113)$$

$$= p_0 + p_1 Y - \frac{1}{2} \log(2) Y^2 + \frac{1}{6} Y^3 - \frac{1}{4} \text{Li}_3(-\exp(-2Y)) \quad (114)$$

where p_0 and p_1 are arbitrary integration constants.

2. The solution to

$$\diamond_{YY}(Y) = \log(\cosh(Y)), \quad \diamond(0) = \diamond_Y(0) = 0 \quad (115)$$

is symmetric with respect to $Y = 0$ and may be expressed in either of the two equivalent forms:

$$\begin{aligned} \diamond\left(Y; -\frac{3}{16}\zeta(3), \frac{\pi^2}{24}\right) &= -\frac{3}{16}\zeta(3) + \frac{\pi^2}{24}Y - \frac{1}{2}\log(2)Y^2 + \frac{1}{6}Y^3 - \frac{1}{4}\text{Li}_3(-\exp(-2Y)) \\ &= -\frac{3}{16}\zeta(3) - \frac{1}{2}\log(2)Y^2 - \frac{1}{8}\text{Li}_3(-\exp(-2Y)) - \frac{1}{8}\text{Li}_3(-\exp(2Y)) \end{aligned} \quad (116)$$

Proof:

$$\log(\cosh(Y)) = -\log(2) + Y + \log(1 + \exp(-2Y)) \quad (117)$$

$$= -\log(2) + Y + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\exp(-2nY)}{n}, \quad Y > 0 \quad (118)$$

Term-by-term indefinite integration gives

$$\begin{aligned} \int^Y \log(\cosh(Y')) dY' &= p_1 - \log(2)Y + \frac{1}{2}Y^2 - \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\exp(-2nY)}{n^2}, \quad Y > 0 \\ &= p_1 - \log(2)Y + \frac{1}{2}Y^2 + \frac{1}{2} \sum_{n=1}^{\infty} \frac{[-\exp(-2Y)]^n}{n^2}, \quad Y > 0 \end{aligned} \quad (119)$$

$$= p_1 - \log(2)Y + \frac{1}{2}Y^2 + \frac{1}{2} \text{Li}_2(-\exp(-2Y)) \quad (120)$$

where p_1 is the arbitrary integration constant.

Integrating again gives

$$\begin{aligned} \int^Y dY'' \int^{Y''} \log(\cosh(Y')) dY' &= \int^Y dY' \left\{ p_1 - \log(2)Y' + \frac{1}{2}Y'^2 + \frac{1}{2} \sum_{n=1}^{\infty} \frac{[-\exp(-2Y')]^n}{n^2} \right\}, \quad Y > 0 \\ &= p_0 + p_1 Y - \frac{1}{2}\log(2)Y^2 + \frac{1}{6}Y^3 - \frac{1}{4} \sum_{n=1}^{\infty} \frac{[-\exp(-2Y)]^n}{n^3}, \quad Y > 0 \\ &= p_0 + p_1 Y - \frac{1}{2}\log(2)Y^2 + \frac{1}{6}Y^3 - \frac{1}{4} \text{Li}_3(-\exp(-2Y)) \end{aligned} \quad (121)$$

where p_0 and p_1 are arbitrary integration constants.

The solution to

$$\diamond_{YY}(Y) = \log(\cosh(Y)), \quad \diamond(0) = \diamond_Y(0) = 0 \quad (122)$$

requires the power series expansion

$$\diamond(Y; p_0, p_1) = p_0 + \frac{3}{16}\zeta(3) + (p_1 - \frac{\pi^2}{24})Y - \frac{1}{2}\log(2)Y^2 + \dots \quad (123)$$

where we have used $\zeta(m) = \sum_{n=1}^{\infty} 1/n^m$ and $\zeta(2) = \pi^2/6$. The solution is

$$\begin{aligned} \diamond \left(Y; -\frac{3}{16}\zeta(3), \frac{\pi^2}{24} \right) &= -\frac{3}{16}\zeta(3) + \frac{\pi^2}{24}Y - \frac{1}{2}\log(2)Y^2 + \frac{1}{6}Y^3 + \dots \\ &= -\frac{3}{16}\zeta(3) - \frac{1}{2}\log(2)Y^2 - \frac{1}{8}\text{Li}_3(-\exp(-2Y)) - \frac{1}{8}\text{Li}_3(-\exp(2Y)) \end{aligned} \quad (124)$$

where the second line follows from the first after (i) recognizing that the first form is symmetric with respect to the origin by invoking Lemma 1 and noting that the boundary condition $d\phi/dY(0) = 0$ has removed the term linear in Y and (ii) applying the formula for the symmetric part of an arbitrary function, $\text{Symm}(f(x)) = (1/2)[f(x) + f(-x)]$. Both forms are identical; the first does not look symmetric in Y , though it is, but this apparently unsymmetric form is convenient for asymptotic matching. The second is self-evidently symmetric about $Y = 0$, but hides the fact that $\diamond(Y)$ has linear and cubic terms as $Y \rightarrow \infty$.

■

A.3 Analytical evaluation of $\mathfrak{U}(Y)$, the iterated integral of $Y^2 \text{sech}^2(Y)$, [another part of the third order solution]

$$\int dY'' \int Y'^2 \text{sech}^2(Y') dY' = \mathfrak{U}(Y; q_0, q_1) \quad (125)$$

where q_0, q_1 are constants of integration.

$$\begin{aligned} \int_0^Y Y'^2 \text{sech}^2(Y') dY' &= q_1 - \frac{1}{12} \frac{\pi^2}{1 + (e^Y)^2} - \frac{1}{12} \frac{\pi^2 (e^Y)^2}{1 + (e^Y)^2} + 2 \frac{Y^2 (e^Y)^2}{1 + (e^Y)^2} \\ &\quad - 2 \frac{Y \ln(1 + e^{2Y})}{1 + (e^Y)^2} - 2 \frac{Y \ln(1 + e^{2Y}) (e^Y)^2}{1 + (e^Y)^2} \\ &\quad - \frac{\text{Li}_2(2, -(e^Y)^2)}{1 + (e^Y)^2} - \frac{\text{Li}_2(2, -(e^Y)^2) (e^Y)^2}{1 + (e^Y)^2} \\ &= \tilde{q}_1 - \frac{\pi^2}{12} + 2Y^2 \frac{\exp(2Y)}{1 + \exp(2Y)} - 2Y \log(1 + \exp(2Y)) - \text{Li}_2(-\exp(2Y)) \end{aligned} \quad (126)$$

where $\tilde{q}_1$ is a constant of integration.

Term-by-term integration and $q_1 \equiv \tilde{q}_1 - (\pi^2/12)$ yield

$$\begin{aligned} \mathfrak{U}(Y; q_0, q_1) &= \int_0^Y \left\{ q_1 + 2Y'^2 \frac{\exp(2Y')}{1 + \exp(2Y')} - 2Y' \log(1 + \exp(2Y')) - \text{Li}_2(-\exp(2Y')) \right\} dY' \\ &= q_0 + q_1 Y + Y^2 \ln(1 + e^{2Y}) + 2Y \text{Li}_2(-\exp(2Y)) - \frac{3}{2} \text{Li}_3(-\exp(2Y)) \end{aligned} \quad (127)$$

where we have used

$$\int_0^Y 2Y'^2 \frac{\exp(2Y')}{1 + \exp(2Y')} dY' = Y^2 \log(1 + \exp(2Y)) + Y \text{Li}_2(-\exp(2Y)) - (1/2) \text{Li}_3(-\exp(2Y))$$

$$\int_0^Y (-2)Y' \log(1 + \exp(2Y')) dY' = Y \text{Li}_2(-\exp(2Y)) - (1/2) \text{Li}_3(-\exp(2Y))$$

$$\int_0^Y (-1) \text{Li}_2(-\exp(2Y')) dY' = (-1/2) \text{Li}_3(-\exp(2Y))$$

We need the specific function that solves the ODE and also two homogenous boundary conditions:

$$\psi_{YY}(Y) = Y^2 \text{sech}^2(Y), \quad \psi(0) = \psi_Y(0) = 0 \quad (128)$$

$$\psi(0; q_0, q_1) = q_0 - \frac{3}{2} \text{Li}_3(-1) \quad (129)$$

$$\text{Li}_3(-1) = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^3} = -(3/4)\zeta(3),$$

$$q_0 = \frac{3}{2} \text{Li}_3(-1) = -\frac{9}{8} \zeta(3) = -1.3523140160545435710747 \quad (130)$$

$$\frac{d\psi}{dY}(0; q_0, q_1) = q_1 - \text{Li}_2(-1) \quad (131)$$

$$\text{Li}_2(-1) = \sum_{n=1}^{\infty} (-1)^n / n^2 = -\pi^2/12. \text{ Therefore}$$

$$q_1 = \text{Li}_2(-1) = -\frac{\pi^2}{12} = -.822467033424113218236207583 \quad (132)$$

$$\psi(0; -\frac{9}{8}, -\frac{\pi^2}{12}) = (3/4)\zeta(3) + \frac{\pi^2}{12} Y + Y^2 \ln(1 + e^{2Y}) + 2Y \text{Li}_2(-\exp(2Y)) - \frac{3}{2} \text{Li}_3(-\exp(2Y))$$

We need the following formulas, easily obtained by asymptotic and power series expansions in Maple, to derive the outer limit of the inner solution.

$$\log(\cosh(Y)) \sim Y - \log(2) + \exp(-2Y) - \frac{1}{2} \exp(-4Y) + O(\exp(-6Y)) \quad (133)$$

$$\text{Li}_2(-\exp(-2Y)) \sim -e^{-2Y} + 1/4 e^{-4Y} - 1/9 e^{-6Y} + \dots \quad (134)$$

$$\text{Li}_2(-\exp(2Y)) \sim -2Y^2 - 1/6 \pi^2 + e^{-2Y} - 1/4 e^{-4Y} + 1/9 e^{-6Y} \quad (135)$$

$$\text{Li}_3(-\exp(-2Y)) \sim -e^{-2Y} + \frac{1}{8} e^{-4Y} - \frac{1}{27} e^{-6Y} \quad (136)$$

$$\text{Li}_3(-\exp(2Y)) \sim -\frac{\pi^2}{3} Y - \frac{4}{3} Y^3 - e^{-2Y} + \frac{1}{8} e^{-4Y} - \frac{1}{27} e^{-6Y} + \dots \quad (137)$$

$$\log(1 + \exp(2Y)) \sim 2Y + \exp(-2Y) - \frac{1}{2} \exp(-4Y) + \frac{1}{3} \exp(-6Y) \quad (138)$$

B Linear-Error Function Identities and Lemma

Table 2 gives the identities which are needed when the mean flow is regularized using the error function.

The solution to

$$V_{YY} = U, \quad V(0) = V_Y(0) = 0 \quad (139)$$

is

$$V(Y) = (1/12) (2Y^3 - 3Y) \text{erf}(Y) + (1/[6\sqrt{\pi}]) Y^2 \exp(-Y^2) \quad (140)$$

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Table 2: Derivatives and Integrals of the $Y \operatorname{erf}(Y)$ Function

U_{YY}	$(4/\sqrt{\pi})(1 - Y^2) \exp(-Y^2)$
U_Y	$\operatorname{erf}(Y) + (2/\sqrt{\pi})Y \exp(-Y^2)$
U	$Y \operatorname{erf}(Y)$
$\int U(Y') dY'$	$A' + (1/2)(Y^2 - 1/2) \operatorname{erf}(Y) + (1/[2\sqrt{\pi}])Y \exp(-Y^2)$
$V(Y) \equiv \int dY'' \int^{Y''} U(Y') dY'$	$A'' + B''Y + (1/12)(2Y^3 - 3Y) \operatorname{erf}(Y) + (1/[6\sqrt{\pi}])Y^2 \exp(-Y^2)$
where A', A'', B'' are arbitrary constants.	

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