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Regularity “in Large” for the $3D$ Salmon’s Planetary Geostrophic Model of Ocean Dynamics

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Abstract: It is well known, by now, that the three-dimensional non-viscous planetary geostrophic model, with vertical hydrostatic balance and horizontal Rayleigh friction/damping, coupled to the heat diffusion and transport, is mathematically ill-posed. This is because the no-normal flow physical boundary condition implicitly produces an additional boundary condition for the temperature at the lateral boundary. This additional boundary condition is different, because of the Coriolis forcing term, than the no-heat-flux physical boundary condition. Consequently, the second order parabolic heat equation is over-determined with two different boundary conditions. In a previous work we proposed one remedy to this problem by introducing a fourth-order artificial hyper-diffusion to the heat transport equation and proved global regularity for the proposed model. A shortcoming of this higher-oder diffusion is the loss of the maximum/minimum principle for the heat equation. Another remedy for this problem was suggested by R. Salmon by introducing an additional Rayleigh-like friction/damping term for the vertical component of the velocity in the hydrostatic balance equation. In this paper we prove the global, for all time and all initial data, well-posedness of strong solutions to the three-dimensional Salmon’s planetary geostrophic model of ocean dynamics. That is, we show global existence, uniqueness and continuous dependence of the strong solutions on initial data for this model. Unlike the 3D viscous PG model, we are still unable to show the uniqueness of the weak solution. Notably, we also demonstrate in what sense the additional damping term, suggested by Salmon, annihilate the ill-posedness in the original system; consequently, it can be viewed as “regularizing" term that can possibly be used to regularize other related systems.

Keywords: planetary geostrophic model, global regularity, ocean dynamics model, global circulation

MSC: 35Q35, 65M70, 86-08,86A10

1 Introduction

The starting point in the derivation of the ocean circulation models is Boussinesq equations which are the Navier–Stokes equations with rotation and a heat transport equation. The global existence of strong solution to the Navier–Stokes equations, which are a particular case of the Boussinesq equations when the temperature is identically zero, is one of the most challenging problems in applied analysis. However, geophysicists take advantage of the shallowness of the oceans and the atmosphere and introduce the hydrostatic balance approximation in the vertical motion. This in turn simplifies the Boussinesq model, and leads to the primi-

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tive equations of ocean and atmosphere dynamics (see, e.g., [14], [15], [16], [18], [20], [22], [34] and references therein). Further, horizontally, approximations based on the fast rotation of the earth, and the shallowness of the atmosphere and ocean imply the smallness of the Rossby number, which consequently lead to the geostrophic balance between the Coriolis force and the horizontal pressure gradient (cf. e.g., [11], [18], [22], [34] and references therein). By taking advantage of these assumptions and other geophysical considerations several intermediate models have been developed and used in numerical studies of weather prediction, long-time climate dynamics and large scale ocean circulation dynamics (see, e.g., [2], [3], [6], [7], [18], [20], [23], [26], [27], [28], [29], [36] and references therein).

The planetary geostrophic (PG) model, the inviscid and adiabatic form of "thermocline" equations, of large scale ocean circulation are derived by standard scaling analysis for gyre–scale oceanic motion (see [17], [19] , [21], [22], [34] and [35]). They are given in their simplest dimensionless β–plane mid-latitude approximation by the system of equations:

$$p_x - fv = 0, \quad p_y + fu = 0, \quad p_z - T = 0, \tag{1}$$

$$u_x + v_y + w_z = 0 \tag{2}$$

$$\partial_t T + uT_x + vT_y + wT_z = \kappa_v T_{zz}\,, \tag{3}$$

in the domain $\Omega = \{(x, y, z) : (x, y) \in M \subset \mathbb{R}^2,\ \text{and } z \in (-h, 0)\}$ and $h > 0$. For convenience, we assume that h is a constant. Here (u, v, w) denotes the velocity field, p is the pressure, and T is the temperature, which are the unknowns. $f = f_0 + \beta y$ is the β–plane mid-latitude approximation of the Coriolis force. The first two equations in (1) represent the geostrophic balance and the third equation represents the hydrostatic balance. The diffusive term, $\kappa_v T_{zz}$ is a leading order approximation to the effect of macro-scale turbulent mixing. Based on physical ground Samelson and Vallis [26] have argued that in closed ocean basin, with the no-normal-flow boundary conditions, this model can be solved only in restricted domains which are bounded away from the lateral boundary, $\partial M \times (-h, 0)$. Thus, it cannot be utilized in the study of the large-scale circulation. Furthermore, it has been pointed out numerically in [8] that arbitrarily small linear disturbances (disturbances that are supported at small spatial scales) will grow arbitrarily fast when the flow becomes baroclinically unstable. This nonphysical growth at small scales is a signature of mathematical ill-posedness of this model near unstable baroclinic mode. Therefore, Samelson and Vallis proposed in [26] various dissipative schemes to overcome these physical and numerical difficulties. In particular, they propose to add either a linear Rayleigh-like drag/friction/damping or a conventional eddy viscosity to the horizontal components of the momentum equations, and a horizontal diffusion in the thermodynamic equation (subject to no-heat-flux at the lateral boundary.) The planetary geostrophic (PG) model with conventional eddy viscosity has been studied mathematically in [4], [24], [25]. In [4] we show the global existence and uniqueness of weak and strong solutions to this 3D viscous PG model. We also provide rigorous estimates, depending on the various physical parameters, for the dimension of its global attractor. In the case where the dissipative scheme for the horizontal momentum is the linear drag Rayleigh friction it is observed that the no-normal-flow at the lateral boundary yields, due to the Coriolis force, an additional boundary condition to, and different from, the no-heat-flux. Therefore, the second order parabolic PDE that governs the temperature (the thermodynamic equation) has, too many boundary conditions to be satisfied, and hence it is over-determined and ill-posed (see, e.g., the detailed discussion regarding this matter in section 2, below, in [5], [26] and the references therein). To remedy this situation it is argued in [26] that one would have to add to the thermodynamic equation a higher order (biharmonic) horizontal diffusion in order to be able to satisfy both physical boundary conditions, i.e., the no-normal-flow and no-heat-flux boundary conditions at the lateral boundary (cf. e.g., [5], [26], [27]). In [5], we introduce, instead, a new PG model with an appropriate artificial horizontal "hyperdiffusion" term, to the heat equation, which involves the Coriolis parameters. Under the two natural physical boundary conditions at the lateral boundary we are able to prove in [5] the global existence and uniqueness of the strong solutions. Moreover, we also show the existence of the finite dimensional global attractor. It is worth mentioning, however, that the shortcoming of adding a higher-order diffusion operator in the temperature evolution equation, which is compatible with the physical boundary conditions, is the loss of the maximum/minimum principle for the temperature; which is a fundamental qualitative property of the temperature.

To overcome the above mentioned non-physical baroclinical instabilities and numerical ill-posedness Salmon introduced in [22] the following alternative planetary geostrophic model in the cylindrical domain Ω:

$$\epsilon u - f v + p_x = 0, \tag{4}$$
$$\epsilon v + f u + p_y = 0, \tag{5}$$
$$\delta w + p_z = T, \tag{6}$$
$$u_x + v_y + w_z = 0 \tag{7}$$
$$\partial_t T - \kappa_h (T_{xx} + T_{yy}) - \kappa_v T_{zz} + u T_x + v T_y + w T_z = Q\,, \tag{8}$$

where ϵ and δ are positive constants representing the linear (Rayleigh friction/damping) damping coefficients, and κ_h is positive constant which stand for the horizontal heat diffusivity, and Q is time independent heat source. We partition the boundary of Ω into:

$$\Gamma_u = \{(x,y,z) \in \overline{\Omega} : z = 0\}, \tag{9}$$
$$\Gamma_b = \{(x,y,z) \in \overline{\Omega} : z = -h\}, \tag{10}$$
$$\Gamma_s = \{(x,y,z) \in \overline{\Omega} : (x,y) \in \partial M,\ -h \le z \le 0\}. \tag{11}$$

System (4)–(8) is equipped with the following boundary conditions – with no-normal flow and non-heat flux on the side walls and the bottom (see, e.g., **[14], [15], [18], [22], [23], [26],[27], [28]**):

$$\text{on } \Gamma_u : w = 0,\ \frac{\partial T}{\partial z} + \alpha T = 0; \tag{12}$$
$$\text{on } \Gamma_b : w = 0,\ \frac{\partial T}{\partial z} = 0; \tag{13}$$
$$\text{on } \Gamma_s : (u,v)\cdot \vec{n} = 0,\ \frac{\partial T}{\partial \vec{n}} = 0, \tag{14}$$

where $\vec{n}$ is the normal vector to the lateral boundary Γ_s. In addition, we supply the system with the initial condition:

$$T(x,y,z,0) = T_0(x,y,z). \tag{15}$$

Observe that when $\delta = 0$ one obtains, formally, the original ill-posed PG model with Rayleigh friction/damping of the horizontal momentum (with coefficient $\epsilon > 0$). Therefore, one can view the additional damping term, δw, in (6) as a "regularizing" term, as it will be argued in section 2.

In this paper we focus on the question of, and prove, the global regularity and well-posedness of the $3D$ Salmon's PG model (4)–(8) for all time and all initial data. We remark that a general discussion concerning the nonlinear system (4)–(15) was presented in [31], but without providing any evidence of its global regularity, a problem that we provide a positive answer for it in this contribution.

The paper is organized as follows. In section 2, we introduce our notations and recall some well-known relevant inequalities. In section 3 we show the short-time existence of strong solutions of system (4)–(8) employing a Galerkin approximation procedure. Section 4 is the main section in which we establish the required estimates for proving the global existence and uniqueness of the strong solutions, and also show their continuous dependence on the initial data.

2 Preliminaries

Let us denote by $L^r(\Omega)$ and $W^{m,r}(\Omega)$, $H^r(\Omega)$ the usual L^r–Lebesgue and Sobolev spaces, respectively (cf., [1]). We denote by

$$\|\phi\|_r = \left(\int_\Omega |\phi(x,y,z)|^r \, dxdydz \right)^{\frac{1}{r}}, \qquad \text{for every } \phi \in L^r(\Omega). \tag{16}$$

We set

$$\widetilde{\mathcal{V}} = \left\{ T \in C^\infty(\overline{\Omega}) : \left.\frac{\partial T}{\partial z}\right|_{z=-h} = 0; \left.\left(\frac{\partial T}{\partial z} + \alpha T\right)\right|_{z=0} = 0; \left.\frac{\partial T}{\partial \vec{n}}\right|_{\Gamma_s} = 0 \right\},$$

and denote by V the closure spaces of $\widetilde{\mathcal{V}}$ in $H^1(\Omega)$ under the H^1−topology. For convenience, we also introduce the following equivalent norm on V:

$$\|\phi\|_V^2 = \kappa_h \|\partial_x \phi(x,y,z)\|_2^2 + \kappa_h \|\partial_y \phi(x,y,z)\|_2^2 + \kappa_v \left(\|\partial_z \phi(x,y,z)\|_2^2 + \alpha \|\phi(z=0)\|_{L^2(M)}^2 \right). \tag{17}$$

The equivalence of this norm on V to the H^1−norm can be justified thanks to the Poincaré inequality (21), below.

Next, we recall the following three-dimensional Sobolev and Ladyzhenskaya inequalities (see, e.g., [1], [9], [10], [13])

$$\|\psi\|_{L^3(\Omega)} \le C_0 \|\psi\|_{L^2(\Omega)}^{1/2} \|\psi\|_{H^1(\Omega)}^{1/2}, \tag{18}$$

$$\|\psi\|_{L^4(\Omega)} \le C_0 \|\psi\|_{L^2(\Omega)}^{1/4} \|\psi\|_{H^1(\Omega)}^{3/4}, \tag{19}$$

$$\|\psi\|_{L^6(\Omega)} \le C_0 \|\psi\|_{H^1(\Omega)}, \tag{20}$$

for every $\psi \in H^1(\Omega)$. Here C_0 is a dimensionless positive constant which might depend on the shape of M and Ω but not on their sizes. We also introduce the following version of Poincaré inequality

$$\|\psi\|_{L^2(\Omega)}^2 \le 2h \|\psi(z=0)\|_{L^2(M)}^2 + h^2 \|\psi_z\|_{L^2(\Omega)}^2, \tag{21}$$

$$\|\psi\|_{L^6(\Omega)}^6 \le 2h \|\psi(z=0)\|_{L^6(M)}^6 + h^2 \|\psi^2 \; \psi_z\|_{L^2(\Omega)}^2. \tag{22}$$

By solving the linear system (4)–(6) we obtain

$$u = -\frac{\epsilon p_x + f p_y}{\epsilon^2 + f^2}, \tag{23}$$

$$v = \frac{f p_x - \epsilon p_y}{\epsilon^2 + f^2}, \tag{24}$$

$$w = \frac{T - p_z}{\delta}. \tag{25}$$

Observe that from the no-normal-flow boundary condition (14) on the lateral boundary, Γ_s, one infers that

$$(u_z, v_z) \cdot \vec{n}|_{\Gamma_s} = 0. \tag{26}$$

As a result of (23)-(25) and (26) one has

$$\frac{\partial T}{\partial \vec{e}}|_{\Gamma_s} = \delta \frac{\partial w}{\partial \vec{e}}|_{\Gamma_s}, \tag{27}$$

where $\vec{e} = \frac{\epsilon\vec{n} + f\vec{k}\times\vec{n}}{\sqrt{\epsilon^2+f^2}}$, and $\vec{k}$ is the unit vector of vertical direction. We remark that the vectors $\vec{e}$ and $\vec{n}$ are parallel on Γ_s if and only if the Coriolis coefficient $f = 0$.

Since we deal here with the case when the Coriolis coefficient $f \neq 0$, it is observed that when $\delta = 0$ equations (14) and (27) imply two different boundary conditions for the temperature on the lateral boundary Γ_s:

$$\frac{\partial T}{\partial \vec{n}}|_{\Gamma_s} = 0 \quad \text{and} \quad \frac{\partial T}{\partial \vec{e}}|_{\Gamma_s} = 0. \tag{28}$$

Consequently, when $\delta = 0$, (28) makes (8), the second-order parabolic equation for the temperature, overdetermined and ill-posed. However, when $\delta > 0$, equation (27) does not generate an additional boundary condition to the no-heat-flux, (14), since the right-hand side $\delta \frac{\partial w}{\partial \vec{e}}|_{\Gamma_s}$ in (27) is not specified in advance, but it adjusts itself dynamically to satisfy (27). Accordingly, one can view the δw term in (6) as a regularizing effect, since it annihilates the ill-posedness situation when $\delta = 0$.

Next, we show how to solve for the pressure term and the effect of $\delta > 0$ in regularizing the pressure. Thanks to (23)-(25) and (7) we have the following elliptic system (since $\delta > 0$) for the pressure

$$-\left[\left(\frac{\epsilon p_x + f p_y}{\epsilon^2 + f^2}\right)_x + \left(\frac{-f p_x + \epsilon p_y}{\epsilon^2 + f^2}\right)_y + \left(\frac{p_z - T}{\delta}\right)_z\right] = 0. \tag{29}$$

Using the boundary conditions (12) and (13) we infer from (23)-(25) the following boundary conditions for the pressure:

$$\text{on } \Gamma_u \text{ and } \Gamma_b : \; p_z = T, \quad \text{and on } \Gamma_s : \; \frac{\partial p}{\partial \vec{e}} = 0, \tag{30}$$

where $\vec{e} = \frac{\epsilon \vec{n} + f \vec{k} \times \vec{n}}{\sqrt{\epsilon^2 + f^2}}$, as in (27).

Notice that by following the techniques developed in **[12]** and **[37]** (for the case of smooth domains, see, for example, **[13]** p. 89, and **[33]**), the three-dimensional second order elliptic boundary–value problem (29)–(30) has a unique solution for every given T; moreover, this solution enjoys the following regularity properties. Taking the $L^2(\Omega)$ inner product of equation (29) with p, integrating by parts and applying the boundary conditions (30) and using the Cauchy–Schwarz inequality, we obtain

$$\int_\Omega \left[\frac{\epsilon}{\epsilon^2 + f^2}\left(p_x^2 + p_y^2\right) + \frac{p_z^2}{\delta}\right] dxdydz = \frac{1}{\delta}\int_\Omega T p_z \, dxdydz \le \frac{1}{\delta}\|T\|_2 \|p_z\|_2. \tag{31}$$

Denote by

$$0 < F_0 = \min f < F_1 = \max f. \tag{32}$$

We observe that the assumption $F_0 > 0$ indicates that we are dealing with a mid-latitude case and away from the equator. By using (32) and applying Young's inequality to (31), we reach

$$\int_\Omega \left[\frac{\epsilon}{\epsilon^2 + F_1^2}\left(p_x^2 + p_y^2\right) + \frac{p_z^2}{2\delta}\right] dxdydz \le \int_\Omega \left[\frac{\epsilon}{\epsilon^2 + f^2}\left(p_x^2 + p_y^2\right) + \frac{p_z^2}{2\delta}\right] dxdydz \le \frac{1}{2\delta}\|T\|_2^2. \tag{33}$$

Furthermore, by (29) and the above estimate, we have

$$\begin{aligned}
\left\|\frac{\epsilon}{\epsilon^2 + f^2}(p_{xx} + p_{yy}) + \frac{p_{zz}}{\delta}\right\|_2 &= \left\|\frac{\beta p_x(\epsilon^2 - f^2) + 2\epsilon\beta f p_y}{(\epsilon^2 + f^2)^2} + \frac{T_z}{\delta}\right\|_2 \\
&\le C\left(\frac{\beta(\|p_x\|_2 + \|p_y\|_2)}{\epsilon^2 + F_0^2} + \left\|\frac{T_z}{\delta}\right\|_2\right) \\
&\le C\left(\frac{\beta(\epsilon + F_1)}{\epsilon^{1/2}\delta^{1/2}(\epsilon^2 + F_0^2)}\|T\|_2 + \frac{\|T_z\|_2}{\delta}\right).
\end{aligned} \tag{34}$$

As a result of the above and (23)-(25), we obtain

$$\|\epsilon u\|_2 + \|\epsilon v\|_2 + \|\delta w\|_2 \le C(\|\nabla p\|_2 + \|T\|_2) \le C\|T\|_2, \tag{35}$$

and

$$\|\epsilon u\|_{H^1(\Omega)} + \|\epsilon v\|_{H^1(\Omega)} + \|\delta w\|_{H^1(\Omega)} \le C(\|\nabla p\|_{H^1(\Omega)} + \|T\|_{H^1(\Omega)}) \le C\|T\|_{H^1(\Omega)}. \tag{36}$$

Definition 1. *Let $T_0 \in V$, and let $\mathcal{T}$ be a fixed positive time. (u, v, w, p, T) is called a strong solution of (4)–(8) on the time interval $[0, \mathcal{T}]$ if*
1)

$$\begin{aligned}
&T \in C([0,\mathcal{T}], V) \cap L^2([0,\mathcal{T}], H^2(\Omega)), \\
&T_t \in L^1([0,\mathcal{T}], L^2(\Omega)), \\
&T_t(z=0) \in L^1([0,\mathcal{T}], H^{-1/2}(M)).
\end{aligned}$$

2) *(u, v, w, p) satisfies* (23)–(30).

3) *Moreover,* (8) *is satisfied in the weak sense, namely, for every $t_0 \in [0, \mathcal{T}]$*

$$
\begin{aligned}
&\int_\Omega T(t)\psi \, dxdydz - \int_\Omega T(t_0)\psi \, dxdydz \\
&+ \int_{t_0}^{t} \left[\int_\Omega (\kappa_h T_x \psi_x + \kappa_h T_y \psi_y + \kappa_v T_z \psi_z) \, dxdydz + \kappa_v \, \alpha \int_M T(z=0)\psi(z=0) \, dxdy \right] ds \qquad (37) \\
&+ \int_{t_0}^{t} \int_\Omega \left[v \cdot \nabla T(s) + w T_z(s) \right] \psi \, dxdydz \, ds = \int_{t_0}^{t} \int_\Omega Q\psi \, dxdydz \, ds,
\end{aligned}
$$

for every $\psi \in V$, and $t \in [t_0, \mathcal{T}]$.

3 Short-time Existence of the Strong Solutions

In this section we will show the short-time existence of the strong solution of system (4)–(8).

Theorem 2. *Let $Q \in L^2(\Omega)$ and $T_0 \in V$ be given. Then there exists a strong solution (u, v, w, p, T) of system* (4)–(8) *on the interval $[0, \mathcal{T}^{***}]$, where $\mathcal{T}^{***}$ is a positive time given in (57), below. Furthermore, $\partial_t T \in L^2([0, \mathcal{T}^{***}]; L^2(\Omega))$ and $\partial_t T(z=0) \in L^2([0, \mathcal{T}^{***}]; H^{-1/2}(M))$; and equation (8) holds as a functional equation in $L^2([0, \mathcal{T}^{***}]; L^2(\Omega))$.*

Proof. We will use a Galerkin like procedure to show the existence of the strong solution for system (4)–(8). First, we will show the existence of the weak solutions. Let $\{\phi_k \in V \cap H^2(\Omega)\}_{k=1}^{\infty}$ and $\{\lambda_k \in \mathbb{R}^+\}_{k=1}^{\infty}$ be the eigenfunctions and their corresponding eigenvalues of the second order elliptic operators $-\kappa_h (T_{xx} + T_{yy}) - \kappa_v T_{zz}$, subject to the boundary conditions (12)–(14) (see, e.g., [13]). The eigenvalues are ordered such that $0 < \lambda_1 \le \lambda_2 \le \cdots$; moreover, $\{\phi_k\}_{k=1}^{\infty}$ is an orthogonal basis of $L^2(\Omega)$. Let $m \in \mathbb{Z}^+$ be fixed and H_m be the linear space generated by $\{\phi_k\}_{k=1}^{m}$. We will denote by $P_m : L^2 \to H_m$, the orthogonal projection in L^2. The Galerkin approximating system of order m that we use for (4)–(8) reads:

$$\epsilon \, u_m - f \, v_m + \partial_x p_m = 0, \qquad (38)$$

$$\epsilon \, v_m + f \, u_m + \partial_y p_m = 0, \qquad (39)$$

$$\delta \, w_m + \partial_z p_m = T_m, \qquad (40)$$

$$\partial_x u_m + \partial_y v_m + \partial_z w_m = 0 \qquad (41)$$

$$\partial_t T_m - \kappa_h (\partial_{xx} T_m + \partial_{yy} T_m) - \kappa_v \partial_{zz} T_m + P_m \left[u_m \partial_x T_m + v_m \partial_y T_m + w_m \partial_z T_m \right] = P_m Q, \qquad (42)$$

$$T_m(x, y, z, 0) = P_m T_0(x, y, z), \qquad (43)$$

where $T_m = \sum_{k=1}^{m} a_k(t)\phi_k(x, y, z)$, and (u_m, v_m, w_m, p_m) is the solution of the system (38)–(41) under boundary condition $w_m|_{z=0} = w_m|_{z=-h} = 0$; $(u_m, v_m) \cdot \vec{n}\big|_{\Gamma_s} = 0$. Based on discussion in the previous section, equation (42) is an ODE system with the unknown $a_k(t)$, $k = 1, \cdots, m$. Furthermore, it is easy to check that the vector field in equation (42) is locally Lipschitz with respect to $a_k(t)$, $k = 1, \cdots, m$, since it is quadratic. Therefore, there is a unique solution $a_k(t)$, $k = 1, \cdots, m$, to equation (42) for a short interval of time $[0, \mathcal{T}_m^{*}]$. Let $[0, \mathcal{T}_m^{**})$ be the maximal interval of existence for system (38)–(43). We will focus our discussion below on the interval $[0, \mathcal{T}_m^{**})$, and will show that $\mathcal{T}_m^{**} = +\infty$.

By taking the $L^2(\Omega)$ inner product of equation (42) with T_m, we obtain

$$\frac{1}{2} \frac{d \|T_m\|_2^2}{dt} + \kappa_h \left(\|\partial_x T_m\|_2^2 + \|\partial_y T_m\|_2^2 \right) + \kappa_v \left(\|\partial_z T_m\|_2^2 + \alpha \|T_m(z=0)\|_2^2 \right) \qquad (44)$$

$$+ \int_\Omega \left[u_m \partial_x T_m + v_m \partial_y T_m + w_m \partial_z T_m \right] T_m \, dxdydz = \int_\Omega Q \, T_m \, dxdydz. \qquad (45)$$

It is easy to show by integrating by parts and by using the relevant boundary conditions (12)–(14) that

$$\int_\Omega [u_m \partial_x T_m + v_m \partial_y T_m + w_m \partial_z T_m]\ T_m\, dxdydz = 0. \tag{46}$$

Furthermore, by the Cauchy–Schwarz inequality and (21) we have

$$\left| \int_\Omega Q\, T_m\ dxdydz \right| \le \|Q\|_2 \|T_m\|_2$$
$$\le \frac{1}{\sqrt{\lambda_1}} \|Q\|_2 \left[\kappa_h \left(\|\partial_x T_m\|_2^2 + \|\partial_y T_m\|_2^2 \right) + \kappa_v \left(\|\partial_z T_m\|_2^2 + \alpha \|T_m(z=0)\|_2^2 \right) \right],$$

where λ_1 is the first eigenvalue discussed above. From the above estimates, we obtain

$$\frac{d\|T_m\|_2^2}{dt} + \kappa_h \left(\|\partial_x T_m\|_2^2 + \|\partial_y T_m\|_2^2 \right) + \kappa_v \left(\|\partial_z T_m\|_2^2 + \alpha \|T_m(z=0)\|_2^2 \right) \le \frac{\|Q\|_2^2}{\lambda_1}. \tag{47}$$

Consequently, we have,

$$\frac{d\|T_m\|_2^2}{dt} + \lambda_1 \|T_m\|_2^2 \le \frac{\|Q\|_2^2}{\lambda_1}.$$

Thanks to Gronwall inequality, we conclude that

$$\|T_m(t)\|_2^2 \le \|T_0\|_2^2\, e^{-\lambda_1 t} + \frac{\|Q\|_2^2}{\lambda_1^2}, \tag{48}$$

for every $t \in [0, \mathcal{T}_m^{**})$. From the above, we conclude that $T_m(t)$ must exist globally, i.e., $\mathcal{T}_m^{**} = +\infty$. Therefore, for any given $\mathcal{T} > 0$ and any $t \in [0, \mathcal{T}]$, we have

$$\|T_m(t)\|_2^2 \le \|T_0\|_2^2\, e^{-\lambda_1 t} + \frac{\|Q\|_2^2}{\lambda_1^2}. \tag{49}$$

Furthermore, by integrating (47) with respect to the time variable over the interval $[0, t]$, for $t \in [0, \mathcal{T}]$, and by (49), we get

$$\int_0^t \left[\kappa_h \left(\|\partial_x T_m(s)\|_2^2 + \|\partial_y T_m(s)\|_2^2 \right) + \kappa_v \left(\|\partial_z T_m(s)\|_2^2 + \alpha \|T_m(z=0)(s)\|_2^2 \right) \right]\ ds$$
$$\le \|T_0\|_2^2 + \frac{\|Q\|_2^2\, t}{\lambda_1}. \tag{50}$$

As a result of all the above we have established that T_m exists globally in time, and that it is uniformly bounded, with respect to m, in the $L^\infty([0, \mathcal{T}]; L^2(\Omega))$ and $L^2([0, \mathcal{T}]; V)$ norms.

Next, and similar to the theory of $3D$ Navier–Stokes equations (see, e.g., [9] and [30]), let us show that $\partial_t T_m$ is uniformly bounded, with respect to m, in the $L^{\frac{4}{3}}([0, \mathcal{T}]; V')$ norm, where V' is the dual space of V. From (42), we have, for every $\psi \in V$

$$\langle \partial_t T_m, \psi \rangle = \left\langle P_m Q + \kappa_h(\partial_{xx} T_m + \partial_{yy} T_m) + \kappa_v \partial_{zz} T_m - P_m \left[u_m \partial_x T_m + v_m \partial_y T_m + w_m \partial_z T_m \right], \psi \right\rangle.$$

Here, $\langle \cdot, \cdot \rangle$ is the dual action of V'. It is clear that

$$|\langle P_m Q, \psi \rangle| \le \|Q\|_2 \|\psi\|_2, \tag{51}$$

and by integration by parts and using boundary condition (12)–(14), we have

$$\left| \left\langle \kappa_h(\partial_{xx} T_m + \partial_{yy} T_m) + \kappa_v \partial_{zz} T_m, \psi \right\rangle \right| \le C \|T_m\|_V\ \|\psi\|_V, \tag{52}$$

recall that $\|\cdot\|_V$ is defined in (17). Next, let us get an estimate for

$$|\langle P_m\left[u_m\partial_x T_m + v_m\partial_y T_m + w_m\partial_z T_m\right], \psi\rangle| = \left|\int_\Omega \left[u_m\partial_x T_m + v_m\partial_y T_m + w_m\partial_z T_m\right]\ \psi_m\ dxdydz\right|,$$

where $\psi_m = P_m\psi$. Thus, by integration by parts and using (41), (35), (36) and relevant boundary conditions, we obtain

$$\begin{aligned}
&|\langle P_m\left[u_m\partial_x T_m + v_m\partial_y T_m + w_m\partial_z T_m\right], \psi\rangle| \\
&= \left|\int_\Omega \left[u_m\partial_x \psi_m + v_m\partial_y \psi_m + w_m\partial_z \psi_m\right]\ T_m\ dxdydz\right| \\
&\le C\left[\|u_m\|_4 + \|v_m\|_4 + \|w_m\|_4\right]\ \|T_m\|_4\ \|\nabla\psi_m\|_2 \qquad (53)\\
&\le C\left(\|u_m\|_2^{1/4}\|u_m\|_{H^1}^{3/4} + \|v_m\|_2^{1/4}\|v_m\|_{H^1}^{3/4} + \|w_m\|_2^{1/4}\|w_m\|_{H^1}^{3/4}\right)\|T_m\|_2^{1/4}\|T_m\|_{H^1}^{3/4}\|\nabla\psi_m\|_2 \\
&\le C\left(\|T_m\|_2^2 + \|T_m\|_2^{1/2}\|T_m\|_V^{3/2}\right)\ \|\nabla\psi\|_2. \qquad (54)
\end{aligned}$$

Here C depends on ϵ and δ. Therefore, by the estimates (51)–(54), we have

$$|\langle\partial_t T_m, \psi\rangle| \le C\left(\|Q\|_2 + \|T_m\|_V + \|T_m\|_2^2 + \|T_m\|_2^{1/2}\|T_m\|_V^{3/2}\right)\|\psi\|_V.$$

Thus, we have

$$\begin{aligned}
&\int_0^t \|\partial_t T_m(t)\|_{V'}^{\frac{4}{3}} dt \le C\int_0^t \left(\|Q\|_2^{4/3} + \|T_m\|_2^{8/3} + (1+\|T_m\|_2^2)^{1/3}\|T_m\|_V^2\right) ds \\
&\le C\left(\|Q\|_2^{4/3} t + \|T_m\|_2^{8/3} t + (1+\|T_m\|_2^2)^{1/3}\int_0^t \|T_m\|_V^2 ds\right) \\
&\le C\left(\|Q\|_2^{\frac{4}{3}} t + (\|T_0\|_2^2\ e^{-\lambda_1\ t} + \frac{\|Q\|_2^2}{\lambda_1^2})^{\frac{4}{3}} t + C(\kappa_h, \kappa_v)(\|T_0\|_2^2\ e^{-\lambda_1\ t} + \frac{\|Q\|_2^2}{\lambda_1^2})^{\frac{1}{3}}(\|T_0\|_2^2 + \frac{\|Q\|_2^2\ t}{\lambda_1^2})\right). \qquad (55)
\end{aligned}$$

Therefore, $\partial_t T_m$ is uniformly bounded, with respect to m, in the $L^{\frac{4}{3}}([0,\mathcal{T}]; V')$ norm. Thanks to (49), (50) and (55), one can apply the Aubin's compactness Theorem (cf., for example, **[9], [30]**) and extract a subsequence $\{T_{m_j}\}$ of $\{T_m\}$ and a subsequence $\{\partial_t T_{m_j}\}$ of $\{\partial_t T_m\}$; which converge to $T \in L^\infty([0,\mathcal{T}]; L^2(\Omega)) \cap L^2([0,\mathcal{T}]; V)$ and $\partial_t T \in L^{\frac{4}{3}}([0,\mathcal{T}]; V')$, respectively, in the following sense:

$$\begin{cases}
T_{m_j} \to T & \text{in } L^2([0,\mathcal{T}]; L^2(\Omega)) \quad \text{strongly;}\\
T_{m_j} \to T & \text{in } L^\infty([0,\mathcal{T}]; L^2(\Omega)) \quad \text{weak-star;}\\
T_{m_j} \to T & \text{in } L^2([0,\mathcal{T}]; H^1(\Omega)) \quad \text{weakly;}\\
\partial_t T_{m_j} \to \partial_t T & \text{in } L^{\frac{4}{3}}([0,\mathcal{T}]; V') \quad \text{weakly.}
\end{cases}$$

Moreover, from (38)–(41) (see also (4)–(7)) we observe that $\{u_m, v_m, w_m\}$ depend linearly on T_m. Therefore, the elliptic estimates (35) and (36) imply, thanks to (49) and (50), uniform bounds, with respect to m, for $\{u_m, v_m, w_m\}$ in $L^\infty([0,\mathcal{T}]; L^2(\Omega))$ and $L^2([0,\mathcal{T}]; H^1(\Omega))$, respectively. Therefore, we can extract a subsequence of $\{u_{m_j}, v_{m_j} w_{m_j}\}$, corresponding to the readily established converging subsequence for the temperature $\{T_{m_j}\}$, which will be also labeled $\{u_{m_j}, v_{m_j} w_{m_j}\}$, that converges to $\{u, v, w\}$ weak-star in $L^\infty([0,\mathcal{T}]; L^2(\Omega))$, and weakly in $L^2([0,\mathcal{T}]; H^1(\Omega))$. By passing to the limit, one can show as in the case of Navier–Stokes equations (see, for example, **[9], [30]**) that T also satisfies (37). In other words, T is a weak solution of the system (4)–(8).

By taking the $L^2(\Omega)$ inner product of equation (42) with $-\kappa_h(\partial_{xx}T_m + \partial_{yy}T_m) - \kappa_v\partial_{zz}T_m$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\kappa_h\left(\|\partial_x T_m\|_2^2 + \|\partial_y T_m\|_2^2\right) + \kappa_v\left(\|\partial_z T_m\|_2^2 + \alpha\|T_m(z=0)\|_2^2\right)\right] + \|\kappa_h(\partial_{xx}T_m + \partial_{yy}T_m) + \kappa_v\partial_{zz}T_m\|_2^2\\
&= \int_\Omega (Q - u_m\partial_x T_m + v_m\partial_y T_m + w_m\partial_z T_m)\left(\kappa_h(\partial_{xx}T_m + \partial_{yy}T_m) + \kappa_v\partial_{zz}T_m\right)\, dxdydz\\
&\le (\|Q\|_2 + \|u_m\|_6\|\partial_x T_m\|_3 + \|v_m\|_6\|\partial_y T_m\|_3 + \|w_m\|_6\|\partial_z T_m\|_3)\ \|\kappa_h(\partial_{xx}T_m + \partial_{yy}T_m) + \kappa_v\partial_{zz}T_m\|_2\\
&\le (\|Q\|_2 + C\|T_m\|_6\|\nabla T_m\|_3)\ \|\kappa_h(\partial_{xx}T_m + \partial_{yy}T_m) + \kappa_v\partial_{zz}T_m\|_2\\
&\le \left(\|Q\|_2 + C\|T_m\|_2^2\right)\ \|\kappa_h(\partial_{xx}T_m + \partial_{yy}T_m) + \kappa_v\partial_{zz}T_m\|_2\\
&\quad + C\left[\kappa_h\left(\|\partial_x T_m\|_2^2 + \|\partial_y T_m\|_2^2\right) + \kappa_v\left(\|\partial_z T_m\|_2^2 + \alpha\|T_m(z=0)\|_2^2\right)\right]^{\frac{3}{4}}\ \|\kappa_h(\partial_{xx}T_m + \partial_{yy}T_m) + \kappa_v\partial_{zz}T_m\|_2^{\frac{3}{2}}.
\end{aligned}$$

Therefore, applying the Cauchy–Schwarz and Young inequalities to the above estimate and using (49), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[\kappa_h\left(\|\partial_x T_m\|_2^2 + \|\partial_y T_m\|_2^2\right) + \kappa_v\left(\|\partial_z T_m\|_2^2 + \alpha\|T_m(z=0)\|_2^2\right)\right] + \|\kappa_h(\partial_{xx}T_m + \partial_{yy}T_m) + \kappa_v\partial_{zz}T_m\|_2^2\\
&\le \|Q\|_2^2 + \frac{C\|Q\|_2^4}{\lambda_1^4} + C\|T_0\|_2^4 + C\left[\kappa_h\left(\|\partial_x T_m\|_2^2 + \|\partial_y T_m\|_2^2\right) + \kappa_v\left(\|\partial_z T_m\|_2^2 + \alpha\|T_m(z=0)\|_2^2\right)\right]^3. \qquad (56)
\end{aligned}$$

Let $\mathrm{M} = 1 + \kappa_h\left(\|\partial_x T_m\|_2^2 + \|\partial_y T_m\|_2^2\right) + \kappa_v\left(\|\partial_z T_m\|_2^2 + \alpha\|T_m(z=0)\|_2^2\right)$. Consequently, we have

$$\frac{d\mathrm{M}}{dt} \le C(1 + \|Q\|_2^4 + \|T_0\|_2^4)\mathrm{M}^3.$$

Thanks to Gronwall inequality, we have

$$\begin{aligned}
&1 + \kappa_h\left(\|\partial_x T_m\|_2^2 + \|\partial_y T_m\|_2^2\right) + \kappa_v\left(\|\partial_z T_m\|_2^2 + \alpha\|T_m(z=0)\|_2^2\right)\\
&\le \frac{1 + \kappa_h\left(\|\partial_x T_0\|_2^2 + \|\partial_y T_0\|_2^2\right) + \kappa_v\left(\|\partial_z T_0\|_2^2 + \alpha\|T_0(z=0)\|_2^2\right)}{\left(1 - C\,t\ \left(1 + \|T_0\|_2^4 + \|Q\|_2^4\right)\ \left[1 + \kappa_h\left(\|\partial_x T_0\|_2^2 + \|\partial_y T_0\|_2^2\right) + \kappa_v\left(\|\partial_z T_0\|_2^2 + \alpha\|T_0(z=0)\|_2^2\right)\right]\right)^{1/2}}.
\end{aligned}$$

Therefore, for every $t \in [0, \mathcal{T}^{***}]$, where

$$\mathcal{T}^{***} := \frac{1}{4C\left(\left(1 + \|T_0\|_2^4 + \|Q\|_2^4\right)\ \left[1 + \kappa_h\left(\|\partial_x T_0\|_2^2 + \|\partial_y T_0\|_2^2\right) + \kappa_v\left(\|\partial_z T_0\|_2^2 + \alpha\|T_0(z=0)\|_2^2\right)\right]\right)}, \qquad (57)$$

we have

$$\begin{aligned}
&\kappa_h\left(\|\partial_x T_m\|_2^2 + \|\partial_y T_m\|_2^2\right) + \kappa_v\left(\|\partial_z T_m\|_2^2 + \alpha\|T_m(z=0)\|_2^2\right)\\
&\le 1 + 2\left[\kappa_h\left(\|\partial_x T_0\|_2^2 + \|\partial_y T_0\|_2^2\right) + \kappa_v\left(\|\partial_z T_0\|_2^2 + \alpha\|T_0(z=0)\|_2^2\right)\right]. \qquad (58)
\end{aligned}$$

Moreover, by integrating (56) we obtain

$$\begin{aligned}
&\int_0^t \|\kappa_h(\partial_{xx}T_m(s) + \partial_{yy}T_m(s)) + \kappa_v\partial_{zz}T_m(s)\|_2^2\, ds\\
&\le \kappa_h\left(\|\partial_x T_0\|_2^2 + \|\partial_y T_0\|_2^2\right) + \kappa_v\left(\|\partial_z T_0\|_2^2 + \alpha\|T_0(z=0)\|_2^2\right) + C[1 + \|Q\|_2^4 + \|T_0\|_2^4]\, t +\\
&\quad + C\left[1 + \kappa_h\left(\|\partial_x T_0\|_2^2 + \|\partial_y T_0\|_2^2\right) + \kappa_v\left(\|\partial_z T_0\|_2^2 + \alpha\|T_0(z=0)\|_2^2\right)\right]^3 t, \qquad t \in [0, \mathcal{T}^{***}]. \qquad (59)
\end{aligned}$$

Notice that T_m exists, globally. What we have just proved is that the $L^2([0, \mathcal{T}^{***}]; H^2(\Omega))$ norm of T_m is bounded uniformly with respect to m. As a result of all the above we have T_m exists, at least, on $[0, \mathcal{T}^{***}]$ and is uniformly bounded, with respect to m, in $L^\infty([0, \mathcal{T}^{***}]; V)$ and $L^2([0, \mathcal{T}^{***}]; H^2(\Omega))$ norms. Furthermore, and as for the theory of the Navier-Stokes equations (see, for example, **[9], [30]**), we can use the above bounds (58) and (59) to show that the $L^2([0, \mathcal{T}^{***}]; L^2(\Omega))$ norm of $\partial_t T_m$ and the $L^2([0, \mathcal{T}^{***}]; H^{-1/2}(M))$ norm of $\partial_t T_m(z = 0)$

are uniformly bounded with respect to m. Passing to the limits, we conclude that there is a strong solution to system (4)–(8), at least, on $[0, \mathcal{T}^{***}]$. Furthermore, this strong solution enjoys the following properties:

$$\partial_t T \in L^2([0, \mathcal{T}^{***}]; L^2(\Omega)) \quad \text{and} \quad \partial_t T(z=0) \in L^2([0, \mathcal{T}^{***}]; H^{-1/2}(M)). \tag{60}$$

The above regularity estimates are sufficient to complete the proof of Theorem 2, following standard techniques from the theory of the Navier–Stokes equations (see, e.g., [9] and [30]). Furthermore, as a consequence of the above estimates, in particular those implying (60), we conclude that equation (8) holds as a functional equation in $L^2([0, \mathcal{T}^{***}]; L^2(\Omega))$. □

4 Global Existence and Uniqueness of the Strong Solutions

In the previous section we have established the short-time existence of the strong solution to system (4)–(8). In this section we will show the global existence and uniqueness, i.e. global regularity, of strong solutions to the system (4)–(8), and their continuous dependence on initial data.

Theorem 3. *Let $Q \in L^2(\Omega)$, $T_0 \in V$ and $\mathcal{T} > 0$, be given. Then there exists a unique strong solution (u, v, w, p, T) of the system (4)–(8), on the interval $[0, \mathcal{T}]$, which depends continuously on the initial data in the sense specified in equation (76) below.*

Proof. Denote by (u, v, w, p, T) the strong solution corresponding to the initial data T_0 with maximal interval of existence $[0, \mathcal{T}_\star)$, that has been established in Theorem 2. We will show that $\mathcal{T}_\star = \infty$. To show this we assume by contradiction that $\mathcal{T}_\star < \infty$. Consequently, it is clear that

$$\limsup_{t\to\mathcal{T}_\star^-} \|T(t)\|_{H^1(\Omega)} = \infty,$$

because, otherwise, and by virtue of Theorem 2, the solution can be extended beyond the maximal time of existence, $\mathcal{T}_\star$. Next, we will show that $\|T(t)\|_{H^1(\Omega)}$ is bounded uniformly on the interval $[0, \mathcal{T}_\star)$. In what follows we will focus our discussion and estimates on the finite maximal interval of existence $[0, \mathcal{T}_\star)$.

4.1 L^2 estimates

As a result of Theorem 2, equation (8) holds in $L^2_{\text{loc}}([0, \mathcal{T}_\star); L^2(\Omega))$, therefore we can take the inner product of equation (8) with T, in $L^2(\Omega)$, and obtain

$$\begin{aligned}&\frac{1}{2}\frac{d\|T\|_2^2}{dt} + \kappa_h\left(\|\partial_x T\|_2^2 + \|\partial_y T\|_2^2\right) + \kappa_v\left(\|\partial_z T\|_2^2 + \alpha\|T(z=0)\|_2^2\right)\\ &= \int_\Omega QT\,dxdydz - \int_\Omega (u\partial_x T + v\partial_y T + w\partial_z T)\,T\,dxdydz.\end{aligned}$$

After integrating by parts we get

$$\int_\Omega (u\partial_x T + v\partial_y T + w\partial_z T)\,T\,dxdydz = 0. \tag{61}$$

As a result of the above we conclude

$$\begin{aligned}&\frac{1}{2}\frac{d\|T\|_2^2}{dt} + \kappa_h\left(\|\partial_x T\|_2^2 + \|\partial_y T\|_2^2\right) + \kappa_v\left(\|\partial_z T\|_2^2 + \alpha\|T(z=0)\|_2^2\right)\\ &= \int_\Omega QT\,dxdydz \le \|Q\|_2\,\|T\|_2.\end{aligned}$$

By the inequality (21), we have

$$\|T\|_{L^2(\Omega)}^2 \le \frac{h^2}{\kappa_v}(1+\frac{2}{h\alpha})\left[\kappa_v\left(\|\partial_x T\|_2^2+\|\partial_y T\|_2^2\right)+\kappa_v\left(\|\partial_z T\|_2^2+\alpha\|T(z=0)\|_2^2\right)\right]. \tag{62}$$

Using (62) and the Cauchy–Schwarz inequality we obtain

$$2\frac{d\|T\|_2^2}{dt}+2\kappa_h\left(\|\partial_x T\|_2^2+\|\partial_y T\|_2^2\right)+\kappa_v\left(\|\partial_z T\|_2^2+\alpha\|T(z=0)\|_2^2\right) \tag{63}$$

$$\le \frac{h^2}{\kappa_v}\left(1+\frac{2}{\alpha h}\right)\|Q\|_2^2. \tag{64}$$

By (62) and thanks to Gronwall inequality the above gives

$$\|T\|_2^2 \le e^{-\frac{\kappa_v t}{4(h^2+2h/\alpha)}}\|T_0\|_2^2+\frac{h^4}{2\kappa_v^2}\left(1+\frac{2}{\alpha h}\right)^2\|Q\|_2^2, \tag{65}$$

for are $t\in[0,\mathcal{T}_\star)$. Moreover, we also have

$$\int_0^t\left[2\kappa_h\left(\|\partial_x T\|_2^2+\|\partial_y T\|_2^2\right)+\kappa_v\left(\|\partial_z T\|_2^2+\alpha\|T(z=0)\|_2^2\right)\right]\,ds$$

$$\le \|T_0\|_2^2+\frac{h^2}{\kappa_v}\left(1+\frac{2}{\alpha h}\right)\|Q\|_2^2\,t, \tag{66}$$

for are $t\in[0,\mathcal{T}_\star)$.

We remark that estimates (65) and (66) also follow directly from (49) and (50), respectively.

4.2 L^6 estimates

Recall from Theorem 2 that $T \in L^\infty_{\text{loc}}([0,\mathcal{T}_\star),H^1(\Omega))\cap L^2_{\text{loc}}([0,\mathcal{T}_\star),H^2(\Omega))$, therefore $|T|^4T \in L^2_{\text{loc}}([0,\mathcal{T}_\star);L^2(\Omega))$. Since by Theorem 2 equation (8) holds in $L^2_{\text{loc}}([0,\mathcal{T}_\star);L^2(\Omega))$ we can take the inner product of the equation (8), in $L^2(\Omega)$, with $|T|^4T$ to get

$$\frac{1}{6}\frac{d\|T\|_6^6}{dt}+5\int_\Omega\left[\kappa_h\left(|\partial_x T|_2^2+|\partial_y T|_2^2\right)+\kappa_v|\partial_z T|_2^2\right]|T|^4\,dxdydz+\alpha\kappa_v\|T(z=0)\|_6^6$$

$$=\int_\Omega Q|T|^4T\,dxdydz-\int_\Omega(uT_x+vT_y+wT_z)\,|T|^4T\,dxdydz.$$

By integration by parts, and using (7) and the boundary conditions (12)-(14) we get

$$\int_\Omega(uT_x+vT_y+wT_z)\,|T|^4T\,dxdydz=0. \tag{67}$$

As a result of the above we conclude

$$\frac{1}{6}\frac{d\|T\|_6^6}{dt}+5\int_\Omega\left[\kappa_h\left(|\partial_x T|_2^2+|\partial_y T|_2^2\right)+\kappa_v|\partial_z T|_2^2\right]|T|^4\,dxdydz+\alpha\kappa_v\|T(z=0)\|_6^6$$

$$=\int_\Omega Q|T|^4T\,dxdydz\le\|Q\|_2\|T\|_{10}^5\le C\|Q\|_2\left(\|T\|_6^2\,\|\nabla T^3\|+\|T\|_6^5\right).$$

By the Cauchy–Schwarz inequality we get

$$\frac{d\|T\|_6^6}{dt}+\int_\Omega\left[\kappa_h\left(|\partial_x T|_2^2+|\partial_y T|_2^2\right)+\kappa_v|\partial_z T|_2^2\right]|T|^4\,dxdydz+\alpha\kappa_v\|T(z=0)\|_6^6$$

$$=\int_\Omega Q|T|^4T\,dxdydz\le C\|Q\|_2^2\|T\|_6^4+\|Q\|_2\|T\|_6^5\le C\|Q\|_2^2\|T\|_6^4+\|T\|_6^6.$$

Thus, from the above and (22), we have

$$\frac{d\|T\|_6^2}{dt} \leq C\|Q\|_2^2 + \|T\|_6^2 \leq C\left[\|Q\|_2^2 + \|T\|_2^2 + \kappa_h\left(\|\partial_x T\|_2^2 + \|\partial_y T\|_2^2\right) + \kappa_v\|\partial_z T\|_2^2\right].$$

By integrating the above inequality and using (65) and (66), we get

$$\|T(t)\|_6^2 \leq C\left[(1 + \|Q\|_2^2)(1+t) + \|T_0\|_{H^1(\Omega)}^2\right]. \tag{68}$$

4.3 H^1 estimates

Recall again that $T \in L^\infty_{\mathrm{loc}}([0, \mathcal{T}_\star), H^1(\Omega)) \cap L^2_{\mathrm{loc}}([0, \mathcal{T}_\star), H^2(\Omega))$, and since, by Theorem 2, equation (8) holds in $L^2_{\mathrm{loc}}([0, \mathcal{T}_\star); L^2(\Omega))$ we can take the inner product of the equation (8) with $-\kappa_h(T_{xx} + T_{yy}) - \kappa_v T_{zz}$, in $L^2(\Omega)$, and use (60) to obtain, thanks to a Lemma of Lions-Magenes concerning the derivative of functions with values in Banach space (cf. Chap. III-p.169- [30]),

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\kappa_h\left(\|\partial_x T\|_2^2 + \|\partial_y T\|_2^2\right) + \kappa_v\left(\|\partial_z T\|_2^2 + \alpha\|T(z=0)\|_2^2\right)\right] + \|\kappa_h(T_{xx} + T_{yy}) + \kappa_v T_{zz}\|_2^2 \\
&= -\int_\Omega Q\left[\kappa_h(T_{xx} + T_{yy}) + \kappa_v T_{zz}\right] dxdydz + \int_\Omega (u\partial_x T + v\partial_y T + w\partial_z T)\left[\kappa_h(T_{xx} + T_{yy}) + \kappa_v T_{zz}\right] dxdydz \\
&\leq \left[\|Q\|_2 + (\|u\|_6 + \|v\|_6 + \|w\|_6)\|\nabla T\|_3\right]\|\kappa_h(T_{xx} + T_{yy}) + \kappa_v T_{zz}\|_2 \\
&\leq \left[\|Q\|_2 + C\|T\|_6^{3/2}\|\kappa_h(T_{xx} + T_{yy}) + \kappa_v T_{zz}\|_2^{1/2}\right]\|\kappa_h(T_{xx} + T_{yy}) + \kappa_v T_{zz}\|_2.
\end{aligned}$$

By the Cauchy–Schwarz and Young's inequalities we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[\kappa_h\left(\|\partial_x T\|_2^2 + \|\partial_y T\|_2^2\right) + \kappa_v\left(\|\partial_z T\|_2^2 + \alpha\|T(z=0)\|_2^2\right)\right] + \|\kappa_h(T_{xx} + T_{yy}) + \kappa_v T_{zz}\|_2^2 \\
&\leq C\|Q\|_2^2 + C\|T\|_6^6.
\end{aligned}$$

By Gronwall, we get

$$\begin{aligned}
&\kappa_h\left(\|\partial_x T(t)\|_2^2 + \|\partial_y T(t)\|_2^2\right) + \kappa_v\left(\|\partial_z T(t)\|_2^2 + \alpha\|T(z=0)(t)\|_2^2\right) \\
&\quad + \int_0^t \|\kappa_h\left(T_{xx}(s) + T_{yy}(s)\right) + \kappa_v T_{zz}(s)\|_2^2\, ds \\
&\leq C(1 + \|Q\|_2^2 + \|T\|_6^6)\, t + \|T_0\|_{H^1(\Omega)}^2 \\
&\leq C(1 + \|Q\|_2^2)\, t + C\left[(1 + \|Q\|_2^2)(1+t) + \|T_0\|_{H^1(\Omega)}^2\right]^3 t + \|T_0\|_{H^1(\Omega)}^2 =: K_v(t).
\end{aligned} \tag{69}$$

Thus,

$$\limsup_{t \to \mathcal{T}_\star^-} \|T\|_{H^1(\Omega)} = K_v(\mathcal{T}_\star).$$

This contradicts the assumption that $\mathcal{T}_\star$ is finite, therefore, $\mathcal{T}_\star = \infty$, and the solution (u, v, w, p, T) exists globally in time.

4.4 Uniqueness of the strong solution and continuous dependence on initial data

Next, we show the continuous dependence on the initial data and the the uniqueness of the strong solutions. Let $(u_1, v_1, w_1, p_1, T_1)$ and $(u_2, v_2, w_2, p_2, T_2)$ be two strong solutions of the system (4)–(8) with corresponding initial data $(T_0)_1$ and $(T_0)_2$, respectively. Denote by $u = u_1 - u_2$, $v = v_1 - v_2$, $w = w_1 - w_2$, $p = p_1 - p_2$ and

$\theta = T_1 - T_2$. It is clear that

$$\epsilon u - f v + p_x = 0, \tag{70}$$

$$\epsilon v + f u + p_y = 0, \tag{71}$$

$$\delta w + p_z = \theta, \tag{72}$$

$$u_x + v_y + w_z = 0 \tag{73}$$

$$\partial_t \theta - \kappa_h (\theta_{xx} + \theta_{yy}) - \kappa_v \theta_{zz} + u_1 \theta_x + v_1 \theta_y + w_1 \theta_z + u \partial_x T_2 + v \partial_y T_2 + w \partial_z T_2 = 0\,, \tag{74}$$

and (u, v, w) and θ satisfy boundary conditions (12)–(14). By Theorem 2 and Theorem 3 equation (74) holds in $L^2([0, \mathcal{T}]; L^2(\Omega))$ and $\theta \in L^\infty([0, \mathcal{T}), H^1(\Omega)) \cap L^2([0, \mathcal{T}), H^2(\Omega))$, for all $\mathcal{T} > 0$. Therefore, by taking the inner product of equation (74) with θ in $L^2(\Omega)$, and using boundary conditions (12)–(14), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d\|\theta\|_2^2}{dt} + \kappa_h \left(\|\partial_x \theta\|_2^2 + \|\partial_y \theta\|_2^2 \right) + \kappa_v \|\partial_z \theta\|_2^2 + \alpha \|\theta(z=0)\|_2^2 \\
&= - \int_\Omega \left[u_1 \theta_x + v_1 \theta_y + w_1 \theta_z + u(T_2)_x + v(T_2)_y + w(T_2)_z \right] \, \theta \, dxdydz.
\end{aligned}$$

By integration by parts and again boundary conditions (12)–(14), we get

$$- \int_\Omega \left[u_1 \theta_x + v_1 \theta_y + w_1 \theta_z \right] \, \theta \, dxdydz = 0. \tag{75}$$

Notice that

$$\begin{aligned}
&\left| \int_\Omega \left[u(T_2)_x + v(T_2)_y + w(T_2)_z \right] \, \theta \, dxdydz \right| \le C \|\nabla T_2\|_2 \left(\|u\|_4 + \|v\|_4 + \|w\|_4 \right) \|\theta\|_4 \\
&\le C \|\nabla T_2\|_2 \left(\|u\|_2^{1/4} \|u\|_{H^1}^{3/4} + \|v\|_2^{1/4} \|v\|_{H^1}^{3/4} + \|w\|_2^{1/4} \|w\|_{H^1}^{3/4} \right) \|\theta\|_2^{1/4} \|\theta\|_{H^1}^{3/4} \\
&\le C \|\nabla T_2\|_2 \|\theta\|_2^{1/2} \|\theta\|_{H^1}^{3/2} \le C \|\nabla T_2\|_2 \left(\|\theta\|_2^2 + \|\theta\|_2^{1/2} \|\nabla \theta\|_2^{3/2} \right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{1}{2}\frac{d\|\theta\|_2^2}{dt} + \kappa_h \left(\|\partial_x \theta\|_2^2 + \|\partial_y \theta\|_2^2 \right) + \kappa_v \|\partial_z \theta\|_2^2 + \alpha \|\theta(z=0)\|_2^2 \\
&\le C \|\nabla T_2\|_2 \left(\|\theta\|_2^2 + \|\theta\|_2^{1/2} \|\nabla \theta\|_2^{3/2} \right).
\end{aligned}$$

By Young's inequality, we get

$$\begin{aligned}
&\frac{d\|\theta\|_2^2}{dt} + \kappa_h \left(\|\partial_x \theta\|_2^2 + \|\partial_y \theta\|_2^2 \right) + \kappa_v \|\partial_z \theta\|_2^2 + \alpha \|\theta(z=0)\|_2^2 \\
&\le C \|\nabla T_2\|_2^4 \|\theta\|_2^2.
\end{aligned}$$

Thanks to Gronwall inequality, we obtain

$$\|\theta(t)\|_2^2 \le \|\theta(t=0)\|_2^2 e^{C \int_0^t \|\nabla T_2(s)\|_2^4 \, ds}.$$

Since T_2 is a strong solution, we have by virtue of (69)

$$\|\theta(t)\|_2^2 \le \|\theta(t=0)\|_2^2 \, e^{C \int_0^t K_v^2(s) ds}, \tag{76}$$

where the value of T_0 in the definition of K_v in (69) is replaced by $T_2(0)$. As a result, the above inequality proves the continuous dependence of the solutions on the initial data. In particular, when $\theta(t=0) = 0$, we have $\theta(t) = 0$, and consequently also $u(t) = v(t) = w(t) = 0$, for all $t \ge 0$. Therefore, the strong solution is unique. □

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