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Stanislas Ouaro and Malick Zoungrana

Multiplicity of solutions to discrete inclusions with the $p(k)$ -Laplace type equations

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Abstract: In this article, we prove the existence and multiplicity of solutions to discrete inclusions with the $p(k)$ -Laplace type equations. We are interested in the existence of three solutions with the aid of linking arguments and using a three critical points theorem, for locally Lipschitz continuous functions.

Keywords: $p(k)$ -Laplace; multiple solutions; discrete inclusions; three critical points theorem; locally Lipschitz continuous functions

MSC: 47A75, 35B38, 35P30, 34L05, 34L30

1 Introduction

In this work, we study the existence and multiplicity of solutions to discrete inclusions of the problem below

$$\begin{cases} -\Delta(a(k-1, \Delta u(k-1))) \in \lambda \partial F(k, u(k)), k \in \mathbb{Z} \\ u(k+m) = u(k), \forall k \in \mathbb{Z} \end{cases} \quad (1.1)$$

where $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator, $u(k) \in \mathbb{R}^n$ for all $k \in \mathbb{Z}$; $a(k, \cdot): \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous function for all $k \in \mathbb{Z}$. Let the function $F: \mathbb{Z} \times \mathbb{R}^n \rightarrow \mathbb{R}$ and $\partial F(k, u)$ denotes the Clarke subdifferential of F with respect to the second variable. Recall that if $f: X \rightarrow \mathbb{R}$ is a locally Lipschitz functional and $x \in X$, the Clarke subdifferential of f at the point x is the nonempty subset $\partial f(x)$ of X^* which is defined by

$$\partial f(x) := \{x^* \in X^* : \langle x^*, v \rangle \leq f^0(x; v), \text{ for all } v \in X\},$$

where $f^0(x; v)$ is the generalized directional derivative of f at the point x along the direction v (to be defined later).

Here, λ is a positive real parameter and $m \geq 2$ is a fixed natural number.

Discrete boundary value problems have been intensively studied in the last decade. The modeling of certain nonlinear problems from economy, biology, physics, mechanics, computer science and finance have led to the rapid development of the theory of difference equations; see the monograph of [2, 3, 8, 32] for an overview on this subject. Equations involving the discrete p -laplacian operator, subjected to classical or less classical boundary conditions, have been widely studied by many authors using various techniques. Recently, many results have been established by applying variational methods. In this direction we mention the papers [6, 17, 18, 22, 24, 29, 31] and the references therein. However, for the discrete anisotropic case, the research has not

Stanislas Ouaro: Université Ouaga I Pr Joseph KI-ZERBO, Unité de Formation et de Recherche en Sciences Exactes et Appliquées, Département de Mathématiques, B.P.7021 Ouagadougou 03, Burkina Faso, E-mail: souaro@univ-ouaga.bf, ouaro@yahoo.fr

Malick Zoungrana: Université Ouaga I Pr Joseph KI-ZERBO, Unité de Formation et de Recherche en Sciences Exactes et Appliquées, Département de Mathématiques, B.P.7021 Ouagadougou 03, Burkina Faso

been so abundant. We just point out that, in this context, some results concerning existence of solutions, their multiplicity and the existence of ground-state solutions were obtained via various critical point methods, see for instance [4, 5, 11, 16, 21, 26–28, 30, 31]. Existence and multiplicity results in the case of discrete inclusions involving the $p(k)$ -Laplace type operator are rather scarce.

Our aim is to establish the existence and multiplicity results for problem (1.1) through variational methods. We will use some abstract tools contained in [5, 8] studying the above anisotropic discrete inclusions. We also note that, for completeness, the existence of infinitely many solutions for anisotropic discrete equations has been studied in [23] by using similar methods. In [22] the authors proved the existence of one nontrivial solution for discrete p -laplacian equations by using a new technique. Finally, for completeness, the interested readers can see the recent article [13] and references therein for more details on discrete anisotropic equations. Our paper is organized as follows: some useful preliminary results are presented in Section 2. In Section 3, we recall the new abstract critical point theorems established in [12], while Section 4 is devoted to the multiplicity results for problem (1.1).

2 Basic definitions and preliminary results

Let $(E, \|\cdot\|)$ be a real Banach space. We denote by E^* the dual space of E , while $\langle \cdot, \cdot \rangle$ stands for the duality pairing between E^* and E .

A function $J : E \rightarrow \mathbb{R}$ is called locally Lipschitz continuous, if for every $u \in E$, there exist a neighbourhood V_u of u and a constant $L_u \geq 0$ such that

$$|J(z) - J(w)| \leq L_u \|z - w\| \quad \text{for all } z, w \in V_u.$$

If $u, z \in E$, we write $J^0(u; z)$ for the generalized directional derivative of J at the point u along the direction z , i.e.

$$J^0(u; z) := \limsup_{w \rightarrow u, t \rightarrow 0^+} \frac{J(w + tz) - J(w)}{t}.$$

The generalized gradient of the function J in u , denoted by $\partial J(u)$, is the set

$$\partial J(u) := \{u^* \in E^* : \langle u^*, z \rangle \leq J^0(u; z), \text{ for all } z \in E\}.$$

The basic properties of generalized directional derivative and generalized gradient were studied in [9, 10].

We recall that if J is continuously Gâteaux differentiable at u , then J is locally Lipschitz at u and $\partial J(u) = \{J'(u)\}$, where $J'(u)$ stands for the first derivative of J at u .

Furthermore, a point u is called a (generalized) critical point of the locally Lipschitz continuous function J , if $0_{E^*} \in \partial J(u)$, i.e.

$$J^0(u; z) \geq 0, \quad \text{for every } z \in E.$$

Clearly, if J is continuously Gâteaux differentiable at u , then u becomes a (classical) critical point of J , that is $J'(u) = 0_{E^*}$.

A locally Lipschitz continuous functional $J : E \rightarrow \mathbb{R}$ is said to fulfil the Palais-Smale (PS) condition if every sequence $\{u_n\}$ in E such that $\{J(u_n)\}$ is bounded and

$$J^0(u_n; u - u_n) \geq -\epsilon_n \|u - u_n\|$$

for all $u \in E$, where $\epsilon_n \rightarrow 0^+$ as $n \rightarrow \infty$, possesses a convergent subsequence.

Definition 2.1. A function f defined on a normed space X to $\overline{\mathbb{R}}$ is said to be coercive over an unbounded part P of X if $\lim_{\|x\| \rightarrow +\infty} f(x) = +\infty$, where $x \in P$. f is said to be anti-coercive if $(-f)$ is coercive.

For a complete overview on the non-smooth calculus we refer the readers to the paper [25]. Furthermore, Let us cite a recent book [20] as a general reference on the subject of our paper.

Our main tool will be the following abstract critical point theorems, for locally Lipschitz continuous functions that we recall here, for completeness, in their general form.

Theorem 2.2. ([33], Theorem 2.3) *Let E be a real Banach space. Assume that $E := E_1 \oplus E_2$, with E_2 finite-dimensional subspace of E . Let $J : E \rightarrow \mathbb{R}$ be a locally Lipschitz continuous functional satisfying the non-smooth (PS) condition and such that*

$$J(u) \leq 0, (\forall u \in \overline{B}(0, \rho) \cap E_2),$$

$$J(u) \geq 0, (\forall u \in \overline{B}(0, \rho) \cap E_1),$$

for some $\rho > 0$.

Assume also that J is bounded from below and $\inf_{u \in E} J(u) < 0$. Then, J has at least two non-zero critical points.

Theorem 2.3. [25] *Let $(E, \|\cdot\|)$ be a real Banach space and let $J : E \rightarrow \mathbb{R}$ be a locally Lipschitz continuous functional satisfying (PS) condition. If there exist $u_1, u_2 \in E$, $u_1 \neq u_2$ and $r \in (0, \|u_2 - u_1\|)$ such that $\inf\{J(u) : \|u - u_1\| = r\} \geq \max\{J(u_1), J(u_2)\}$ and we denote by Γ the family of continuous paths $\gamma : [0, 1] \rightarrow E$ joining u_1 and u_2 , then*

$c := \inf_{\gamma \in \Gamma} \max_{s \in [0, 1]} J(\gamma(s)) \geq \max\{J(u_1), J(u_2)\}$ is a critical value for E and $K_c - \{u_1, u_2\} \neq \emptyset$, where K_c is the set of critical points at the level c .

We now make the following assumptions on the data.

(H₁) : $\exists A : \mathbb{Z} \times \mathbb{R}^n \rightarrow \mathbb{R}$ with $a(k, \xi) = \frac{\partial A(k, \xi)}{\partial \xi}$, $\forall k \in \mathbb{Z}$ and $A(k, 0) = 0$ for all $k \in \mathbb{Z}$ and $\xi \in \mathbb{R}^n$.

(H₂) : $\exists C_1 > 0$ such that $|a(k, \xi)| \leq C_1(1 + |\xi|^{p(k)-1})$, $\forall k \in \mathbb{Z}$ and $\forall \xi \in \mathbb{R}^n$.

(H₃) : $(a(k, \xi) - a(k, \eta)) \cdot (\xi - \eta) > 0$, $\forall (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$ such that $\xi \neq \eta$ and $\forall k \in \mathbb{Z}$.

(H₄) : $|\xi|^{p(k)} \leq a(k, \xi) \cdot \xi \leq p(k)A(k, \xi)$, $\xi \in \mathbb{R}^n$ and $p : \mathbb{Z} \rightarrow (2, +\infty)$, where p is an m -periodic function, i.e. $p(k+m) = p(k)$, $\forall k \in \mathbb{Z}$.

(H₅) : a is an m -periodic function with respect to k , i.e. $a(m+k, \xi) = a(k, \xi)$ for all $(k, \xi) \in \mathbb{Z} \times \mathbb{R}^n$.

(H₆) : F is an m -periodic function with respect to k , i.e. $F(k, u) = F(k+m, u)$ for all $(k, u) \in \mathbb{Z} \times \mathbb{R}^n$.

(H₇) : $F(k, \cdot)$ is locally Lipschitz continuous for all $k \in \mathbb{Z}$.

(H₈) : $F(k, 0) = 0$ for all $k \in \mathbb{Z}$.

Example 2.4. As examples of functions satisfying assumptions (H₁) – (H₈), we can give the following.

1. $a(k, \xi) = |\xi|^{p(k)-2} \cdot \xi$, for $k \in \mathbb{Z}$ and $\xi \in \mathbb{R}^n$.
2. $a(k, \xi) = (1 + |\xi|^2)^{(p(k)-2)/2} \cdot \xi$, for $k \in \mathbb{Z}$ and $\xi \in \mathbb{R}^n$.
3. $F : \mathbb{Z} \times \mathbb{R}^n$ given by

$$F(k, t) \begin{cases} 16t^4 & \text{if } |t| \leq \frac{1}{2}, \\ -4|t| + 3 & \text{if } |t| \in (\frac{1}{2}, 1), \\ t^4 + |t-1| - 2 & \text{if } |t| \geq 1. \end{cases}$$

3 Three critical points theorem for locally Lipschitz functionals

In this section we recall the tools used in [7, 9, 12, 15].

Theorem 3.1. Let (X, τ) be an Hausdorff space and $\Phi, J : X \rightarrow \mathbb{R}$ be functionals. Let M be the set (possibly empty) of all the global minimizers of J and define

$$\alpha := \inf_{x \in X} \Phi(x)$$

and

$$\beta := \begin{cases} \inf_{x \in M} \Phi(x) & \text{if } M \neq \emptyset \\ \sup_{x \in X} \Phi(x) & \text{if } M = \emptyset. \end{cases}$$

Let $\alpha < \beta$ and assume that the set

$$\{x \in X : \Phi(x) + \sigma J(x) \leq \rho\},$$

for every $\sigma > 0$ and every $\rho \in \mathbb{R}$, is sequentially compact (if not empty). Then at least one of the following conditions holds.

(a) There exists a continuous mapping $h : (\alpha, \beta) \rightarrow X$ with the following property: for every $t \in (\alpha, \beta)$,

$$\Phi(h(t)) = t$$

and for every $x \in \Phi^{-1}(t)$ with $x \neq h(t)$,

$$J(x) > J(h(t)).$$

(b) There exists $\lambda > 0$ such that the functional $\Phi + \lambda J$ admits at least two global minimizers in X .

Theorem 3.2. Let E be a finite dimensional real Banach space. Let $\mu : E \rightarrow \mathbb{R}$ be a coercive C^1 functional such that $\mu(0) = 0$ and let $J : E \rightarrow \mathbb{R}$ be locally Lipschitz. Let $s > 0$ and $0 < r < s$ be fixed. Assume that

$$(b_1) \quad \lim_{\mu(u) \rightarrow +\infty} \inf \frac{J(u)}{\mu(u)} \geq 0;$$

$$(b_2) \quad \inf_{u \in E} J(u) < \inf_{\mu(u) \leq s} J(u);$$

$$(b_3) \quad J(0) \leq \inf_{r \leq \mu(u) \leq s} J(u).$$

Then, there exists $\lambda > 0$ such that the functional $\mu + \lambda J$ has at least three critical points in E , at least two of which are non-trivial.

Theorem 3.3. Let E be a finite dimensional real Banach space. Let $\mu : E \rightarrow \mathbb{R}$ be a coercive C^1 functional such that $\mu(0) = 0$ and let $J : E \rightarrow \mathbb{R}$ be a locally Lipschitz functional bounded from below. Let $s > 0$ and $0 < r < s$ be fixed constants. Assume moreover that conditions (b_2) and (b_3) hold. Then there exists $\lambda > 0$ such that the functional $\mu + \lambda J$ has at least three critical points in E .

4 Variational framework and auxiliary results

From now on, we will use the following notations.

$$p^+ := \max_{k \in \mathbb{Z}[1, m]} p(k) \text{ and } p^- := \min_{k \in \mathbb{Z}[1, m]} p(k),$$

where $\mathbb{Z}[a, b] := \{a, a+1, \dots, b\}$, with $a, b \in \mathbb{N}$ such that $a \leq b$.

Define the space

$$H_m = \left\{ u = \{u(k)\}_{k \in \mathbb{Z}} : u(k) \in \mathbb{R}^n, u(k+m) = u(k), k \in \mathbb{Z} \right\},$$

which equipped with the Euclidean norm

$$\|u\|_e := \left(\sum_{k=1}^m |u(k)|^2 \right)^{\frac{1}{2}}$$

becomes an Hilbert space.

Put for any $a \in \mathbb{R}^n$,

$$W_a := \left\{ u = \{u(k)\}_{k \in \mathbb{Z}} : u(k) = a, k \in \mathbb{Z} \right\} \text{ and } Y_a := W_a^\perp.$$

Thus W_a consists of constant sequences and we have an orthogonal decomposition

$$H_m = Y_a \oplus W_a, \text{ for any } a \in \mathbb{R}^n.$$

The energy functional corresponding to (1.1) is

$$J_m(u) = \sum_{k=1}^m A(k-1, \Delta u(k-1)) - \lambda \sum_{k=1}^m F(k, u(k)).$$

Definition 4.1. A solution of problem (1.1) is a function $u \in H_m$ such that

$$-\sum_{k=1}^m \Delta(a(k-1; \Delta u(k-1)))h(k) - \lambda \sum_{k=1}^m F^0(k; u(k))h(k) \geq 0,$$

for all $h \in H_m$ with $h(k) \geq 0$ and for all $k \in \mathbb{Z}$.

Lemma 4.2. Let $u \in H_m$ be a critical point of J_m . Then, u satisfies problem (1.1).

Proof.

It is obvious that J_m is locally Lipschitz continuous. Assume that $u \in H_m$ is a critical point of J_m . Then, for any $h \in H_m$ we obtain $J^0(u; h) \geq 0$, which means that

$$\sum_{k=1}^m a(k-1; \Delta u(k-1)) \Delta h(k-1) - \lambda \sum_{k=1}^m F^0(k; u(k))h(k) \geq 0.$$

Using Abel's summation by parts formula, (H_5) and as $h \in H_m$, it follows that

$$-\sum_{k=1}^m \Delta(a(k-1; \Delta u(k-1)))h(k) - \lambda \sum_{k=1}^m F^0(k, u(k), h(k)) \geq 0.$$

This means that $0_{H_m^*} \in \partial J(u)$ and thus u satisfies (1.1).

Now, we recall some auxiliary results, which we use later on, see [14].

Lemma 4.3. The following properties hold.

(a₁) For every $s > 0$,

$$\sum_{k=1}^m |u(k)|^s \leq m \|u\|_e^s, \text{ for all } u \in H_m.$$

(a₂) For every $s \geq 2$,

$$\sum_{k=1}^m |u(k)|^s \geq m^{\frac{(2-s)}{2}} \|u\|_e^s, \text{ for all } u \in H_m.$$

(a₃) For all $u \in H_m$,

$$\sum_{k=1}^m |\Delta u(k-1)|^{p(k-1)} \leq m \left(2^{p^+} \|u\|_e^{p^+} + 1 \right).$$

5 Multiple solutions to problem (1.1) by applying Theorem 2.2

Assume that F satisfies additionally the following.

(a_4) There exist m -periodic functions $s : \mathbb{Z} \rightarrow [2; +\infty)$, $\alpha_1 : \mathbb{Z} \rightarrow (0, +\infty)$ and a function $\alpha_2 : \mathbb{Z} \rightarrow \mathbb{R}$ for which

$$F(k, u) \geq \alpha_1(k)|u|^{s(k)} + \alpha_2(k),$$

for all $k \in \mathbb{Z}$ and $u \in \mathbb{R}^n$ such that $|u| \geq M$, where $M \geq 1$ is fixed and sufficiently large.

Put $s^- := \min_{k \in \mathbb{Z}[1, m]} s(k)$; $\alpha_1^- := \min_{k \in \mathbb{Z}[1, m]} \alpha_1(k)$; $\alpha_2^- := \min_{k \in \mathbb{Z}[1, m]} \alpha_2(k)$, where s , α_1 , α_2 are functions defined above.

Assume further that

(a_5) $\lim_{|u| \rightarrow 0} \frac{F(k, u)}{|u|^{s^-}} = 0$ uniformly in $k \in \mathbb{Z}$.

(a_6) There exist constants $\omega_0, \omega_1, \omega_2 > 0$, with $\omega_2 > \omega_1 > \omega_0$, such that for all $k \in \mathbb{Z}$:

(a_6) $F(k, u) \geq 0$ with $|u| \leq \omega_0$;

(a_6) $F(k, u) < 0$ with $\omega_1 \leq |u| \leq \omega_2$.

Example 5.1. As examples of functions satisfying assumptions (H_6) – (H_8) and (a_4) – (a_6) we can give the following, where $m \geq 2$ is a fixed even natural number.

1. $F : \mathbb{Z} \times \mathbb{R}^n$ given by

$$F(k, t) := \begin{cases} 16t^4 & \text{if } |t| \leq \frac{1}{2}, \\ -4|t| + 3 & \text{if } |t| \in (\frac{1}{2}, 1), \\ t^4 + |t - 1| - 2 & \text{if } |t| \geq 1. \end{cases}$$

and

2. $s : \mathbb{Z} \rightarrow [2, +\infty)$ such that

$$s(k) := \begin{cases} 4 & \text{if } k = 2l, \\ 2 & \text{if } k = 2l + 1; l \in \mathbb{Z}. \end{cases}$$

Lemma 5.2. Assume that conditions (H_1), (H_2), (a_2), (a_3) and (a_4) hold with $s^- > p^+$. Then, the functional J_m is anti-coercive on H_m , for all $\lambda > 0$.

Proof.

Using (H_2), (a_2), (a_3) and (a_4) we obtain

$$\begin{aligned} J_m(u) &\leq mC_1 + \left(C_1 + \frac{C_1}{p^-}\right)m \left(2^{p^+} \|u\|_e^{p^+} + 1\right) - \lambda \alpha_1^- m^{\frac{2-s^-}{2}} \|u\|_e^{s^-} - \lambda \alpha_2^- m \\ &\leq \left(C_1 + \frac{C_1}{p^-}\right) 2^{p^+} m \|u\|_e^{p^+} - \lambda \alpha_1^- m^{\frac{2-s^-}{2}} \|u\|_e^{s^-} + \left[(2C_1 + \frac{C_1}{p^-}) - \lambda \alpha_2^-\right] m. \end{aligned}$$

Since $s^- > p^+$, we see that J_m is anti-coercive on H_m .

Lemma 5.3. Assume that conditions (H_5), (H_6) and (a_4) hold with $s^- = p^+$. Then the functional J_m is anti-coercive on H_m , for any $\lambda \in (\lambda_0, +\infty)$, where $\lambda_0 = \frac{C_1(p^- + 1)2^{p^+} m^{\frac{p^+}{2}}}{p^- \alpha_1^{-1}}$.

Proof.

Based on the proof of Lemma 5.1, we get

$$J_m \leq \left[\left(C_1 + \frac{C_1}{p^-}\right) 2^{p^+} m - \lambda \alpha_1^- m^{\frac{2-s^-}{2}}\right] \|u\|_e^{p^+} + \left[(2C_1 + \frac{C_1}{p^-}) - \lambda \alpha_2^-\right] m.$$

Thus, J_m is anti-coercive on H_m for any $\lambda \in (\lambda_0, +\infty)$.

Note that for any $p^+ \geq 1$, the functional $\|\cdot\|_{p^+} : Y_a \rightarrow \mathbb{R}$ defined by

$$\|u\|_{p^+} := \left(\sum_{k=1}^m |\Delta u(k-1)|^{p^+} \right)^{\frac{1}{p^+}}$$

is a norm on Y_a , while it is obviously not a norm on H_m . Since all norms on Y_a are equivalent, therefore there exists a constant $\zeta > 0$ such that

$$\sum_{k=1}^m |\Delta u(k-1)|^{p^+} \geq \zeta \|u\|_e^{p^+}, \quad (5.1)$$

for all $u \in Y_a$

Theorem 5.4. Assume that conditions (H_1) – (H_8) and (a_4) , (a_5) , (a_6) hold with $s^- > p^+$. Let $\lambda > 0$ be fixed. Then problem (1.1) has at least three m -periodic solutions, at least two of which are non-trivial.

Proof.

Choose a positive real number ϵ satisfying

$$\epsilon \leq \frac{\zeta}{\lambda m p^+}.$$

By (a_5) and (a_6) there exists $\rho \in (0, \omega_0)$ with $\omega_0 < \frac{1}{4}$ such that

$$F(k, u) \leq \epsilon |u|^{s^-} \text{ for } |u| \leq \rho. \quad (5.2)$$

If $u \in Y_a$ with $\|u\| \leq \rho$ then $|u(k)| \leq \rho$ for all $k \in \mathbb{Z}$.

From hypothesis (H_4) and using (5.1), it follows that

$$\sum_{k=1}^m A(k-1, \Delta u(k-1)) \geq \frac{1}{p^+} \zeta \|u\|_e^{p^+}.$$

By using (5.2) we have

$$-\lambda \sum_{k=1}^m F(k, u) \geq -\lambda \epsilon \sum_{k=1}^m |u(k)|^{s^-}.$$

Consequently,

$$J_m(u) \geq \frac{1}{p^+} \zeta \|u\|_e^{p^+} - \lambda \epsilon \sum_{k=1}^m |u(k)|^{s^-} \geq \|u\|_e^{s^-} \left(\frac{1}{p^+} \zeta - \lambda \epsilon m \right) \geq 0.$$

Thus, the above relation together with (H_1) and (H_8) yields

$$J_m(u) \geq J_m(0) \text{ for all } u \in Y_a \text{ with } \|u\|_e \leq \rho.$$

Note also that for every $u \in W_a$ we have $\Delta u(k-1) = 0$ for all $k \in \mathbb{Z}$ and then, according to (H_1) , $A(k, 0) = 0$; so,

$$J_m(u) = -\lambda \sum_{k=1}^m F(k, u(k)),$$

for all $u \in W_a, \forall a \in \mathbb{R}^n$. If $u \in W_a$ with $\|u\|_e \leq \rho$ then $|u(k)| \leq \omega_0$ for all $k \in \mathbb{Z}$. Thus, by (a_4) and (H_8) it follows that

$$J_m(u) \leq J_m(0) \text{ for all } u \in W_a \text{ with } \|u\|_e \leq \rho, \text{ where } a \in \mathbb{R}^n.$$

Let $\Psi_m = -J_m$. Using Lemma 5.1 we deduce that Ψ_m satisfies the (PS) condition. Note that Ψ_m is bounded from below, moreover as Ψ_m is coercive and continuous it admits a minimizer. Using (b_2) we obtain

$$\inf_{u \in H_m} \Psi(u) = - \sup_{u \in H_m} J(u) < 0.$$

We have shown that assumptions of Theorem 2.2 are satisfied, so Ψ_m has at least three critical points, at least two of them are non-zero critical points. By Lemma 4.2 these are non-trivial m -periodic solutions of problem (1.1).

In the case $s^- = p^+$ we have the following special case.

Corollary 5.5. Assume that conditions (H_1) – (H_8) and $(a_4), (a_5), (a_6)$ hold with $s^- = p^+$. Let $\lambda \in (\lambda_0, +\infty)$. Then, problem (1.1) has at least three m -periodic solutions, at least two of which are non-trivial.

6 Multiple solutions to problem (1.1) by three critical points theorem

In this paragraph we use the Theorem 3.3 to show the existence of multiple solutions for the problem (1.1)

Let functionals $\mu, J : H_m \rightarrow \mathbb{R}$ be defined by

$$\mu(u) := \sum_{k=1}^m A(k-1, \Delta u(k-1)) \quad \text{and} \quad J(u) := - \sum_{k=1}^m F(k, u(k)).$$

Then, we see that $J_m = \mu + \lambda J$.

Assume that F has the following properties.

(H_9) : There exists a constant $C \in \mathbb{R}$ such that

$$F(k, u) \leq C \quad \text{for all } (k, u) \in \mathbb{Z} \times \mathbb{R}^n.$$

(H_{10}) : There exist numbers $\tau_1, \tau_2, \tau_3 > 0$, $\tau_3 > \tau_2 > \tau_1$ such that for all $k \in \mathbb{Z}$:

$$(H_{10}) F(k, u) < 0 \text{ with } 0 < |u| \leq \tau_1,$$

$$(H_{10}) F(k, u) > 0 \text{ with } \tau_2 \leq |u| \leq \tau_3.$$

Example 6.1. As example of function satisfying assumptions (H_6) – (H_{10}) we can give the following, where $m \geq 2$ is a fixed even natural number.

1. $F : \mathbb{Z} \times \mathbb{R}^n$ given by

$$F(k, t) := \begin{cases} -\sin |t| & \text{if } |t| \leq \pi, \\ |t - \pi| & \text{if } \pi < |t| < 2\pi, \\ \pi & \text{if } |t| \geq 2\pi. \end{cases}$$

In this paragraph we will consider the problem (1.1) on Y_a .

Remark 6.2. The functional μ is not coercive on H_m , but it is coercive on Y_a , for $a \in \mathbb{R}^n$. Indeed, given a sequence $(u_n)_{n \in \mathbb{N}}$ such that $u_n(k) = a$ for all $n \in \mathbb{N}$ and all $k \in \mathbb{Z}$, we see that

$$\mu(u_n) = \sum_{k=1}^m A(k-1, \Delta u_n(k-1)) = \sum_{k=1}^m A(k-1, 0) = 0.$$

Letting $a \rightarrow +\infty$, we see that μ is not coercive on H_m . Furthermore, by using (5.1) we get

$$\sum_{k=1}^m A(k-1, \Delta u(k-1)) \geq \frac{1}{p^+} \zeta \|u\|_e^{p^+}.$$

It is inferred that μ is coercive on Y_a .

Let us note that the solutions obtained being in Y_a , for $a \in \mathbb{R}^n$; there are therefore no constant solutions, however there may still be a zero solution.

Theorem 6.3. *Assume that conditions $(H_1) - (H_{10})$ hold. Then, there exists $\lambda > 0$ such that problem (1.1) has at least three solutions in Y_a , at least two of which are necessarily non-zero.*

Proof.

Based on previous results, μ is coercive on Y_a and C^1 , therefore, μ verifies assumptions of Theorem 3.3.

According to the hypothesis (H_9) ,

$$F(k, u(k)) \leq C \implies J(u) \geq -mC > -\infty.$$

Hence J is bounded from below.

By (H_{10}) it follows that

$$-F(k, u(k)) < 0,$$

for all $u \in Y_a$ such that $\tau_2 \leq |u(k)| \leq \tau_3$ for all $k \in \mathbb{Z}$. We thus deduce that there exists a point $u \in Y_a$ such that $J(u) < 0$. Consequently,

$$\inf_{u \in Y_a} J(u) < 0.$$

By (H_{10}) and since μ is continuous, coercive, convex, non-negative and $\mu(0) = 0$, we get that there are $s, r > 0$, such that $J(u) > 0$ for $r \leq \mu(u) \leq s$. Therefore, (b_2) is satisfied.

Now, by (H_8) and (H_{10}) we thus obtain

$$J(0) = 0 < \inf_{r \leq \mu(u) \leq s} J(u).$$

Hence, condition (b_3) is satisfied. Thus, by Theorem 3.3, we see that there exists $\lambda > 0$ such that the functional J_m has at least three critical points on Y_a . Since by Lemma 4.2 critical points of J_m are solutions of problem (1.1), hence, the result follows.

For numbers $r_0, s_0 > 0$ we put

$$r' := \inf\{\|u\|_{\max} : \mu(u) \geq r_0\} \text{ and } s' := \sup\{\|u\|_{\max} : \mu(u) \leq s_0\},$$

where

$$\|u\|_{\max} = \max_{k \in \mathbb{Z}[1, m]} |u(k)|.$$

Hence,

$$\{u \in Y_a : \mu(u) \geq r_0\} \subset \{u \in Y_a : \|u\|_{\max} \geq r'\}$$

and

$$\{u \in Y_a : \mu(u) \leq s_0\} \subset \{u \in Y_a : \|u\|_{\max} \leq s'\}.$$

Assume that F satisfies the following.

(H_{11}) : There exist constants $r_0, s_0 > 0$, $r_0 < s_0$ such that the following relations hold:

$$(H_{11}') \sup_{|u| < s'} F(k_0, u) < \sup_{u \in \mathbb{R}^n} F(k, u) \text{ for some } k_0 \in \mathbb{Z}[1, m];$$

$$(H_{11}'') F(k, u) \leq 0 \text{ for all } (k, u) \in \mathbb{Z}[1, m] \times \mathbb{R}^n \text{ with } r' \leq |u| \leq s'.$$

Example 6.4. As examples of functions satisfying assumptions $(H_6) - (H_9)$ and (H_{11}) we can give the following, where $m \geq 2$ is a fixed even natural number.

1. $F : \mathbb{Z} \times \mathbb{R}^n$ given by

$$F(k, t) := \begin{cases} \frac{3}{5}\pi|t| & \text{if } |t| \leq \frac{5}{6}\pi, \\ \sin |t| & \text{if } \frac{5}{6}\pi < |t| < 2\pi, \\ \sin |t|a(k) & \text{if } |t| \geq 2\pi, \end{cases}$$

where $a : \mathbb{Z} \rightarrow \mathbb{R}$ is defined by

$$a(k) := \begin{cases} \frac{1}{2} & \text{if } k = 2l, \\ 1 & \text{if } k = 2l + 1; l \in \mathbb{Z}. \end{cases}$$

Remark 6.5. Note that (H'_{11}) is equivalent to following hypothesis

$$\sum_{k=1}^m \sup_{|t| \leq s'} F(k, t) < \sum_{k=1}^m \sup_{t \in \mathbb{R}} F(k, t); \quad (6.1)$$

in fact from (6.1) it is easy to get (H'_{11}) , since

$$\sup_{|t| \leq s'} F(k, t) \leq \sup_{t \in \mathbb{R}} F(k, t) \quad \text{for any } s' > 0 \text{ and for any } k \in \mathbb{Z}[1, m]. \quad (6.2)$$

On the other hand, since (6.1) is equivalent to the following.

$\left(\sup_{t \in \mathbb{R}} F(1, t) - \sup_{|t| < s'} F(1, t) \right) + \left(\sup_{t \in \mathbb{R}} F(2, t) - \sup_{|t| < s'} F(2, t) \right) + \dots + \left(\sup_{t \in \mathbb{R}} F(m, t) - \sup_{|t| < s'} F(m, t) \right) > 0$, we get (6.1), noting again (6.2). Assumption (6.1) was used in the context of anisotropic problems in [15].

Theorem 6.6. Let $r_0, s_0 > 0$, $r_0 < s_0$. Assume that conditions $(H_1) - (H_9)$ and (H_{11}) hold. Then there exists $\lambda > 0$ such that problem (1.1) has at least three solutions on Y_a , at least two of which are non-trivial.

Proof.

We show that J_m verifies assumptions of Theorem 3.3. Clearly, μ is coercive and is of class C^1 on Y_a , $\mu(0) = 0$ and J is bounded from below. By (H'_{11}) , we deduce that

$$\begin{aligned} \inf_{u \in Y_a} J(u) &= - \sum_{k=1}^m \sup_{\{i \in \mathbb{Z}[1, m], u(i) \in \mathbb{R}^n\}} F(k, u(i)) \\ &< - \sum_{k=1}^m \sup_{\{i \in \mathbb{Z}[1, m], u(i) \leq s'\}} F(k, u(i)) = \inf_{\|u\|_{\max} \leq s'} J(u) \leq \inf_{\mu(u) \leq s_0} J(u). \end{aligned}$$

Thus, (b_2) is satisfied. For any $u \in Y_a$ with $r_0 \leq \mu(u) \leq s_0$, we have $r' \leq \|u\|_{\max} \leq s'$. By (H'_{11}) , we obtain

$$\begin{aligned} \inf_{r_0 \leq \mu(u) \leq s_0} J(u) &\geq \inf_{r' \leq \|u\|_{\max} \leq s'} J(u) \\ &= - \sum_{k=1}^m \sup_{\{i \in \mathbb{Z}[1, m] : \max_{i \in \mathbb{Z}[1, m]} |u(i)| \leq s'\}} F(k, u(i)) \geq 0 = J(0). \end{aligned}$$

Therefore, (b_3) is satisfied. Finally, by Theorem 3.3 there exists $\lambda > 0$ such that the functional J_m has at least three critical points in Y_a . Consequently, by Lemma 4.2, problem (1.1) has at least three solutions. Note that without the hypothesis (H'_{11}) it is possible to obtain a type of small solutions, lying near the origin. Let us consider the following remark.

Remark 6.7. Let $k_0 \in \mathbb{Z}[1, m]$ be fixed. Since $F(k_0, t) \leq C$ for some $C > 0$, so

$$\sup_{t \in \mathbb{R}^n} F(k_0, t) = \alpha \leq C.$$

If $\alpha > 0$ we can reason as follows. Since $t \mapsto F(k_0, t)$ is continuous and since $F(k_0, 0) = 0$, there exists sufficiently small positive real $s' > 0$ depending on α such that

$$-\frac{\alpha}{2} \leq F(k_0, t) \leq \frac{\alpha}{2}.$$

Thus,

$$\sup_{|t| \leq s'} F(k_0, t) < \sup_{t \in \mathbb{R}^n} F(k_0, t).$$

From the above remark we have the following result.

Corollary 6.8. Assume that conditions $(H_1) - (H_9)$ and (H_{11}'') hold. Let

$$\sup_{t \in \mathbb{R}^n} F(k_0, t) > 0,$$

for some $k_0 \in \mathbb{Z}[1, m]$. Then there exists $\lambda > 0$ such that problem (1.1) has at least three solutions on Y_a , for $a \in \mathbb{R}^n$, at least two of which are non-trivial.

We can also replace (H_{11}'') with (H_{10}') provided that τ_1 is sufficiently large. Indeed, we can reason as follows. Let $\|\cdot\|_{Y_a} : Y_a \rightarrow \mathbb{R}$ a functional defined in the following way

$$\|u\|_{Y_a} = \max_{k \in \mathbb{Z}[1, m]} |\Delta u(k-1)|.$$

It's clear that $\|\cdot\|_{Y_a}$ is a norm on Y_a while it is obviously not a norm on H_m . Since all norms on Y_a , for $a \in \mathbb{R}^n$ are equivalent, therefore there exists a constant $\zeta_0 > 0$ such that

$$\|u\|_{\max} \leq \zeta_0 \|u\|_{Y_a}. \quad (6.3)$$

Remark 6.9. Let $s_0 > \frac{1}{p^+}$, $0 < r_0 < s_0$ and $\tau_0 = \zeta_0(s_0 p^+)^{\frac{1}{p^+}}$. If (H_{10}') is satisfied with $\tau_1 \geq \tau_0$, then (H_{11}'') is also satisfied with $s' = \tau_0$. The inequality $\mu(u) \leq s_0$ is equivalent to

$$\sum_{k=1}^m A(k-1, \Delta u(k-1)) \leq s_0.$$

Hence,

$$A(k-1, \Delta u(k-1)) \leq s_0,$$

for every $k \in \mathbb{Z}[1, m]$.

Since $A(k-1, \Delta u(k-1)) \geq \frac{1}{p(k-1)} |\Delta u(k-1)|^{p(k-1)}$, we thus obtain

$$\frac{1}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} \leq s_0,$$

for every $k \in \mathbb{Z}[1, m]$.

Since $s_0 > \frac{1}{p^+}$, for every $k \in \mathbb{Z}[1, m]$, we obtain

$$|\Delta u(k-1)| \leq (s_0 p^+)^{\frac{1}{p^+}}.$$

By (6.3), we have

$$\|u\|_{\max} \leq \zeta_0 (s_0 p^+)^{\frac{1}{p^+}}$$

and

$$s' = \sup\{\|u\|_{\max} : \mu(u) \leq s_0\} = \zeta_0 (s_0 p^+)^{\frac{1}{p^+}}.$$

Further

$$\{u \in Y_a : \mu(u) \leq s_0\} \subset \{u \in Y_a : |u| \leq \zeta_0 (s_0 p^+)^{\frac{1}{p^+}}\} \subset \{u \in Y_a : |u| \leq \tau_1\}.$$

Corollary 6.10. Let $s_0 > \frac{1}{p^+}$, $0 < r_0 < s_0$ and $\tau_0 = \zeta_0(s_0 p^+)^{\frac{1}{p^+}}$. Assume that conditions $(H_1) - (H_9)$ and (H'_{11}) hold. If moreover (H'_{10}) is satisfied with $\tau_1 \geq \tau_0$, then there exists $\lambda > 0$ such that problem (1.1) has at least three solutions on Y_a , for $a \in \mathbb{R}^n$, at least two of which are non-trivial.

Analogously we obtain the following.

Corollary 6.11. Let $s_0 \leq \frac{1}{p^+}$, $0 < r_0 < s_0$ and $\tau_0 = \zeta_0(s_0 p^+)^{\frac{1}{p^+}}$. Assume that conditions $(H_1) - (H_9)$ and (H'_{11}) hold. If moreover (H'_{10}) is satisfied with $\tau_1 \geq \tau_0$, then there exists $\lambda > 0$ such that problem (1.1) has at least three solutions on Y_a , for $a \in \mathbb{R}^n$, at least two of which are non-trivial.

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