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The reconstruction of an equation of visco-elasticity

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Abstract: We are concerned with the reconstruction of two coefficients of an integro-differential equation modeling the deformation of materials with memory. We show that we can explicitly reconstruct the memory, the source terms and the diffusion constant from two observations only.

1 Introduction

Consider the inverse problem of recovering the function h , which models the memory effect, the control p , and the diffusion constant k , as treated by F. Colombo, D. Guidetti and V. Vespri, see [2],

$$\begin{cases} u_t(x, t) = k\Delta u(x, t) + \int_0^t h(t-s)\Delta u(x, s)ds + p(t)g(x) & x \in \Omega \subset \mathbb{R}^d, \quad d \geq 2 \\ u(x, 0) = a(x) \\ \partial_n u(x, t) = 0 & \text{where } x \in \partial\Omega \end{cases} \quad (1)$$

from the knowledge of two given observations

$$G_i(t) = \int_{\Omega} u(x, t)\phi_i(x)dx \quad \text{for } t \geq 0 \text{ and } i = 1, 2 \quad (2)$$

where ϕ_i is to be chosen appropriately later. We now can state the problem:

Problem: Given the functions $a, g \in L^2(\Omega)$ reconstruct $\{k, h(t), p(t)\}$ from two observations $G_i(t)$.

This note is a follow up of the work in [2] where global existence of the solution of the direct problem was shown, and so (2) holds. Since the uniqueness of the inverse problem of recovering $\{k, h(t), p(t)\}$ was proved to follow from two observations, see problem 1.2 in [2], it remains to find a reconstruction procedure which is the objective of this note. We shall do so with fewer and weaker conditions as our explicit procedure requires two simple G_i . As material science is a rapidly growing area, it calls for new and simple computational methods that helps design materials with new properties by getting explicit results from reverse engineering, [2–4].

2 Preliminaries

This note discusses the reconstruction aspects of the functions $\{k, h, p\}$ from two measurements only, see Problem 1.2 in [2]. The method is simple and makes use of the spectral theory of the Laplacian under Neumann boundary conditions which we recall now. Let us agree to denote by λ_n and φ_n its eigenvalues and eigenfunctions

$$\begin{cases} -\Delta\varphi_n(x) = \lambda_n\varphi_n(x) & x \in \Omega \\ \partial_n\varphi_n(x) = 0 & \text{if } x \in \partial\Omega. \end{cases}$$

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Recall that $\lambda_1 = 0$, $\lambda_n \rightarrow \infty$ and that φ_n is an eigenbasis of $L^2(\Omega)$, which we normalize by $\|\varphi_n\| = 1$. We assume that the boundary $\partial\Omega$ is smooth and

$$h, p \in L^1(0, \infty) \text{ are unknown but we are given } a, g \in L^2(\Omega). \quad (3)$$

The fact that g is known means that all its Fourier coefficients $g_n = \int_{\Omega} g(x) \varphi_n(x) dx$ are also given. We now prove the following lemma

Lemma 1. *If $0 \leq g \neq \text{constant}$ in $L^2(\Omega)$ then at least two Fourier coefficients are non zero, $g_1 \neq 0$ and $g_n \neq 0$ with $n \geq 2$.*

Proof. Since $0 \leq g \neq 0$ then $g_1 = |\Omega|^{-1/2} \int_{\Omega} g(x) dx > 0$. Next $g(x) = \sum g_n \varphi_n(x)$ and if all $g_n = 0$ for $n \geq 2$ then $g(x) = g_1 |\Omega|^{-1} = \text{constant}$ which is impossible. $\square$.

The unique solution of (1), which is shown to exist globally in [2], can be written as

$$u(x, t) = \sum_{n \geq 1} c_n(t) \varphi_n(x) \quad (4)$$

which converges in $L^2(\Omega)$ and

$$c_n(t) = \int_{\Omega} u(x, t) \varphi_n(x) dx. \quad (5)$$

In order to define the observation G_i , by (2), we shall choose ϕ_i so to filter and observe one c_i at a time only from the solution (4). To this end multiply (1) by φ_n and integrate to get a sequence of equations involving the Fourier coefficients c_n

$$\int_{\Omega} u_t(x, t) \varphi_n(x) dx = k \int_{\Omega} \Delta u(x, t) \varphi_n(x) dx + \int_{\Omega} \int_0^t h(t-s) \Delta u(x, s) ds \varphi_n(x) dx + p(t) \int_{\Omega} g(x) \varphi_n(x) dx$$

which reduces to the system

$$c'_n(t) = -\lambda_n k c_n(t) - \lambda_n \int_0^t h(t-s) c_n(s) ds + p(t) g_n \quad (6)$$

where we recall that $c_n(0) = \int_{\Omega} a(x) \varphi_n(x) dx$ and $g_n = \int_{\Omega} g(x) \varphi_n(x) dx$ are known but k, h and p are still unknown.

We first start by finding the function p . To this end we use, $\phi_1(x) := \varphi_1(x) = |\Omega|^{-1/2}$, and (5), to obtain

$$G_1(t) = |\Omega|^{-1/2} \int_{\Omega} u(x, t) dx = c_1(t) \quad \text{and so} \quad G_1(0) = |\Omega|^{-1/2} \int_{\Omega} a(x) dx$$

From (6) and the fact $\lambda_1 = 0$, we get

$$c'_1(t) = p(t) g_1$$

and since $g_1 \neq 0$, we get

$$p(t) = \frac{1}{g_1} G'_1(t) = \frac{1}{\int_{\Omega} g(x) dx} \frac{d}{dt} \int_{\Omega} u(x, t) dx. \quad (7)$$

Going back to (6), use $g_n \neq 0$, and $t = 0$, to get

$$c'_n(0) = -\lambda_n k c_n(0) + p(0) g_n$$

and thus

$$k = \left(p(0) g_n - c'_n(0) \right) / \lambda_n c_n(0) = \left(p(0) g_n - G'_n(0) \right) / \lambda_n G_n(0). \quad (8)$$

This means that we need to choose the corresponding $c_n(0) \neq 0$. However this is easily achieved by taking $a(x) = \sum_{n \geq 1} \varphi_n(x)/n \in L^2(\Omega)$.

Finally to find h , where we need to recall the global existence of the solution u , [2], means that equation (6) holds for $t \in (0, \infty)$ and so we can use the Laplace transform, and with $h \in L^1(0, \infty)$, we get

$$s \mathcal{L}(c_n)(s) = -\lambda_n k \mathcal{L}(c_n)(s) - \lambda_n \mathcal{L}(h)(s) \mathcal{L}(c_n)(s) + \mathcal{L}(p)(s) g_n + c_n(0). \quad (9)$$

and so

$$\lambda_n \mathcal{L}(h)(s) = -s - \lambda_n k + \frac{\mathcal{L}(p)(s) g_n + c_n(0)}{\mathcal{L}(c_n)(s)}$$

while we have from (7), $\mathcal{L}(p)(s) = \frac{s \mathcal{L}(G_1)(s) - G_1(0)}{g_1}$. It follows then

$$\lambda_n \mathcal{L}(h)(s) = s \left(\frac{g_n \mathcal{L}(G_1)(s) - g_1 \mathcal{L}(c_n)(s)}{g_1 \mathcal{L}(c_n)(s)} \right) + \frac{g_1 c_n(0) - g_n G_1(0)}{g_1 \mathcal{L}(c_n)(s)} - \lambda_n k$$

It remains to take the inverse Laplace transform to find h . Thus we have proved the main result of this note.

Proposition 2. Assume that $0 \leq g(x) = c$ in $L^2(\Omega)$, then we can reconstruct $\{k, h, p\}$ using two measurements G_i for $i = 1, 2$ given by (2).

Remark: It is clear that the global existence of the solution u means that $\mathcal{L}(h)$ is well defined. As for its inverse Laplace transform, either $\mathcal{L}(h)$ is found in tables or one has to use the many available inversion formulae to approximate h , see [1, 5].

Also it is known that from the physics point of view that $h(t) \leq 0$. This fact can easily be verified by Bernstein's theorem, [6, theorem 12f], that it is necessary and sufficient,

$$(-1)^n \frac{d^n}{ds^n} \mathcal{L}(h)(s) = \int_0^\infty e^{-st} t^n h(t) dt \leq 0 \quad \text{for any } n \geq 0, \text{ and } s > 0.$$

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