EXISTENCE AND ASYMPTOTIC PERIODICITY OF SOLUTIONS FOR NEUTRAL INTEGRO-DIFFERENTIAL EVOLUTION EQUATIONS WITH INFINITE DELAY

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ABSTRACT. In this work, making use of the theory of resolvent operators and Banach fixed point theorem, we first discuss the existence and regularity of mild solutions for neutral partial functional integro-differential equations with infinite delay. We assume that the linear part of the considered equation generates a resolvent operator and the nonlinear function satisfies Lipschitz conditions. Then we investigate the asymptotic periodicity of mild solutions under asymptotic periodic assumption on the nonlinear function. The obtained results extend somewhat the related conclusions in literature. In the end, an example is presented to illustrate the obtained results.

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1. Introduction

In this paper, we study the existence, regularity and asymptotic periodicity of solutions for neutral functional integro-differential evolution equations with infinite delay of the form

$$\begin{cases} \frac{d}{dt} \left[x(t) + \int_0^t N(t-s)x(s)ds \right] = A \left[x(t) + \int_0^t F(t-s)x(s)ds \right] + f(t, x_t), & t \geq 0, \\ x_0 = \phi \in \mathcal{B}, \end{cases} \quad (1.1)$$

where $x(\cdot)$ is the state variable taking values in a Banach space X and the notation x_t represents the functional defined by $x_t: (-\infty, 0] \rightarrow X$, $x_t(\theta) = x(t + \theta)$, belonging to some abstract phase space $\mathcal{B}$ which is defined axiomatically. The operator $(A, D(A))$ is the infinitesimal generator of an analytic semigroup on X , while $F(t): D(A) \rightarrow D(A)$ and $N(t): X \rightarrow X$ are two families of bounded linear operators. The nonlinear function $f(\cdot, \cdot)$ is to be specified later.

Neutral partial integro-differential equations without or with time delay arise in various areas. For example, the following heat conduction system, see [3, 6, 7, 30], has been frequently used to

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describe the evolution of the temperature in different types of materials with fading memory,

$$\begin{cases} \frac{\partial}{\partial t} \left[u(t, x) + \int_{-\infty}^t k_1(t-s)u(s, x)ds \right] = c\Delta u(t, x) + \int_{-\infty}^t k_2(t-s)\Delta u(s, x)ds \\ \quad + f(t, u(t, x)), \quad t \geq 0, \\ u(t, x) = 0, \quad x \in \partial\Omega. \end{cases} \quad (1.2)$$

In this system, $\Omega \subset \mathbb{R}^n$ is open, bounded and has smooth boundary, $(t, x) \in [0, \infty) \times \Omega$, $u(t, x)$ represents the temperature in x at time t , c is a physical constant and $k_i: \mathbb{R} \rightarrow \mathbb{R}$, $i = 1, 2$, are functions of the internal energy and the heat flux relaxation respectively, and the internal energy here the heat flux terms are described as functionals of u and Δu . For more models of partial integro-differential equations and related applications we refer to the paper [2] and the monographs [22, 38]. One of the important way to deal with partial integro-differential equations is to transform them into (neutral) abstract integro-differential evolution equations in Banach space, and then study the abstract systems by applying theory of resolvent operators. Actually, by assuming the solution u is known on $(-\infty, 0]$. we can transform (1.2) into the abstract neutral integro-differential evolution equation of the following form

$$\frac{d}{dt} \left[u(t) + F(t, u_t) \right] = Au(t) + \int_0^t B(t-s)u(s)ds + f(t, u_t) \quad \text{for } t \geq 0.$$

In [10], it was illustrated that the equation

$$\frac{du(t)}{dt} = A \left[u(t) + \int_{-\infty}^t K(t-s)u(s)ds \right] + \int_{-\infty}^t F(t-s)u(s)ds \quad \text{for } t \geq 0$$

can be regarded as the abstract formulation of the model proposed and studied by Coleman and Gurtin [8], Gurtin and Pipkin [20], and Miller [31] for the heat flow in a rigid isotropic viscoelastic material in the elastic case, and the model in thermoviscoelasticity considered by Leugering [28].

An important approach to study the above abstract equations is the theory of resolvent operators founded in [17]–[19]. Grimmer et al. proved in [17]–[19] that there is a (unique) resolvent operator $(R(t))_{t \geq 0}$ associated to the linear homogeneous equation

$$\begin{cases} v'(t) = Av(t) + \int_0^t \gamma(t-s)v(s)ds & \text{for } t \geq 0, \\ v(0) = v_0 \in X. \end{cases}$$

Based on it the authors obtained the representation, existence, and uniqueness of mild solutions of the following inhomogeneous integro-differential evolution equation in Banach space X

$$\begin{cases} v'(t) = Av(t) + \int_0^t \gamma(t-s)v(s)ds + f(t) & \text{for } t \geq 0, \\ v(0) = v_0 \in X, \end{cases} \quad (1.3)$$

via the resolvent operator, where $f: \mathbb{R}^+ \rightarrow X$ is a continuous function. Actually, the resolvent operator, replacing the role of C_0 -semigroup for evolution equations, plays an essential role in solving Eq. (1.3) in weak and strict senses. Based on these important works, in these years much works on various topics, such as existence, regularity and stability of solutions and control problems, for semilinear integro-differential evolution equations has been done by many authors

through applying the theory of resolvent operator, see [4, 5, 11, 14–16, 32, 40] and the references therein.

On the other hand, existence of periodic solutions is one of the most interesting and important topics in the qualitative theory of differential equations due to their significance in practical sciences. Many contributions on the existence of periodic solutions for differential equations have been made. Meanwhile, the concept of asymptotically ω -periodic solutions is more general than periodic solutions and, from view of applications, asymptotically periodic systems describe the models more realistically and accurately than periodic ones. The notion of S -asymptotically ω -periodic functions was introduced by Henríquez et al. in [25, 26] (i.e. a bounded continuous function $f(\cdot)$ is called to be S -asymptotically ω -periodic if there exists $\omega > 0$ such that $\lim_{t \rightarrow \infty} \|f(t + \omega) - f(t)\| = 0$) and it is related to and more general than that of asymptotic ω -periodic functions. Moreover, the authors have established there a relationship between S -asymptotically ω -periodic functions and the class of asymptotically ω -periodic functions. In these years this topic is attracting much attention increasing interest of many researchers, some recent works of this area can be found in [1, 9, 12, 24, 29, 35, 37, 39] among others. Particularly, Dos Santos and Henríquez [37] investigated existence of S -asymptotically ω -periodic solutions for neutral integro-differential equations of the following form

$$\begin{cases} \frac{d}{dt} \left[x(t) + \int_0^t N(t-s)x(s)ds \right] = Ax(t) + \int_0^t B(t-s)x(s)ds + f(t, x(t)) & \text{for } t \geq 0, \\ x(0) = x_0, \end{cases}$$

via the theory of resolvent operators associated to the linear homogeneous equation

$$\begin{cases} \frac{d}{dt} \left[x(t) + \int_0^t N(t-s)x(s)ds \right] = Ax(t) + \int_0^t B(t-s)x(s)ds & \text{for } t \geq 0, \\ x(0) = x_0, \end{cases} \quad (1.4)$$

which was founded in [36]. In [36] the authors first established the theory of resolvent operators for (1.4) and then they discussed the existence and uniqueness of solutions for a semilinear neutral integro-differential equation with infinite delay.

To the best of our knowledge, the regularity and asymptotic periodicity of mild solutions for the neutral partial functional integro-differential equation (1.1) are untreated topics in the literature up to know. The purpose of this paper is to explore the existence, regularity and S -asymptotic periodicity of solutions for the equation (1.1). The main tool in our discussion is also the theory of resolvent operators for neutral linear integro-differential systems of (1.1) established in [36]. We will represent the global mild solutions of Eq. (1.1) by the resolvent operator $R(t)$ and analyze existence and uniqueness of solutions by using theory of resolvent operator and Banach fixed point theorem. Following this we further study the regularity of mild solutions for Eq. (1.1) under some smoothness conditions. It is seen that the results obtained in this part are better than those in [14, 16, 27] due to the application of the newly theory of resolvent operators. More precisely, other than in [14, 16, 27], we do not impose here any smoothness condition on the neutral term. At last, we study the existence and uniqueness of S -asymptotically ω -periodic solutions assuming the nonlinear function is so. The results are proved by utilizing Schauder's fixed point theorem and Banach fixed point theorem, respectively, for the different cases that the resolvent operator is compact and non-compact. We point out here that, compared to [29, 37], the conditions for the existence of S -asymptotically ω -periodic solutions are somewhat weaker (see Theorem 4.1 and

Theorem 4.2). Clearly, our obtained results in this article extend and develop the existing results in literature mentioned above.

Subsequently the paper is organized as follows. In Section 2, we state briefly the basic theory of analytic semigroup and resolvent operators. In Section 3 we discuss the existence and regularity of global mild solutions of (1.1). The existence and uniqueness of S -asymptotically ω -periodic solutions is then studied in Section 4. Finally, in Section 5, we provide an example to illustrate the applications of the obtained results.

2. Preliminaries

In this section, we present some preliminary facts on theory of analytic semigroup and resolvent operators to be used in this paper.

Let $(X, \|\cdot\|)$ and $(Z, \|\cdot\|)$ be two Banach spaces, we denote by $\mathcal{L}(X, Z)$ the Banach space of bounded linear operators from X into Z endowed with the general operator norm and we abbreviate this notation to $\mathcal{L}(X)$ when $X = Z$. For a linear operator $A: D(A) \subseteq X \rightarrow X$, let Y be the Banach space $(D(A), \|\cdot\|_1)$ with the graph norm $\|x\|_1 = \|Ax\| + \|x\|$, for $x \in D(A)$. Hereafter $C([0, T], X)$ denotes the Banach space consisting of continuous functions from $[0, T]$ to X with the norm

$$\|x\|_C = \sup_{0 \leq t \leq T} \|x(t)\|, \quad x \in C([0, T], X).$$

$C_b([0, \infty); X)$ denotes the Banach space consisting of bounded and continuous functions from $[0, \infty)$ to X with $\|\cdot\|_\infty$ defined by $\|x\|_\infty = \sup_{t \geq 0} \|x(t)\|$.

Next we present briefly the basic theory of resolvent operators for the following neutral integro-differential equation associated to Eq. (1.1).

$$\begin{cases} \frac{d}{dt} \left[x(t) + \int_0^t N(t-s)x(s)ds \right] = A \left[x(t) + \int_0^t F(t-s)x(s)ds \right], \\ x(0) = x_0 \in X. \end{cases} \quad (2.1)$$

DEFINITION 1. A one-parameter family of bounded linear operators $(R(t))_{t \geq 0}$ on X is called a resolvent operator for Eq. (2.1) if it verifies the following conditions.

- (i) The function $R(\cdot): [0, \infty) \rightarrow \mathcal{L}(X)$ is strongly continuous, and there exist $M_0 \geq 1$ and $\omega \in \mathbb{R}$, such that $\|R(t)\| \leq M_0 e^{\omega t}$ for $t \geq 0$;
- (ii) $R(0) = I$;
- (iii) For $x \in Y$, $R(\cdot)x \in C^1((0, \infty), X) \cap C([0, \infty), Y)$, and

$$\frac{d}{dt} \left[R(t)x + \int_0^t N(t-s)R(s)x ds \right] = A \left[R(t)x + \int_0^t F(t-s)R(s)x ds \right], \quad (2.2)$$

$$\frac{d}{dt} \left[R(t)x + \int_0^t R(t-s)N(s)x ds \right] = R(t)Ax + \int_0^t R(t-s)AF(s)x ds, \quad (2.3)$$

for each $t \geq 0$.

We assume for the operators in Eq. (1.1) (or (2.1)) that the following conditions are fulfilled.

(V₁) There are constants $M_0 > 0$ and $\theta \in (\frac{\pi}{2}, \pi)$ such that

$$\rho(A) \supseteq \Lambda_\theta = \{\lambda \in \mathbb{C} : |\arg(\lambda)| < \theta\} \text{ and } \|R(\lambda, A)\| \leq M_0|\lambda|^{-1}, \lambda \in \Lambda_\theta.$$

Hence in this case the operator A generates an analytic semigroup $(T(t))_{t \geq 0}$ on X (see [13, 34]).

(V₂) The function $N : [0, \infty) \rightarrow \mathcal{L}(X)$ is strongly continuous and $\hat{N}(\lambda)x$ is absolutely convergent for $x \in X$ and $\operatorname{Re}(\lambda) > 0$. There exist $\beta > 0$ and an analytical extension of $\hat{N}(\lambda)$ (still denoted by $\hat{N}(\lambda)$) to Λ_θ such that $\|\hat{N}(\lambda)x\| \leq N_0|\lambda|^{-\beta}\|x\|_1$ for each $\lambda \in \Lambda_\theta$ and $x \in D(A)$. Here N_0 is a constant, the notation $\hat{f}(\lambda)$ represents the Laplace transform of $f(t)$.

(V₃) For all $t \geq 0$, $F(t) \in \mathcal{L}(X)$. $F(t) : X \rightarrow D(A)$, and $AF(\cdot)x$ is strongly measurable on $(0, \infty)$ for any $x \in X$. There exists a $b(\cdot) \in L^1_{\text{loc}}(R^+)$ such that $\hat{b}(\lambda)$ exists for $\operatorname{Re} \lambda > 0$ and $\|AF(t)x\| \leq b(t)\|x\|_1$ for all $t > 0$ and $x \in D(A)$. Moreover, the operator valued function $\hat{F} : \Lambda_{\frac{\pi}{2}} \rightarrow \mathcal{L}(Y)$ has an analytical extension (still denoted by $\hat{F}$) to Λ_θ such that $\|A\hat{F}(\lambda)x\| \leq \|\hat{F}(\lambda)\|\|x\|_1$ for all $x \in D(A)$, and $\|A\hat{F}(\lambda)\| \rightarrow 0$ as $|\lambda| \rightarrow \infty$.

(V₄) There exist a subspace $D \subseteq D(A)$ dense in Y and positive constants $C_i, i = 1, 2$, such that $A(D) \subseteq D(A)$, $\hat{F}(\lambda)(D) \subseteq D$, $\hat{N}(\lambda)(D) \subseteq D(A)$, and $\|A\hat{F}(\lambda)x\|_1 \leq C_1\|x\|_1$, $\|\hat{N}(\lambda)x\|_1 \leq C_2|\lambda|^{-\beta}\|x\|_1$ for every $x \in D$ and $\lambda \in \Lambda_\theta$.

Observe that the above assumptions guarantee that $AF(\cdot)$ verifies the conditions of the family $B(\cdot)$ in [36]. Hence, according to [36], we infer readily that, under the hypotheses (V₁)–(V₄) there is a resolvent operator $R(t)$ for the linear part of (1.1) (i.e. (2.1)), which is given by

$$R(t) = \begin{cases} \frac{1}{2\pi i} \int_{\Gamma_{r,\vartheta}} e^{\lambda t} G(\lambda) d\lambda, & t > 0, \\ I, & t = 0, \end{cases}$$

where $G(\lambda) = (\lambda I + \lambda \hat{N}(\lambda) - A - A\hat{F}(\lambda))^{-1}$ satisfying $\|G(\lambda)\| \leq M_0|\lambda|^{-1}$ for $\lambda \in \Lambda_{r,\vartheta}$ and some $M_0 > 0$. We put $\Omega(G) = \{\lambda \in \sum_F \cap \sum_N : \lambda I + \lambda \hat{N}(\lambda) - A - A\hat{F}(\lambda) \text{ is invertible and } G(\lambda) \in \mathcal{L}(X)\}$, $\sigma(G) = \mathbb{C} \setminus \Omega(G)$ and $W_G = \sup\{\operatorname{Re}(\lambda) : \lambda \in \sigma(G)\}$. Here the notations $\sum_F$ and $\sum_N$ stand for the domain of maximal extension of the operators $\hat{F}(\cdot)$ and $\hat{N}(\cdot)$ on $\mathbb{C}$, respectively. The region $\Lambda_{r,\vartheta} = \{\lambda \in \mathbb{C} \setminus \{0\} : |\lambda| > r, |\arg(\lambda)| < \vartheta\}$, and $\Gamma_{r,\vartheta} = \bigcup_{i=1}^3 \Gamma_{r,\vartheta}^i$, the curves $\Gamma_{r,\vartheta}^i, i = 1, 2, 3$, are given respectively by

$$\begin{aligned} \Gamma_{r,\vartheta}^1 &= \{te^{i\vartheta} : t \geq r\}, \\ \Gamma_{r,\vartheta}^2 &= \{re^{i\xi} : -\vartheta \leq \xi \leq \vartheta\}, \\ \Gamma_{r,\vartheta}^3 &= \{te^{-i\vartheta} : t \geq r\}, \end{aligned}$$

for $r > 0, \vartheta \in (\frac{\pi}{2}, \theta)$ being fixed numbers. These curves are oriented so that $\operatorname{Im}(\lambda)$ is increasing.

The following theorem founded in [37] will be used in this paper.

THEOREM 2.1. *If $W_G < 0$, then there exist constants $M \geq 1$ and $\gamma > 0$ such that*

$$\|R(t)\| \leq Me^{-\gamma t}, \quad t \geq 0. \quad (2.4)$$

In this paper, we will use an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [21] and follow the terminology used in [23]. Thus, $\mathcal{B}$ is a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$. We assume that $\mathcal{B}$ satisfies the following axioms:

(A) If $x: (-\infty, \sigma + a) \rightarrow X$, $a > 0$ is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + a)$ the following conditions hold:

- (i) x_t is in $\mathcal{B}$.
- (ii) $\|x(t)\| \leq H\|x_t\|_{\mathcal{B}}$.
- (iii) $\|x_t\|_{\mathcal{B}} \leq K(t - \sigma) \sup\{\|x(s)\| : \sigma \leq s \leq t\} + M(t - \sigma)\|x_\sigma\|_{\mathcal{B}}$.

Here $H \geq 0$ is a constant, $K, M: [0, +\infty) \rightarrow [0, +\infty)$, $K(\cdot)$ is continuous and $M(\cdot)$ is locally bounded, and $H, K(\cdot), M(\cdot)$ are independent of $x(\cdot)$.

(A₁) For the function $x(\cdot)$ in (A), the function $t \rightarrow x_t$ is continuous from $[\sigma, \sigma + a)$ into $\mathcal{B}$.

(B) The space $\mathcal{B}$ is complete.

For our later discussion we also need some additional properties of the space $\mathcal{B}$. Let C_{00} be the space of continuous functions from $(-\infty, 0]$ into X with compact support. Then from [23: Proposition 2.1], $C_{00} \subseteq \mathcal{B}$. We assume that the phase space $\mathcal{B}$ satisfies additionally the following axioms:

- (C₁) If $\{\phi^n\}$ is a Cauchy sequence in $\mathcal{B}$ and converges compactly to ϕ on $(-\infty, 0]$, then $\phi \in \mathcal{B}$ and $\lim_{n \rightarrow +\infty} \|\phi^n - \phi\|_{\mathcal{B}} = 0$.
- (C₂) If a uniformly bounded sequence $\{\phi^n(\theta)\}$ in C_{00} converges to a function $\phi(\theta)$ compactly on $(-\infty, 0]$, then $\phi \in \mathcal{B}$ and $\lim_{n \rightarrow +\infty} \|\phi^n - \phi\|_{\mathcal{B}} = 0$.

Thus we have that

LEMMA 2.1. *If the phase space $\mathcal{B}$ satisfies axiom (C₂), then $C_b((-\infty, 0], X)$ is continuously imbedded into $\mathcal{B}$, and there exists a constant $J > 0$ such that $\|\phi\|_{\mathcal{B}} \leq J\|\phi\|_{\infty'}$ for all $\phi \in C_b((-\infty, 0], X)$.*

In addition, from [33], one has that

LEMMA 2.2. *Let $\mathcal{B}$ satisfy axiom (C₁) and $f: [0, a] \rightarrow \mathcal{B}$, be a continuous function such that $f(t)(\theta)$ is continuous for $(t, \theta) \in [0, a] \times (-\infty, 0]$. Then*

$$\left(\int_0^a f(t) dt \right) (\theta) = \int_0^a f(t)(\theta) dt \quad \text{for } \theta \in (-\infty, 0].$$

Finally we introduce the concept of fading memory spaces. To do this, let $\mathcal{B}_0 = \{\phi \in \mathcal{B} : \phi(0) = 0\}$ which is a closed subspace of the space $\mathcal{B}$, and let the operator $S_0(t)$ on $\mathcal{B}_0$ be defined by

$$[S_0(t)\phi](\theta) = \begin{cases} 0 & -t \leq \theta \leq 0, \\ \phi(t + \theta) & \theta < -t, \end{cases}$$

for $\phi \in \mathcal{B}_0$. Clearly, the operator family $S_0(t)$, $t \geq 0$, is a C_0 -semigroup on $\mathcal{B}_0$.

DEFINITION 2. The space $\mathcal{B}$ is called a fading memory space if it additionally verifies Axioms (C₂), and $S_0(t)\phi \rightarrow 0$ as $t \rightarrow \infty$ for each $\phi \in \mathcal{B}_0$.

Remark 1. According to [23], if $\mathcal{B}$ is a fading memory space, then there exist $K^*, M^* > 0$, such that $K(\cdot)$ and $M(\cdot)$ in (A)-(iii) verify $K^* = \sup_{t \geq 0} \|K(t)\|$ and $M^* = \sup_{t \geq 0} \|M(t)\|$. In this condition, we can choose $J = K^*$, here J is the constant in Lemma 2.1.

3. Existence and regularity of solutions

The purpose of this section is to study the existence, uniqueness and regularity of mild solutions for the equation (1.1) on $(-\infty, \infty)$.

3.1. Existence of solutions

First we discuss in this subsection the existence and uniqueness of mild solutions for Eq. (1.1). The mild solutions of Eq. (1.1) are defined as:

DEFINITION 3. A function $x: \mathbb{R} \rightarrow X$ is said to be mild solution of Eq. (1.1), if $x_0 = \phi$, $x(\cdot)$ is continuous on $[0, \infty)$ and satisfies the integral equation

$$x(t) = R(t)\phi(0) + \int_0^t R(t-s)f(s, x_s)ds \quad \text{for } t \geq 0.$$

To guarantee the existence of mild solutions, we impose the following restrictions on the function $f(\cdot, \cdot)$ in Eq. (1.1).

(H_1) The function $f: [0, \infty) \times \mathcal{B} \rightarrow X$ is continuous and there is a constant $L_1 > 0$ such that

$$\|f(t, \phi_1) - f(t, \phi_2)\| \leq L_1 \|\phi_1 - \phi_2\|_{\mathcal{B}}$$

for any $t \in [0, \infty)$, $\phi_1, \phi_2 \in \mathcal{B}$, and the inequality

$$\|f(t, \phi)\| \leq L_1 \|\phi\|_{\mathcal{B}}, \quad t \in [0, \infty), \quad \phi \in \mathcal{B} \quad (3.1)$$

holds.

THEOREM 3.1. Suppose that $\mathcal{B}$ is a fading memory space, and the assumptions (H_1) and (2.4) hold true. Then, for any $\phi \in \mathcal{B}$, the Cauchy problem (1.1) has a unique mild solution on $(-\infty, \infty)$.

Proof. Let $T > 0$ be small enough such that $K^*L_1MT < 1$. We define $\mathbb{D}_\phi = \{x \in C([0, T], X) : x(0) = \phi(0)\}$ and consider the operator $\Gamma: \mathbb{D}_\phi \rightarrow \mathbb{D}_\phi$ given by

$$(\Gamma x)(t) = R(t)\phi(0) + \int_0^t R(t-s)f(s, x_s)ds.$$

Here we identify x with its extension to $(-\infty, 0]$, by $x(\theta) = \phi(\theta)$ for $\theta \leq 0$. Clearly, $\Gamma(\mathbb{D}_\phi) \subset \mathbb{D}_\phi$.

We can show easily that Γ is a contraction on $\mathbb{D}_\phi$. In fact, let $x, y \in \mathbb{D}_\phi$, we find

$$\begin{aligned} \|(\Gamma x)(t) - (\Gamma y)(t)\| &\leq L_1 M \int_0^t \|x_s - y_s\|_{\mathcal{B}} ds \\ &\leq K^* L_1 M T \|x - y\|_C, \end{aligned}$$

for all $0 \leq t \leq T$, that is

$$\|\Gamma x - \Gamma y\|_C \leq K^* L_1 M T \|x - y\|_C.$$

Therefore, we conclude that there exists a fixed point $x(\cdot)$ for Γ on $C([0, T], X)$ by Banach fixed point theorem. Now we put

$$x^1(t) = \begin{cases} x(t) & t \in [0, T]; \\ \phi(t) & t \in (-\infty, 0], \end{cases}$$

then $x^1(t)$ is clearly a mild solution of problem (1.1) on $(-\infty, T]$.

Next we extend the solutions of Eq. (1.1) to $(-\infty, 2T]$. To do so, we define $\phi^1 = x_T^1$ and consider the equation

$$\begin{cases} \frac{d}{dt} \left[y(t) + \int_0^t N(t-s)y(s)ds \right] = A \left[y(t) + \int_0^t F(t-s)y(s)ds \right] + f(t, y_t), & t \in [T, 2T], \\ y_T = \phi^1 \in \mathcal{B}, \end{cases} \quad (3.2)$$

The mild solutions of System (3.2) are given by

$$\begin{cases} y(t) = R(t-T)\phi^1(0) + \int_T^t R(t-s)f(s, y_s)ds, & t \in [T, 2T], \\ y_T = \phi^1. \end{cases}$$

Here we also identify y with its extension to $(-\infty, T]$, which is defined by $y(T+\theta) = \phi^1(\theta)$ for $\theta \leq 0$.

Put $\mathbb{D}_\phi'' = \{y \in C([T, 2T], X) : y(T) = \phi^1(0)\}$ and define the operator Γ_1 on $\mathbb{D}_\phi''$ by

$$(\Gamma_1 y)(t) = R(t-T)\phi^1(0) + \int_T^t R(t-s)f(s, y_s)ds, \quad t \in [T, 2T].$$

Using Banach fixed point theorem again, we can obtain as above a fixed point $y(t)$ of Γ_1 on $C([T, 2T], X)$. Hence, the function

$$x^2(t) = \begin{cases} y(t) & t \in [T, 2T], \\ x^1(t) & t \in (-\infty, T], \end{cases}$$

is a mild solution of Eq. (1.1) on $(-\infty, 2T]$. Thus, by induction, we get the solution

$$x^{n+1}(t) = \begin{cases} x(t) & t \in [nT, (n+1)T], \\ x^n(t) & t \in (-\infty, nT], \end{cases}$$

of Eq. (1.1) on $(-\infty, (n+1)T]$ for all $n \geq 1$. Therefore, we infer that there is a mild solution of Eq. (1.1) on $(-\infty, \infty)$ for any $\phi \in \mathcal{B}$.

Now, we verify that the uniqueness of mild solutions for Eq. (1.1) by using Gronwall's lemma. Let $u(t, \phi)$ and $v(t, \phi)$ be two solutions of Eq. (1.1) through $(0, \phi)$, then, for each $t \in [0, \infty)$,

$$\begin{aligned} \|u(t, \phi) - v(t, \phi)\| &\leq \int_0^t \|R(t-s)\| \|f(s, u_s) - f(s, v_s)\| ds \\ &\leq K^* L_1 M \int_0^t e^{-\gamma(t-s)} \sup_{0 \leq \tau \leq s} \|u(\tau) - v(\tau)\| ds \end{aligned}$$

which implies that

$$\sup_{0 \leq s \leq t} \|u(s) - v(s)\| \leq K^* L_1 M \int_0^t e^{-\gamma(t-s)} \sup_{0 \leq \tau \leq s} \|u(\tau) - v(\tau)\| ds.$$

Applying Gronwall's lemma, we infer $u(t) = v(t)$ for t in $[0, \infty)$. Therefore, $u(t, \phi) = v(t, \phi)$ for $t \in (-\infty, \infty)$. $\square$

3.2. Regularity of solutions

We devote in this subsection to the regularity of the global mild solutions for Eq. (1.1) obtained above, that is, we will provide sufficient conditions to allow the differentiability of the global mild solutions of (1.1). Let us give the definition of classical solutions of Eq. (1.1) as below.

DEFINITION 4. A function $x(\cdot): \mathbb{R} \rightarrow D(A)$ is said to be a classical solution of Eq. (1.1), if

- (1) $x \in C([0, \infty), D(A)) \cap C^1((0, \infty), X)$;
- (2) x satisfies that $x_0 = \phi$ and

$$\frac{d}{dt} \left[x(t) + \int_0^t N(t-s)x(s)ds \right] = A \left[x(t) + \int_0^t F(t-s)x(s)ds \right] + f(t, x_t) \quad \text{for } t \in [0, \infty).$$

The main result of this subsection is that:

THEOREM 3.2. Let $\mathcal{B}$ be a fading memory space satisfying axiom (C_1) and suppose that the condition (H_1) and (2.4) are fulfilled. Assume further that the function $f \in C^1([0, \infty) \times \mathcal{B}, X)$, and the partial derivatives $D_1 f(\cdot, \cdot)$, $D_2 f(\cdot, \cdot)$ are both Lipschitz continuous with respect to the second variables, i.e., there exists $l_i > 0$ such that

$$\|D_i f(t, x_1) - D_i f(t, x_2)\| \leq l_i \|\phi_1 - \phi_2\|_{\mathcal{B}}, \quad (3.3)$$

for any $t \in [0, \infty)$, $\phi_1, \phi_2 \in \mathcal{B}$, $i = 1, 2$. Then, for any initial function $\phi \in \mathcal{B}$ which is continuously differentiable with $\phi(0) \in D(A)$, the mild solution $x(\cdot)$ given by Theorem 3.1 is also a classical solution of Eq. (1.1).

Proof. Let $T > 0$ (being arbitrary), from Theorem 3.1 the unique mild solution $x(\cdot)$ of Eq. (1.1) through $(0, \phi)$ is given by

$$x(t) = R(t)\phi(0) + \int_0^t R(t-s)f(s, x_s)ds. \quad (3.4)$$

Proceeding as in the proof of Theorem 3.1, one can easily show that there exists a unique solution $y(\cdot) \in C([0, T], X)$ to the following linear integral equation on $(-\infty, T]$,

$$y(t) = \begin{cases} R(t) \left[A\phi(0) + f(0, \phi) \right] - \frac{d}{dt} \int_0^t R(t-s)N(s)\phi(0)ds + \int_0^t R(t-s)AF(s)\phi(0)ds \\ \quad + \int_0^t R(t-s) \left[D_1 f(s, x_s) + D_2 f(s, x_s)y_s \right] ds, & t \in [0, T], \\ \phi'(t), & t \in (-\infty, 0]. \end{cases} \quad (3.5)$$

We now introduce the function $z(t)$ as

$$z(t) = \begin{cases} \phi(0) + \int_0^t y(s)ds, & t \in [0, T], \\ \phi(t), & t \in (-\infty, 0]. \end{cases} \quad (3.6)$$

By Lemma 2.2, we get that

$$z_t = \phi + \int_0^t y_s ds, \quad t \in [0, T], \quad (3.7)$$

and we will show $x(\cdot) = z(\cdot)$ on $[0, T]$, which implies immediately that $x(\cdot)$ is continuously differentiable on $[0, T]$ as $z(\cdot)$ is so. Indeed, it follows from (3.5) and (3.6) that

$$\begin{aligned} z(t) &= \phi(0) + \int_0^t R(s) [A\phi(0) + f(0, \phi)] ds - \int_0^t R(t-s)N(s)\phi(0)ds \\ &\quad + \int_0^t \int_0^s R(s-\tau)AF(\tau)\phi(0)d\tau ds + \int_0^t \int_0^s R(s-\tau) \left[D_1f(\tau, x_\tau) + D_2f(\tau, x_\tau)y_\tau \right] d\tau ds. \end{aligned} \quad (3.8)$$

From (3.7), the function $t \rightarrow z_t$ and $t \rightarrow \int_0^t R(t-s)f(s, z_s)ds$ are continuously differentiable, consequently, we see that

$$\begin{aligned} \frac{d}{dt} \int_0^t R(t-s)f(s, z_s)ds &= \frac{d}{dt} \int_0^t R(s)f(t-s, z_{t-s})ds \\ &= R(t)f(0, \phi) + \int_0^t R(t-s) \left[D_1f(s, z_s) + D_2f(s, z_s)y_s \right] ds, \end{aligned}$$

which implies

$$\int_0^t R(s)f(0, \phi)ds = \int_0^t R(t-s)f(s, z_s)ds - \int_0^t \int_0^s R(s-\tau) \left[D_1f(\tau, z_\tau) + D_2f(\tau, z_\tau)y_\tau \right] d\tau ds. \quad (3.9)$$

And, on the other hand, put $x = \phi(0)$ in (2.3) and integrating from 0 to t , we find

$$\phi(0) + \int_0^t R(s)A\phi(0)ds = R(t)\phi(0) + \int_0^t R(t-s)N(s)\phi(0)ds - \int_0^t \int_0^s R(s-\tau)AF(\tau)\phi(0)d\tau ds. \quad (3.10)$$

Substituting (3.9) and (3.10) into (3.8), we then obtain that

$$\begin{aligned} z(t) &= R(t)\phi(0) + \int_0^t R(t-s)f(s, z_s)ds + \int_0^t \int_0^s R(s-\tau) \left[D_1f(\tau, x_\tau) - D_1f(\tau, z_\tau) \right] d\tau ds \\ &\quad + \int_0^t \int_0^s R(s-\tau) \left[D_2f(\tau, x_\tau) - D_2f(\tau, z_\tau) \right] y_\tau d\tau ds. \end{aligned}$$

Hence, it follows from (3.4) that

$$\begin{aligned} z(t) - x(t) &= \int_0^t R(t-s) \left(f(s, z_s) - f(s, x_s) \right) ds \\ &\quad + \int_0^t \int_0^s R(s-\tau) \left[D_1f(\tau, x_\tau) - D_1f(\tau, z_\tau) \right] d\tau ds \\ &\quad + \int_0^t \int_0^s R(s-\tau) \left[D_2f(\tau, x_\tau) - D_2f(\tau, z_\tau) \right] y_\tau d\tau ds. \end{aligned}$$

So, from the conditions (H_1) and (3.3) it yields that

$$\begin{aligned}
 \|x(t) - z(t)\| &\leq \int_0^t \|R(t-s)\| \|f(s, z_s) - f(s, x_s)\| ds \\
 &\quad + \int_0^t \int_0^s \|R(s-\tau)\| \|D_1 f(\tau, x_\tau) - D_1 f(\tau, z_\tau)\| d\tau ds \\
 &\quad + \int_0^t \int_0^s \|R(s-\tau)\| \|D_2 f(\tau, x_\tau) - D_2 f(\tau, z_\tau)\| \|y_\tau\|_{\mathcal{B}} d\tau ds \\
 &\leq \left(K^* L_1 M + K^* l_1 M T + (K^* \|y\|_C + M^* \|\phi'\|_{\mathcal{B}}) K^* l_2 M T \right) \int_0^t \sup_{0 \leq \tau \leq s} \|x(\tau) - z(\tau)\| ds,
 \end{aligned}$$

or

$$\begin{aligned}
 \sup_{0 \leq s \leq t} \|x(s) - z(s)\| &\leq \left(K^* L_1 M + K^* l_1 M T + (K^* \|y\|_C + M^* \|\phi'\|_{\mathcal{B}}) K^* l_2 M T \right) \\
 &\quad \times \int_0^t \sup_{0 \leq \tau \leq s} \|x(\tau) - z(\tau)\| ds.
 \end{aligned}$$

Then, applying Gronwall lemma, we conclude that $x(t) = z(t)$ for all $t \in [0, T]$ and $x(t)$ is continuously differentiable on $[0, T]$, and hence the following formula holds

$$x'(t) = R'(t)\phi(0) + f(t, x_t) + \int_0^t R'(t-s)f(s, x_s)ds. \quad (3.11)$$

In order to prove $x(\cdot)$ is a classical solution of Eq. (1.1) on $[0, T]$, we only need to testify that it satisfies Eq. (1.1). Using (3.4), we see that

$$\begin{aligned}
 &\frac{d}{dt} \left[x(t) + \int_0^t N(t-s)x(s)ds \right] \\
 &= \frac{d}{dt} \left[R(t)\phi(0) + \int_0^t R(t-s)f(s, x_s)ds + \int_0^t N(t-s) \left[R(s)\phi(0) + \int_0^s R(s-\tau)f(\tau, x_\tau)d\tau \right] ds \right] \\
 &= \frac{d}{dt} \left[R(t)\phi(0) + \int_0^t N(t-s)R(s)\phi(0)ds \right] \\
 &\quad + \frac{d}{dt} \left[\int_0^t R(t-s)f(s, x_s)ds + \int_0^t N(t-s) \int_0^s R(s-\tau)f(\tau, x_\tau)d\tau ds \right] \\
 &:= I_1 + I_2.
 \end{aligned}$$

From (2.2) it follows that

$$I_1 = A \left[R(t)\phi(0) + \int_0^t F(t-s)R(s)\phi(0)ds \right]. \quad (3.12)$$

As for I_2 , we write

$$I_2 = \frac{d}{dt} \left[\int_0^t R(t-s)f(s, x_s)ds \right] + \frac{d}{dt} \left[\int_0^t N(t-s) \int_0^s R(s-\tau)f(\tau, x_\tau)d\tau ds \right] := I_{21} + I_{22}. \quad (3.13)$$

According to (3.11), we have

$$I_{21} = f(t, x_t) + \int_0^t R'(t-s)f(s, x_s)ds, \quad (3.14)$$

and, by (2.2) again,

$$\begin{aligned} I_{22} &= \frac{d}{dt} \left(\int_0^t \int_\tau^t N(t-s)R(s-\tau)f(\tau, x_\tau)dsd\tau \right) \\ &= \frac{d}{dt} \left(\int_0^t \int_0^{t-\tau} N(t-\tau-s)R(s)f(\tau, x_\tau)dsd\tau \right) \\ &= A \int_0^t R(t-s)f(s, x_s)ds - \int_0^t R'(t-s)f(s, x_s)ds \\ &\quad + A \int_0^t F(t-s) \int_0^s R(s-\tau)f(\tau, x_\tau)d\tau ds. \end{aligned} \quad (3.15)$$

We substitute (3.14) and (3.15) into (3.13) yield that

$$I_2 = A \int_0^t R(t-s)f(s, x_s)ds + A \int_0^t F(t-s) \int_0^s R(s-\tau)f(\tau, x_\tau)d\tau ds + f(t, x_t). \quad (3.16)$$

Combining (3.12) and (3.16) we then obtain

$$\begin{aligned} \frac{d}{dt} \left[x(t) + \int_0^t N(t-s)x(s)ds \right] &= AR(t)\phi(0) + A \int_0^t F(t-s)R(s)\phi(0)ds + A \int_0^t R(t-s)f(s, x_s)ds \\ &\quad + A \int_0^t F(t-s) \int_0^s R(s-\tau)f(\tau, x_\tau)d\tau ds + f(t, x_t) \\ &= A \left(R(t)\phi(0) + \int_0^t R(t-s)f(s, x_s)ds \right) \\ &\quad + A \int_0^t F(t-s) \left(R(s)\phi(0) + \int_0^s R(s-\tau)f(\tau, x_\tau)d\tau \right) ds + f(t, x_t) \\ &= A \left[x(t) + \int_0^t F(t-s)x(s)ds \right] + f(t, x_t). \end{aligned}$$

Finally, as T is arbitrary, we conclude that $x(\cdot)$ is a classical solution of Eq. (1.1) on $[0, +\infty)$, which is the desired result. The proof is completed now. $\square$

4. Asymptotic periodicity of solutions

The goal of this section is to establish the existence and uniqueness of S -asymptotically ω -periodic mild solutions for the equation (1.1). Initially we introduce some concepts and properties related to S -asymptotically ω -periodic functions.

DEFINITION 5. A function $f \in C_b([0, \infty), X)$ is called to be S -asymptotically ω -periodic if there exists $\omega > 0$ such that

$$\lim_{t \rightarrow \infty} \|f(t + \omega) - f(t)\| = 0.$$

We denote by $SAP_\omega(X)$ the set of all S -asymptotically ω -periodic functions from $[0, \infty)$ to X . Note that $SAP_\omega(X)$ is a Banach space with the sup-norm $\|\cdot\|_\infty$.

DEFINITION 6. A continuous function $f: [0, \infty) \times X \rightarrow X$ is said to be uniformly S -asymptotically ω -periodic on bounded sets if for each bounded subset K of X , the set $\{f(t, x) : (t, x) \in [0, \infty) \times K\}$ is bounded, and $\lim_{t \rightarrow \infty} (f(t + \omega, x) - f(t, x)) = 0$ uniformly in $x \in K$.

DEFINITION 7. A continuous function $f: [0, \infty) \times X \rightarrow X$ is said to be asymptotically uniformly continuous on bounded sets if for every $\varepsilon > 0$ and any bounded set $K \subseteq X$, there exist constants $L_{\varepsilon, K} \geq 0$ and $\delta = \delta_{\varepsilon, K} > 0$ such that $\|f(t, x) - f(t, y)\| \leq \varepsilon$ for all $t \geq L_{\varepsilon, K}$ and $x, y \in K$ with $\|x - y\| \leq \delta_{\varepsilon, K}$.

The following two results, established respectively in [25] and [26], will play an essential role in proving the S -asymptotically ω -periodicity of solutions for Eq. (1.1).

LEMMA 4.1 (cf. [25: Lemma 4.1]). *Let X, Y be two Banach spaces, and $f: [0, \infty) \times X \rightarrow Y$ be a function uniformly S -asymptotically ω -periodic on bounded sets and asymptotically uniformly continuous on bounded sets. If $x \in SAP_\omega(X)$. Then*

$$\lim_{t \rightarrow \infty} (f(t + \omega, x(t + \omega)) - f(t, x(t))) = 0.$$

LEMMA 4.2 (cf. [26: Lemma 2.10]). *Assume that $\mathcal{B}$ is a fading memory space. Let $x: \mathbb{R} \rightarrow X$ be a function with $x_0 \in \mathcal{B}$ and $x|_{[0, \infty)} \in SAP_\omega(X)$. Then the function $t \rightarrow x_t$ belongs to $SAP_\omega(\mathcal{B})$.*

Next we give the definition of S -asymptotically ω -periodic mild solutions for Eq. (1.1).

DEFINITION 8. A bounded function $x: \mathbb{R} \rightarrow X$ is said to be S -asymptotically ω -periodic mild solution of Eq. (1.1), if $x_0 = \phi$, $x|_{[0, \infty)} \in SAP_\omega(X)$ and it verifies the integral equation

$$x(t) = R(t)\phi(0) + \int_0^t R(t-s)f(s, x_s)ds \quad \text{for } t \geq 0.$$

Now, we establish the first result of this part.

THEOREM 4.1. *Let $\mathcal{B}$ be a fading memory space and $f \in C([0, \infty) \times \mathcal{B}, X)$ be uniformly S -asymptotically ω -periodic and asymptotically uniformly continuous on bounded sets. Assume that $R(t)$ is a compact operator for each $t > 0$, and the conditions (2.4) and (3.1) are satisfied. Also, there exists a positive function $W \in L^1([0, \infty))$ such that*

$$\sup_{\|\psi\|_{\mathcal{B}} \leq \rho} \|f(t, \psi)\| \leq W(\rho) \quad (4.1)$$

for all $t \geq 0$ and $\psi \in \mathcal{B}$. Then, for any initial function $\phi \in \mathcal{B}$, Eq. (1.1) has an S -asymptotically ω -periodic mild solution provided that

$$\overline{\lim}_{\rho \rightarrow \infty} \left(\rho - \frac{MW(\rho_1)}{\gamma} \right) = \infty, \quad (4.2)$$

where $\rho_1 := K^* \rho + M^* \|\phi\|_{\mathcal{B}}$.

PROOF. Let $SAP_{\omega}^0(X) = \{x \in SAP_{\omega}(X) : x(0) = \phi(0)\}$ and the operator $\Gamma : SAP_{\omega}^0(X) \rightarrow C([0, \infty), X)$ be defined by

$$(\Gamma x)(t) = R(t)\phi(0) + \int_0^t R(t-s)f(s, x_s)ds. \quad (4.3)$$

Here we identify x with its extension to $(-\infty, 0]$, by $x(\theta) = \phi(\theta)$ for $\theta \leq 0$. By virtue of (2.4) and (3.1), we find that, for $x \in SAP_{\omega}^0(X)$, $f(\cdot, x_{(\cdot)})$ is bounded and hence $\Gamma x \in C_b([0, \infty), X)$. Moreover, from Lemma 4.1 and 4.2, we have that $f(\cdot, x_{(\cdot)}) \in SAP_{\omega}(X)$. So, for any $\varepsilon > 0$, there exists a constant $L_{\varepsilon} > 0$ such that $\|f(t + \omega, x_{t+\omega}) - f(t, x_t)\| \leq \varepsilon$ for all $t \geq L_{\varepsilon}$.

We show at first that $\Gamma x \in SAP_{\omega}^0(X)$. Utilizing the conditions (2.4) and (3.1) again, we obtain that

$$\begin{aligned} \|(\Gamma x)(t + \omega) - (\Gamma x)(t)\| &\leq \|R(t + \omega)\phi(0)\| + \|R(t)\phi(0)\| + \int_0^{\omega} \|R(t + \omega - s)\| \|f(s, x_s)\| ds \\ &\quad + \int_0^{L_{\varepsilon}} \|R(t - s)\| \|(f(s + \omega, x_{s+\omega}) - f(s, x_s))\| ds \\ &\quad + \int_{L_{\varepsilon}}^t \|R(t - s)\| \|(f(s + \omega, x_{s+\omega}) - f(s, x_s))\| ds \\ &\leq Me^{-\gamma(t+\omega)} H \|\phi\|_{\mathcal{B}} + Me^{-\gamma t} H \|\phi\|_{\mathcal{B}} + M \|f\|_{\infty} \int_0^{\omega} e^{-\gamma(t+\omega-s)} ds \\ &\quad + M \int_0^{L_{\varepsilon}} e^{-\gamma(t-s)} \|f(s + \omega, x_{s+\omega}) - f(s, x_s)\| ds \\ &\quad + M \int_{L_{\varepsilon}}^t e^{-\gamma(t-s)} \|f(s + \omega, x_{s+\omega}) - f(s, x_s)\| ds \\ &\leq 2MHe^{-\gamma t} \|\phi\|_{\mathcal{B}} + \frac{M\|f\|_{\infty}}{\gamma} e^{-\gamma t} + \frac{2M\|f\|_{\infty}}{\gamma} e^{-\gamma(t-L_{\varepsilon})} + \frac{M}{\gamma} \varepsilon, \end{aligned}$$

where $\|f\|_{\infty} = \sup_{t \geq 0} \|f(t, x_t)\| < \infty$. This implies that $\Gamma x \in SAP_{\omega}^0(X)$.

We denote $B_\rho(0, SAP_\omega^0(X)) = \{x(\cdot) \in SAP_\omega^0(X) : \|x\|_\infty \leq \rho\}$. It is clear that $B_\rho(0, SAP_\omega^0(X))$ is a bounded closed convex subset of $SAP_\omega^0(X)$. Next we prove that the operator Γ has a fix point on $B_{\rho_0}(0, SAP_\omega^0(X))$ for some $\rho_0 > 0$ (determined below) by applying Schauder fixed point theorem. To do so, we certify Γ is a completely continuous operator on $B_{\rho_0}(0, SAP_\omega^0(X))$.

We claim that there exists $\rho_0 > 0$ such that $\Gamma(B_{\rho_0}(0, SAP_\omega^0(X))) \subset B_{\rho_0}(0, SAP_\omega^0(X))$. Indeed, if it is not true, then for each $\rho > 0$, there exists a function $x_\rho(\cdot) \in B_\rho(0, SAP_\omega^0(X))$, but $\Gamma x_\rho(\cdot) \notin B_\rho(0, SAP_\omega^0(X))$, that is $\|(\Gamma x_\rho)(t)\| > \rho$ for some $t := t(\rho) \in [0, \infty)$ (dependent on ρ). On the other hand, however, in view of (4.1) and $\|x_t\|_{\mathcal{B}} \leq \rho_1$, we find that

$$\begin{aligned} \rho < \|(\Gamma x_\rho)(t)\| &\leq MH\|\phi\|_{\mathcal{B}} + M \int_0^t e^{-\gamma(t-s)} W(\rho_1) ds, \\ &\leq MH\|\phi\|_{\mathcal{B}} + \frac{MW(\rho_1)}{\gamma}, \end{aligned}$$

or

$$\rho - \frac{MW(\rho_1)}{\gamma} < MH\|\phi\|_{\mathcal{B}},$$

which contradicts (4.2). Hence there is a $\rho_0 > 0$ such that $\Gamma(B_{\rho_0}(0, SAP_\omega^0(X))) \subset B_{\rho_0}(0, SAP_\omega^0(X))$.

To show the complete continuity of the operator Γ , we first note that it is clearly continuous on $B_{\rho_0}(0, SAP_\omega^0(X))$ by the dominated convergence theorem.

Then we show that $\Gamma(B_{\rho_0}(0, SAP_\omega^0(X))) = \{\Gamma x : x \in B_{\rho_0}(0, SAP_\omega^0(X))\}$ is equicontinuous in $SAP_\omega^0(X)$. To do so, let $t_1, t_2 \in (0, \infty)$ and $\varepsilon > 0$ with $0 < \varepsilon < t_1 < t_2 < \infty$, then for any $x \in B_{\rho_0}(0, SAP_\omega^0(X))$,

$$\begin{aligned} \|(\Gamma x)(t_1) - (\Gamma x)(t_2)\| &\leq \| [R(t_2) - R(t_1)] \varphi(0) \| + \left\| \int_0^{t_1-\varepsilon} [R(t_2-s) - R(t_1-s)] f(s, x_s) ds \right\| \\ &\quad + \left\| \int_{t_1-\varepsilon}^{t_1} [R(t_2-s) - R(t_1-s)] f(s, x_s) ds \right\| + \left\| \int_{t_1}^{t_2} R(t_2-s) f(s, x_s) ds \right\| \\ &\leq \| [R(t_2) - R(t_1)] \varphi(0) \| + \int_0^{t_1-\varepsilon} \| R(t_2-s) - R(t_1-s) \| W(\rho_1) ds \\ &\quad + 2MW(\rho_1)\varepsilon + MW(\rho_1)|t_2 - t_1|. \end{aligned}$$

As $t_2 \rightarrow t_1$ and ε sufficiently small, the right-hand side of the above inequality tends to zero independently of $x \in B_{\rho_0}(0, SAP_\omega^0(X))$, since $R(t)$ is strong continuous and the compactness of $R(t)$ for $t > 0$ implies the continuity in the uniform operator topology on $(0, \infty)$. Thus, Γ maps $B_{\rho_0}(0, SAP_\omega^0(X))$ into an equicontinuous family of functions on $[0, \infty)$ (note $(\Gamma x)(0) = \phi(0)$).

Now, we prove that $(\Gamma B_{\rho_0}(0, SAP_\omega^0(X)))(t) = \{(\Gamma x)(t) : x \in B_{\rho_0}(0, SAP_\omega^0(X))\}$ is relatively compact in X for each $t \in [0, \infty)$. Obviously, $(\Gamma B_{\rho_0}(0, SAP_\omega^0(X)))(0)$ is relatively compact in X . Let $t \in (0, \infty)$ be fixed, $0 < \varepsilon < t$, for $x \in B_{\rho_0}(0, SAP_\omega^0(X))$, we define

$$(\Gamma_\varepsilon x)(t) = R(t)\phi(0) + \int_0^{t-\varepsilon} R(t-s)f(s, x_s)ds.$$

Since $R(t)(t > 0)$ is compact, we infer that $(\Gamma_\varepsilon B_{\rho_0}(0, SAP_\omega^0(X)))(t)$ is relatively compact in X . Furthermore, for $x \in B_{\rho_0}(0, SAP_\omega^0(X))$, we have

$$\|(\Gamma x)(t) - (\Gamma_\varepsilon x)(t)\| \leq \int_{t-\varepsilon}^t \|R(t-s)f(s, x_s)\| ds \leq MW(\rho_1)\varepsilon,$$

which implies that $(\Gamma B_{\rho_0}(0, SAP_\omega^0(X)))(t)$ is totally bounded, i.e, relatively compact in X . From the Arzelà–Ascoli theorem, we deduce that Γ is a completely continuous map. Hence, from Schauder's fixed point theorem there exists a fixed point $x(\cdot)$ on $SAP_\omega^0(X)$ for the operator Γ . Now, set

$$x^*(t) = \begin{cases} x(t), & t \in [0, \infty), \\ \phi(t), & t \in (-\infty, 0], \end{cases}$$

then $x^*(t)$ is clearly a S -asymptotically ω -periodic mild solution of problem (1.1) on $(-\infty, \infty)$. The proof is finished. $\square$

If $(R(t))_{t \geq 0}$ doesn't verify the compactness condition, we can prove the existence and uniqueness of S asymptotically ω -periodic solutions by Banach fixed point theorem under lipschitz conditions. That is,

(H_2) The function $f: [0, \infty) \times \mathcal{B} \rightarrow X$ is continuous and there exist a measurable bounded function $L_2 \in L^1([0, \infty), \mathbb{R}^+)$ such that

$$\|f(t, \phi_1) - f(t, \phi_2)\| \leq L_2(t)\|\phi_1 - \phi_2\|_{\mathcal{B}}, \quad t \in [0, \infty), \phi_1, \phi_2 \in \mathcal{B}.$$

THEOREM 4.2. *Let $\mathcal{B}$ be a fading memory space and f be uniformly S -asymptotically ω -periodic on bounded sets. Suppose that (2.4) and the assumption (H_2) hold. Then, for any $\phi \in \mathcal{B}$, the Cauchy problem (1.1) has a unique S -asymptotically ω -periodic mild solution on $(-\infty, \infty)$.*

Proof. Let $SAP_\omega^0(X) = \{x \in SAP_\omega(X) : x(0) = \phi(0)\}$ and the operator $\Gamma: SAP_\omega^0(X) \rightarrow C_b([0, \infty), X)$ given by (4.3). By (2.4) and the condition (H_2), we have that $f(\cdot, \cdot)$ is a bounded function. Furthermore, $\|\Gamma x\|_\infty \leq MH\|\phi\|_{\mathcal{B}} + \frac{M\|f\|_\infty}{\gamma}$, where $\|f\|_\infty = \sup_{t \geq 0} \|f(t, x_t)\|$. In addition,

from Lemma 4.1 and 4.2, we also infer that $f(\cdot, x(\cdot)) \in SAP_\omega(X)$. Hence, for arbitrary $\varepsilon > 0$, there exists constant $L_\varepsilon > 0$ such that $\|f(t + \omega, x_{t+\omega}) - f(t, x_t)\| \leq \varepsilon$ for all $t \geq L_\varepsilon$.

As in the proofs of the previous theorem, we can readily prove that $\Gamma x \in SAP_\omega^0(X)$. In the sequel we show that Γ^n is a contractive map. Let $x, y \in SAP_\omega^0(X)$, due to the condition (H_2) and (2.4) we first have that

$$\begin{aligned} \|(\Gamma x)(t) - (\Gamma y)(t)\| &\leq M \int_0^t L_2(s)\|x_s - y_s\|_{\mathcal{B}} ds \\ &\leq K^* M \|L_2\|_{L^1} \|x - y\|_\infty, \end{aligned} \tag{4.4}$$

for all $0 \leq t < \infty$. Put $u = \Gamma x$ and $v = \Gamma y$, from (4.4) we further get that

$$\begin{aligned} \|(\Gamma^2 x)(t) - (\Gamma^2 y)(t)\| &= \|(\Gamma u)(t) - (\Gamma v)(t)\| \\ &\leq K^* M \int_0^t L_2(s) \sup_{0 \leq \tau \leq s} \|u(\tau) - v(\tau)\| ds \\ &\leq \frac{(K^* M \|L_2\|_{L^1})^2}{2!} \|x - y\|_\infty, \end{aligned}$$

for each $t \in [0, \infty)$. Proceeding inductively, we then obtain that

$$\|(\Gamma^n x)(t) - (\Gamma^n y)(t)\| \leq \frac{(K^* M \|L_2\|_{L^1})^n}{n!} \|x - y\|_\infty.$$

Noting that $\frac{(K^* M \|L_2\|_{L^1})^n}{n!} < 1$ for n large enough, we deduce Γ^n is contractive and hence by Banach fixed point principle Γ has a fixed point in $SAP_\omega^0(X)$. Let

$$x^*(t) = \begin{cases} x(t) & t \in [0, \infty); \\ \phi(t) & t \in (-\infty, 0], \end{cases}$$

then $x^*(t)$ is a S -asymptotically ω -periodic solution of problem (1.1) on $(-\infty, \infty)$.

Finally, the uniqueness of S -asymptotically ω -periodic mild solutions for Eq. (1.1) can be readily verified by using Gronwall lemma. The proof is completed. $\square$

5. An example

In this section, we apply the obtained abstract results to investigate the existence and asymptotic periodicity of solutions for the following neutral partial functional integro-differential equation which arises in the theory of viscoelastic materials (see [8, 10, 20, 31]).

$$\left\{ \begin{aligned} \frac{\partial}{\partial t} \left[z(t, x) - \int_0^t (t-s) e^{-(t-s)} z(s, x) ds \right] &= \frac{\partial^2}{\partial x^2} \left[z(t, x) + \int_0^t e^{-(t-s)} z(s, x) ds \right] \\ &\quad + a(t) \int_{-\infty}^t \int_0^\pi h(s-t, x, z(s, y)) dy ds, \quad 0 \leq x \leq \pi, t \geq 0, \\ z(t, 0) = z(t, \pi) &= 0, \quad t \in [0, \infty), \\ z(\theta, x) &= \phi(\theta, x), \quad \theta \in (-\infty, 0], x \in [0, \pi], \end{aligned} \right. \quad (5.1)$$

where the functions $a(\cdot)$, $f_1(\cdot, \cdot)$ and $\phi(\cdot, \cdot)$ will be described below.

We first need to rewrite this system into the form of Eq. (1.1). For this, let the space $X = L^2([0, \pi], \mathbb{R})$ and the operator $(A, D(A))$ be defined by

$$Az = z''$$

with the domain

$$D(A) = \{z(\cdot) \in X : z', z'' \in X \text{ and } z(0) = z(\pi) = 0\}.$$

Then A generates a strongly continuous semigroup $(T(t))_{t \geq 0}$ which is compact, analytic, and self-adjoint. Moreover, A has a discrete spectrum, the eigenvalues are $-n^2, n \in \mathbb{N}$, with the corresponding normalized eigenvectors $\xi_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$, $n = 1, 2, \dots$. And, for $z \in D(A)$,

$$Az = \sum_{n=1}^{\infty} -n^2 \langle z, \xi_n \rangle \xi_n.$$

Here we choose the phase space as $\mathcal{B} = \mathcal{C}_g^0$, where the space $\mathcal{C}_g^0$ is defined in the following way. Let $g: (-\infty, 0] \rightarrow (0, \infty)$ be a continuous function satisfying $g(\theta) \rightarrow \infty$ as $\theta \rightarrow -\infty$, and the function $G(t) = \sup_{\theta \leq -t} \frac{g(t+\theta)}{g(\theta)}$ is locally bounded for $t \geq 0$, then

$$\mathcal{C}_g^0 = \left\{ \phi \in C((-\infty, 0], X) : \lim_{\theta \rightarrow -\infty} \frac{\|\phi(\theta)\|}{g(\theta)} = 0 \right\},$$

and its norm is given by

$$\|\phi\|_g = \sup_{\theta \leq 0} \frac{\|\phi(\theta)\|}{g(\theta)},$$

for $\phi \in C_g^0$. According to [23], it is known that C_g^0 satisfies the axioms (A), (A_1) , (B), (C_1) and (C_2) , and in this situation we can take $H = 1$, $K(t) = 1$ and $M(t) = G(t)$. Moreover, if $G(\cdot)$ is a bounded function, then C_g^0 is a fading memory space.

We assume that the following conditions hold:

- (a₁) The function $h: (-\infty, 0] \times [0, \pi] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable and there exists a measurable function $b_1(\cdot, \cdot): (-\infty, 0] \times [0, \pi] \rightarrow \mathbb{R}^+$ such that

$$|h(\theta, x, y_1) - h(\theta, x, y_2)| \leq b_1(\theta, x)|y_1 - y_2|,$$

and

$$|h(\theta, x, y)| \leq b_1(\theta, x)|y|,$$

for $\theta \in (-\infty, 0]$, $x \in [0, \pi]$ and $y_1, y_2, y \in \mathbb{R}$. Moreover, there holds

$$L := \int_0^\pi \left(\int_{-\infty}^0 g(\theta) b_1(\theta, x) d\theta \right)^2 dx < \infty. \quad (5.2)$$

- (a₂) The initial function ϕ defined by $\phi(\theta)(x) = \phi(\theta, x)$ belongs to C_g^0 .

We now take $u(t)(x) = z(t, x)$, and define the functions $N(t): X \rightarrow X$, $F(t): D(A) \subseteq X \rightarrow D(A)$ and $f: [0, \infty) \times C_g^0 \rightarrow X$, respectively, as

$$\begin{aligned} (F(t)u)(x) &= e^{-t}u(x), \\ (N(t)u)(x) &= -te^{-t}u(x), \end{aligned}$$

for $u(\cdot) \in X$ and

$$f(t, \phi)(x) = a(t) \int_{-\infty}^0 \int_0^\pi h(\theta, x, \phi(\theta)(y)) dy d\theta \quad \text{for } \phi \in C_g^0.$$

Then, with these notations, system (5.1) are rewritten into the form of (1.1). From [37], it is easy to see that the conditions (V_2) – (V_4) are all verified with $\hat{N}(\lambda) = -\frac{1}{(\lambda+1)^2}I$, $\hat{F}(\lambda) = \frac{1}{\lambda+1}I$ in this situation. As a result, the corresponding linear system of (5.1) has an operator $(R(t))_{t \geq 0}$, which is given by

$$R(t)z = \begin{cases} \frac{1}{2\pi i} \int_{\Gamma_{r,\vartheta}} e^{\lambda t} (\lambda I + \lambda \hat{N}(\lambda) - A - A\hat{F}(\lambda))^{-1} z d\lambda, & t > 0, z \in X, \\ I, & t = 0, \end{cases}$$

where $\Gamma_{r,\vartheta}$ is as described in Section 2. Moreover, by virtue of the property of $(\lambda I - A)^{-1}$, it is seen that $R(t)$ is compact for all $t > 0$ since the semigroup $(T(t))_{t \geq 0}$ is so. Additionally, employing the similar arguments as in [37], we can certify that $W_G = -\frac{1}{2}$ and hence there exist constants $M \geq 1$ and $\gamma > 0$ such that $\|R(t)\| \leq Me^{-\gamma t}$ for all $t \geq 0$.

In the sequel we assume that $G(\cdot)$ is a bounded function so that C_g^0 is a fading memory space. We have that,

- (1) If the function $a(\cdot) \in C^1(\mathbb{R}^+, \mathbb{R})$ and it is S -asymptotically ω -periodic, then the assumption (a_1) implies clearly that the function f satisfies the conditions of Theorem 3.1 and Theorem 3.2. Particularly, from (5.2) and direct computations it is easy to find that the Lipschitz constant L_1 in (H_1) for this case given by $L_1 = \|a\|_\infty \sqrt{\pi L}$ since $a(\cdot)$ is continuous and bounded due to its S -asymptotic periodicity. Therefore, by virtue of Theorems 3.1 and 3.2, for any initial function ϕ which is continuously differentiable on $[0, +\infty)$ with $\phi(0, x) \in D(A)$, the system (5.1) admits a unique classical solution on $(-\infty, +\infty)$. Furthermore, from Theorem 4.1 the solutions are meanwhile S -asymptotically periodic provided that (4.2) is satisfied (here $W(\rho) = L_1 \rho$).
- (2) Let now $a(\cdot)$ be S -asymptotically ω -periodic on $[0, +\infty)$ and belongs to $L^1([0, \infty), \mathbb{R}^+)$. Then (H_2) holds true with $L_2(t) = a(t)\sqrt{\pi L}$. As a consequence, Eq. (5.1) admits a unique S -asymptotically ω -periodic solution for any $\phi \in C_g^0$.

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