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Weak homoclinic solutions of anisotropic discrete nonlinear system with variable exponent

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Abstract: We prove the existence of weak solutions for an anisotropic homoclinic discrete nonlinear system. Suitable Hilbert spaces and norms are constructed. The proof of the main result is based on a minimization method. We also extend the problem by using generalized penalty and source functions.

Keywords: critical point theory, discrete $p(\cdot)$ -Laplacian system, weak homoclinic solutions

MSC: 47A75; 35B38; 35P30; 34L05; 34L30; 39A70; 35B20

1 Introduction

In this paper, we investigate the existence of weak solutions for the following anisotropic nonlinear discrete system.

For $i = 1, \dots, n$

$$\begin{cases} -\Delta(a(k-1)a(k-1, \Delta u_i(k-1))) + |u_i(k)|^{p(k)-2}u_i(k) = f_i(k, u(k)), & k \in \mathbb{Z} \\ \lim_{|k| \rightarrow +\infty} u_i(k) = 0, \end{cases} \quad (1.1)$$

where $\Delta u_i(k) = u_i(k+1) - u_i(k)$ is the forward difference operator and a, f_i are functions to be defined later. The difference equations is the discrete counterpart of PDEs and are usually studied in connection with numerical analysis. In this way, the main operator in problem (1.1)

$$-\Delta(a(k-1, \Delta u_i(k-1)))$$

can be seen as a discrete counterpart of the anisotropic operator

$$-\frac{\partial}{\partial x_j} a \left(x, \frac{\partial}{\partial x_j} u_i \right).$$

In the recent years, increasing attention has been paid to the study of differential and partial differential equations involving variable exponent conditions (see [2, 4, 7, 13, 15–17, 19]). The interest in the study of

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these problems was stimulated by their applications in elastic mechanics, in fluid dynamics and calculus of variations. For information on modeling physical phenomena by equations involving the $p(x)$ -growth condition, we refer the reader to ([10–12]) and the references therein.

The study of homoclinic connections for boundary value problems has had a certain impulse in recent years, motivated by applications in various biological, physical and chemical models, such as phase-transition, physical processes in which the variable transits from an unstable equilibrium to a stable one, or front-propagation in reaction-diffusion equations.

The purpose of this paper is to prove the existence of solutions to the problem (1.1) under appropriate assumptions on α , a and f . We adapt the classical minimization methods used for the study of anisotropic PDEs to prove the existence of solution of problem (1.1). Note that we examine anisotropic difference system on unbounded discrete interval, typically, on the whole set $\mathbb{Z}$, with asymptotic conditions of homoclinic type.

Remark that in the reference [8] the authors studied a particular case, where $\alpha \equiv 1$, $u(\cdot) \in \mathbb{R}$ and the function f does not depend on the solution u . The result we present in this work is more general. Indeed, we consider a system of n equations where the source function f depends on the solution $u(\cdot) = (u_1(\cdot), \dots, u_n(\cdot)) \in \mathbb{R}^n$. Also we make an extension of the main problem where we observe a competition phenomena between the functions α and σ . Our approach is critical points theorem, namely the idea of the proof is to transfer the problem of the existence of solution for (1.1) into the problem of existence of a minimizer for some associated energy functional.

The remaining part of this paper is organized as follows. Section 2 is devoted to mathematical preliminaries. The main existence result is proved in Section 3. In the Section 4, we give an extension of our system.

2 Mathematical background

In this section, we will introduce some notations, definitions and preliminary facts which are used throughout this paper.

To construct appropriate function spaces and apply critical point theory in order to investigate the existence of solutions for system (1.1), we introduce the following basic notations and results which will be used in the proofs of our main results.

We assume that the function $p : \mathbb{Z} \rightarrow (2, +\infty)$ and we denoted by

$$p^- = \min_{k \in \mathbb{Z}} p(k) \quad \text{and} \quad p^+ = \max_{k \in \mathbb{Z}} p(k).$$

For any $i \in \mathbb{Z}[1, n] = \{1, \dots, n\}$ we introduce the spaces

$$H_i^{p(\cdot)} = \left\{ u_i : \mathbb{Z} \rightarrow \mathbb{R}; \quad \rho_p(\cdot)(u_i) := \sum_{k \in \mathbb{Z}} |u_i(k)|^{p(k)} < +\infty \right\}$$

and

$$H_i^{1,p(\cdot)} = \left\{ u_i : \mathbb{Z} \rightarrow \mathbb{R}; \quad \rho_{1,p(\cdot)}(u_i) := \sum_{k \in \mathbb{Z}} |u_i(k)|^{p(k)} + \sum_{k \in \mathbb{Z}} |\Delta u_i(k)|^{p(k)} < +\infty \right\}.$$

On the space $H_i^{p(\cdot)}$, we introduce the Luxemburg norm

$$|u_i|_{p(\cdot)} := \inf \left\{ \lambda > 0; \quad \sum_{k \in \mathbb{Z}} \left| \frac{u_i(k)}{\lambda} \right|^{p(k)} \leq 1 \right\}.$$

Then

$$\begin{aligned} |u_i|_{1,p(\cdot)} &:= \inf \left\{ \lambda > 0; \quad \sum_{k \in \mathbb{Z}} \left| \frac{u_i(k)}{\lambda} \right|^{p(k)} + \sum_{k \in \mathbb{Z}} \left| \frac{\Delta u_i(k)}{\lambda} \right|^{p(k)} \leq 1 \right\} \\ &:= |u_i|_{p(\cdot)} + |\Delta u_i|_{p(\cdot)} \end{aligned}$$

is a norm on the space $H_i^{1,p(\cdot)}$.

We define the space

$$H = \left\{ u = (u_1, \dots, u_n) : \mathbb{Z} \rightarrow \mathbb{R}^n; \varrho_{1,p(\cdot)} := \sum_{i=1}^n \sum_{k \in \mathbb{Z}} |u_i(k)|^{p(k)} + \sum_{i=1}^n \sum_{k \in \mathbb{Z}} |\Delta u_i(k)|^{p(k)} < +\infty \right\}$$

with the norm

$$\begin{aligned} \|u\|_{1,p(\cdot)} &= \sum_{i=1}^n \|u_i\|_{1,p(\cdot)} \\ &= \sum_{i=1}^n \|u_i\|_{p(\cdot)} + \sum_{i=1}^n \|\Delta u_i\|_{p(\cdot)} \\ &= \|u\|_{p(\cdot)} + \|\Delta u\|_{p(\cdot)}. \end{aligned}$$

For the data a and f_i , for any $k \in \mathbb{Z}$, we assume the following.

$$(H_1) : \begin{cases} a(k, \cdot) : \mathbb{R} \rightarrow \mathbb{R}, \text{ and there exists } A(k, \cdot) : \mathbb{R} \rightarrow \mathbb{R} \text{ which satisfies } A(k, 0) = 0 \\ \text{and } a(k, \xi) = \frac{\partial}{\partial \xi} A(k, \xi). \end{cases} \quad (2.1)$$

(H_2) : For all $\xi, \eta \in \mathbb{R}$ with $\xi \neq \eta$ we have

$$(a(k, \xi) - a(k, \eta)) \cdot (\xi - \eta) > 0. \quad (2.2)$$

(H_3) : There exists a positive constant C_1 such that:

$$|a(k, \tau_i)| \leq C_1 \left(j_i(k) + |\tau_i|^{p(k)-1} \right) \quad (2.3)$$

for all $\tau_i \in \mathbb{R}$, where $j_i \in l^{p'(\cdot)}$ with $\frac{1}{p(k)} + \frac{1}{p'(k)} = 1$ for $i \in \mathbb{Z}[1, n]$.

(H_4) : For any $\xi \in \mathbb{R}$, we have

$$|\xi|^{p(k)} \leq a(k, \xi) \xi \leq p(k) A(k, \xi). \quad (2.4)$$

(H_5) : For $i \in \mathbb{Z}[1, n]$ the function $f_i : \mathbb{Z} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is Carathéodory; namely, $f_i(k, \cdot)$ is continuous for $k \in \mathbb{Z}$ and $f_i(\cdot, \xi)$ is measurable for every $\xi \in \mathbb{R}^n$ and satisfy the following conditions: there exists the functions $b_i(\cdot) : \mathbb{Z} \rightarrow (0, +\infty)$ such that

$$|f_i(k, \xi)| \leq b_i(k) |\xi_i|^{p(k)-1}. \quad (2.5)$$

We define F_i such that for each $i \in \mathbb{Z}[1, n]$, there exists $h_i \in \mathbb{R}^n$ such that

$$\nabla F_i(k, u)(h_i) = f_i(k, u) \quad \forall u \in H. \quad (2.6)$$

By (2.5) there exists $(\beta_i(\cdot))_{1 \leq i \leq n} : \mathbb{Z} \rightarrow (0, +\infty)$ such that

$$|F_i(k, u)| \leq \beta_i(k) |u_i(k)|^{p(k)} \quad \text{for all } i \in \mathbb{Z}[1, n] \quad (2.7)$$

where

$$0 < \underline{\beta} = \inf_{\{(k,i) \in \mathbb{Z} \times \mathbb{Z}[1,n]\}} \beta_i(k) \leq \sup_{\{(k,i) \in \mathbb{Z} \times \mathbb{Z}[1,n]\}} \beta_i(k) = \bar{\beta} < +\infty. \quad (2.8)$$

(H_6) : The function $\alpha \in \mathbb{Z} \rightarrow (0, +\infty)$ is such that for all $k \in \mathbb{Z}$

$$0 < \underline{\alpha} = \inf_{k \in \mathbb{Z}} \alpha(k) \leq \sup_{k \in \mathbb{Z}} \alpha(k) = \bar{\alpha} < +\infty. \quad (2.9)$$

Example 2.1.

There are many functions satisfying both $(H_1) - (H_6)$. Let us mention the following.

- $A(k, \xi) = \frac{1}{p(k)} \left((1 + |\xi|^2)^{p(k)/2} - 1 \right)$, where $a(k, \xi) = (1 + |\xi|^2)^{(p(k)-2)/2} \xi$, $\forall k \in \mathbb{Z}, \xi \in \mathbb{R}$;
- $f_i(k, \xi) = |\xi_i|^{p(k)-1}$, $\forall (k, i) \in \mathbb{Z} \times \mathbb{Z}[1, n]$ and $\xi = (\xi_1, \dots, \xi_n)$;
- $\alpha(k) = 1$, $\forall k \in \mathbb{Z}$;
- Let $(h_1, h_2, \dots, h_n)$ be the canonical base of $\mathbb{R}^n$. We have $\nabla F_i(k, \xi)(h_i) = f_i(k, \xi)$ where $F_i(k, \xi) = \frac{1}{p(k)} |\xi_i|^{p(k)}$, $\forall (k, i) \in \mathbb{Z} \times \mathbb{Z}[1, n]$ and $\xi = (\xi_1, \dots, \xi_n)$.

Remark 2.1. Let $u_i \in H_i^{1,p(\cdot)}$. Then $\lim_{|k| \rightarrow +\infty} u_i(k) = 0$.

Indeed, if $u_i \in H_i^{1,p(\cdot)}$,

$$\sum_{k \in \mathbb{Z}} |u_i(k)|^{p(k)} < \infty.$$

Let $\mathbb{Z} = \mathbb{Z}_< \cup \mathbb{Z}_\geq$ with $\mathbb{Z}_< = \{k \in \mathbb{Z}; |u_i(k)| < 1\}$ and $\mathbb{Z}_\geq = \{k \in \mathbb{Z}; |u_i(k)| \geq 1\}$. $\mathbb{Z}_\geq$ is necessary a finite set and $|u_i(k)| < \infty$ for any $k \in \mathbb{Z}_\geq$ since $u_i \in H_i^{1,p(\cdot)}$. As $\mathbb{Z}_\geq$ is a finite set, then $\sum_{k \in \mathbb{Z}_\geq} |u_i(k)|^{p^+} < \infty$.

Therefore

$$\sum_{k \in \mathbb{Z}} |u_i(k)|^{p^+} = \sum_{k \in \mathbb{Z}_<} |u_i(k)|^{p^+} + \sum_{k \in \mathbb{Z}_\geq} |u_i(k)|^{p^+} < \infty.$$

Thus $\lim_{|k| \rightarrow +\infty} u_i(k) = 0$.

Proposition 2.1. (see [8]) If $u \in H$ and $p^+ < +\infty$, then the following properties hold for $i = 1, \dots, n$:

- (a) $\|u\|_{1,p(\cdot)} > 1 \Rightarrow \|u\|_{1,p(\cdot)}^{p^-} \leq \varrho_{1,p(\cdot)}(u) \leq \|u\|_{1,p(\cdot)}^{p^+}$;
- (b) $\|u\|_{1,p(\cdot)} < 1 \Rightarrow \|u\|_{1,p(\cdot)}^{p^+} \leq \varrho_{1,p(\cdot)}(u) \leq \|u\|_{1,p(\cdot)}^{p^-}$.

Theorem 2.1. (see [8]) Let $u_i \in H_i^{p(\cdot)}$ and $v_i \in H_i^{p'(\cdot)}$ with $\frac{1}{p(k)} + \frac{1}{p'(k)} = 1$, $\forall k \in \mathbb{Z}$. Then

$$\sum_{k \in \mathbb{Z}} |u_i(k)v_i(k)| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) |u_i|_{p(\cdot)} |v_i|_{p'(\cdot)} \quad \text{for } i \in \mathbb{Z}[1, n]. \quad (2.10)$$

3 Existence of weak solutions

In this section, we state and prove our main results in this paper. Hence, we first define the weak solution of problem (1.1).

Definition 3.1. A weak solution of problem (1.1) is a function $u \in H$ such that

$$\begin{aligned} \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \alpha(k-1) a(k-1, \Delta u_i(k-1)) \Delta v_i(k-1) &+ \sum_{i=1}^n \sum_{k \in \mathbb{Z}} |u_i(k)|^{p(k)-2} u_i(k) v_i(k) \\ &= \sum_{i=1}^n \sum_{k \in \mathbb{Z}} f_i(k, u(k)) v_i(k) \end{aligned} \quad (3.1)$$

for all $v \in H$.

Note that, since H is a finite dimensional space, the weak solution coincides with the classical solution of the problem (1.1).

Theorem 3.1. Assume that $(H_1) - (H_6)$ holds. Then, there exists a weak solution of the problem (1.1).

To prove this we define the energy functional $J : H \rightarrow \mathbb{R}$ by

$$J(u) = \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \alpha(k-1) A(k-1, \Delta u_i(k-1)) + \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \frac{1}{p(k)} |u_i(k)|^{p(k)} - \sum_{i=1}^n \sum_{k \in \mathbb{Z}} F_i(k, u(k)). \quad (3.2)$$

Lemma 3.1. The functional J is well defined on H and is of class $C^1(H, \mathbb{R})$ with the derivative given by

$$\begin{aligned} \langle J'(u), v \rangle &= \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \alpha(k-1) a(k-1, \Delta u_i(k-1)) \Delta v_i(k-1) + \sum_{i=1}^n \sum_{k \in \mathbb{Z}} |u_i(k)|^{p(k)-2} u_i(k) v_i(k) \\ &\quad - \sum_{i=1}^n \sum_{k \in \mathbb{Z}} f_i(k, u(k)) v_i(k), \end{aligned} \quad (3.3)$$

for all $u, v \in H$.

Indeed, let

$$I(u) = \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \alpha(k-1) A(k-1, \Delta u_i(k-1)), \quad \Lambda(u) = \sum_{i=1}^n \sum_{k \in \mathbb{Z}} F_i(k, u(k))$$

and

$$L(u) = \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \frac{1}{p(k)} |u_i(k)|^{p(k)}.$$

We have by using assumptions (2.1), (2.3) and (2.10) that

$$\begin{aligned} |I(u)| &\leq \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \left| \alpha(k-1) A(k-1, \Delta u_i(k-1)) \right| \\ &\leq \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \bar{\alpha} C_1 \left(j_i(k-1) + \frac{1}{p(k-1)} |\Delta u_i(k-1)|^{p(k-1)-1} \right) |\Delta u_i(k-1)| \\ &\leq \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \bar{\alpha} C_1 j_i(k-1) |\Delta u_i(k-1)| + \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \frac{\bar{\alpha} C_1}{p(k-1)} |\Delta u_i(k-1)|^{p(k-1)} \\ &\leq \bar{\alpha} C_1 \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) \sum_{i=1}^n \|j_i\|_{p'(\cdot)} \|u_i\|_{p(\cdot)} + \frac{\bar{\alpha} C_1}{p^-} \sum_{i=1}^n \sum_{k \in \mathbb{Z}} |\Delta u_i(k-1)|^{p(k-1)} \\ &\leq \bar{\alpha} C_1 \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) \|j\|_{p'(\cdot)} \|u\|_{p(\cdot)} + \frac{\bar{\alpha} C_1}{p^-} \sum_{i=1}^n \sum_{k \in \mathbb{Z}} |\Delta u_i(k-1)|^{p(k-1)} \\ &< +\infty. \end{aligned}$$

By using (2.7) and (2.8)

$$\begin{aligned}
 |\Lambda(u)| &= \left| \sum_{i=1}^n \sum_{k \in \mathbb{Z}} F_i(k, u(k)) \right| \\
 &\leq \sum_{i=1}^n \sum_{k \in \mathbb{Z}} |F_i(k, u(k))| \\
 &\leq \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \beta_i(k) |u_i(k)|^{p(k)} \\
 &\leq \bar{\beta} \sum_{i=1}^n \sum_{k \in \mathbb{Z}} |u_i(k)|^{p(k)} \\
 &< +\infty.
 \end{aligned}$$

We have

$$\begin{aligned}
 |L(u)| &= \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \frac{1}{p(k)} |u_i(k)|^{p(k)} \\
 &\leq \frac{1}{p^-} \sum_{i=1}^n \sum_{k \in \mathbb{Z}} |u_i(k)|^{p(k)} < +\infty.
 \end{aligned}$$

The energy functional J is well defined on H .

Using the method applied in [9], Lemma 3.4, it is not difficult to see that the functional I , L derivative are give by

$$\begin{aligned}
 \langle I'(u), v \rangle &= \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \alpha(k-1) a(k-1, \Delta u_i(k-1)) \Delta v_i(k-1); \\
 \langle L'(u), v \rangle &= \sum_{i=1}^n \sum_{k \in \mathbb{Z}} |u_i(k)|^{p(k)-2} u_i(k) v_i(k)
 \end{aligned}$$

On the other hand, for all $u, v \in H$, there exists $h_i \in \mathbb{R}^n$ such that

$$\begin{aligned}
 \langle \Lambda'(u), v \rangle &= \lim_{t \rightarrow 0^+} \frac{\Lambda(u + tv) - \Lambda(u)}{t} \\
 &= \lim_{t \rightarrow 0^+} \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \frac{F_i(k, u(k) + tv(k)) - F_i(k, u(k))}{t} \\
 &= \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \lim_{t \rightarrow 0^+} \frac{F_i(k, u(k) + tv(k)) - F_i(k, u(k))}{t} \\
 &= \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \nabla F_i(k, u(k))(h_i) \\
 &= \sum_{i=1}^n \sum_{k \in \mathbb{Z}} f_i(k, u(k)).
 \end{aligned}$$

The functional J is clearly of class C^1

□

Lemma 3.2. *The functional J is lower semi-continuous.*

The functional Λ is completely continuous and weakly lower semi-continuous. We have to prove the semi-continuity of I .

The functional A is convex with respect to the second variable according (H_1) and (H_2) . Thus, it is enough to show that I is lower semi-continuous. For this, we fix $u \in H$ and $\varepsilon > 0$. Since I is convex; using (2.3) and (2.10) we deduce that, for any $v \in H$.

$$\begin{aligned}
 I(v) &\geq I(u) + \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \alpha(k-1) a(k-1, \Delta u_i(k-1)) (\Delta v_i(k-1) - \Delta u_i(k-1)) \\
 &\geq I(u) - \sum_{i=1}^n \sum_{k \in \mathbb{Z}} |\alpha(k-1)| |a(k-1, \Delta u_i(k-1))| |\Delta v_i(k-1) - \Delta u_i(k-1)| \\
 &\geq I(u) - \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \bar{\alpha} C_1 \left(j_i(k-1) + |\Delta u_i(k-1)|^{p(k-1)-1} \right) |\Delta v_i(k-1) - \Delta u_i(k-1)| \\
 &\geq I(u) - \bar{\alpha} C_1 \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) \sum_{i=1}^n \|g_i\|_{p'(\cdot)} \|\Delta(v_i - u_i)\|_{p(\cdot)} \\
 &\geq I(u) - \bar{\alpha} C_1 \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) \max_{1 \leq i \leq n} (g_i) \|v - u\|_{1, p(\cdot)},
 \end{aligned}$$

where $g_i(k) = j_i(k) + |\Delta u_i(k)|^{p(k)-1}$.

Finally

$$I(v) \geq I(u) - S(T, u) \|v - u\| \geq I(u) - \varepsilon, \quad (3.4)$$

for all $v \in H$ with $\|v - u\| < \delta = \frac{\varepsilon}{S(T, u)}$, where

$$S(T, u) = \left(1 + \bar{\alpha} C_1 \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) \max_{1 \leq i \leq n} (g_i) \right).$$

We conclude that J is weakly lower semi-continuous.

Proposition 3.1. *The functional J is coercive and bounded from below.*

Indeed, according to (2.4), (2.7) and (2.8) we have

$$\begin{aligned}
 J(u) &= \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \alpha(k-1) A(k-1, \Delta u_i(k-1)) + \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \frac{1}{p(k)} |u_i(k)|^{p(k)} - \sum_{i=1}^n \sum_{k \in \mathbb{Z}} F_i(k, u(k)) \\
 &\geq \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \alpha(k-1) A(k-1, \Delta u_i(k-1)) + \frac{1}{p^+} \sum_{i=1}^n \sum_{k \in \mathbb{Z}} |u_i(k)|^{p(k)} - \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \beta_i(k) |u_i(k)|^{p(k)} \\
 &\geq \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \frac{\alpha(k-1)}{p(k-1)} |\Delta u_i(k-1)|^{p(k-1)} + \frac{\alpha}{p^+} \sum_{i=1}^n \sum_{k \in \mathbb{Z}} |u_i(k)|^{p(k)} - \bar{\beta} \sum_{i=1}^n \sum_{k \in \mathbb{Z}} |u_i(k)|^{p(k)} \\
 &\geq \frac{\alpha}{p^+} \left(\sum_{i=1}^n \sum_{k \in \mathbb{Z}} |\Delta u_i(k-1)|^{p(k-1)} + \sum_{i=1}^n \sum_{k \in \mathbb{Z}} |u_i(k)|^{p(k)} \right) - \bar{\beta} \sum_{i=1}^n \sum_{k \in \mathbb{Z}} |u_i(k)|^{p(k)} \\
 &\geq \frac{\alpha}{p^+} \varrho_{1, p(\cdot)}(u) - \bar{\beta} \sum_{i=1}^n \sum_{k \in \mathbb{Z}} |u_i(k)|^{p(k)} \\
 &\geq \frac{\alpha}{p^+} \varrho_{1, p(\cdot)}(u) - M.
 \end{aligned}$$

To prove the coerciveness of the functional J , we may assume that $\|u\|_{1,p(\cdot)} > 1$. We deduce from the above inequality (a) of Proposition 2.1, that

$$J(u) \geq \frac{\alpha}{p^+} \|u\|_{1,p(\cdot)}^{p^+} - M.$$

Namely J is coercive.

Besides, for $\|u\|_{1,p(\cdot)} \leq 1$, using inequality (b) of Proposition 2.1, we have

$$\begin{aligned} J(u) &\geq \frac{\alpha}{p^+} \varrho_{1,p(\cdot)}(u) - M \\ &\geq \frac{\alpha}{p^+} \|u\|_{1,p(\cdot)}^{p^+} - M \\ &\geq -M \end{aligned}$$

Thus J is bounded from below □

Since J is weakly lower semi-continuous, bounded from below and coercive on H , using the relation between critical points of J and problem (1.1), we deduce that J has a minimizer which is a weak solution of problem (1.1).

4 An extension

In this section we are going to show that the existence result obtained for System (1.1) can be extended to more general discrete boundary value of the form

$$\begin{cases} -\Delta (\alpha(k-1)a(k-1, \Delta u_i(k-1))) + \sigma_i(k)\phi(k, u_i(k)) = \delta_i(k)f_i(k, u(k)), & k \in \mathbb{Z} \\ \lim_{|k| \rightarrow +\infty} u_i(k) = 0, \end{cases} \quad (4.1)$$

for any $i \in \mathbb{Z}[1, n]$. We shall add the following assumption.

(H₇): $\sigma_i : \mathbb{Z} \rightarrow \mathbb{R}$ and $\delta_i : \mathbb{Z} \rightarrow \mathbb{R}$ are such that $\sigma_i(k) \geq \sigma_0 > 0$ for $(k, i) \in \mathbb{Z} \times \mathbb{Z}[1, n]$ and $0 < \delta_i(k) \leq \sup_{\{(k,i) \in \mathbb{Z} \times \mathbb{Z}[1,n]\}} |\delta_i(k)| = \delta_0$.

(H₈): $\phi(k, t) = |t|^{p(k)-2}t$ for $(k, t) \in \mathbb{Z} \times \mathbb{R}$.

Definition 4.1. A weak solution of problem (4.1) is a function $u \in H$ such that

$$\begin{aligned} \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \alpha(k-1)a(k-1, \Delta u_i(k-1))\Delta v_i(k-1) &+ \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \sigma_i(k)|u_i(k)|^{p(k)-2}u_i(k)v_i(k) \\ &= \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \delta_i(k)f_i(k, u(k))v_i(k) \end{aligned}$$

for all $v \in H$.

Theorem 4.1. Under the assumptions (H₁)- (H₈) the problem (4.1) have a least one weak solution in H .

Indeed, for $u \in H$ we define the energy functional corresponding to system (4.1) by

$$\begin{aligned} J(u) &= \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \alpha(k-1) A(k-1, \Delta u_i(k-1)) + \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \frac{\sigma_i(k)}{p(k)} |u_i(k)|^{p(k)} \\ &\quad - \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \delta_i(k) F_i(k, u(k)). \end{aligned}$$

Obviously, J is class $C^1(H, \mathbb{R})$ and is weakly lower semi-continuous, and we show that

$$\begin{aligned} \langle J'(u), v \rangle &= \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \alpha(k-1) a(k-1, \Delta u_i(k-1)) \Delta v_i(k-1) \\ &\quad + \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \sigma_i(k) |u_i(k)|^{p(k)-2} u_i(k) v_i(k) - \sum_{i=1}^n \sum_{k \in \mathbb{Z}} \delta_i(k) f_i(k, u(k)) v_i(k) \end{aligned}$$

for all $u, v \in H$.

This implies that the weak solution of system (4.1) coincide with the critical points of the functional J . It suffice to prove that J is bounded below and coercive in order to complete the proof.

Indeed, we have

$$\begin{aligned} J(u) &\geq \frac{\alpha}{p^+} \sum_{i=1}^n \sum_{k \in \mathbb{Z}} |\Delta u_i(k-1)|^{p(k-1)} + \frac{\sigma_0}{p^+} \sum_{i=1}^n \sum_{k \in \mathbb{Z}} |u_i(k)|^{p(k)} - \delta_0 \bar{\beta} \sum_{i=1}^n \sum_{k \in \mathbb{Z}} |u_i(k)|^{p(k)} \\ &\geq \min \left(\frac{\alpha}{p^+}; \frac{\sigma_0}{p^+} \right) \left[\sum_{i=1}^n \sum_{k \in \mathbb{Z}} |\Delta u_i(k)|^{p(k)} + \sum_{i=1}^n \sum_{k \in \mathbb{Z}} |u_i(k)|^{p(k)} \right] - \delta_0 \bar{\beta} \sum_{i=1}^n \sum_{k \in \mathbb{Z}} |u_i(k)|^{p(k)} \\ &\geq \min \left(\frac{\alpha}{p^+}; \frac{\sigma_0}{p^+} \right) \varrho_{1,p(\cdot)}(u) - \delta_0 \bar{\beta} M. \end{aligned}$$

To prove that J is coercive, we can take $\|u\|_{1,p(\cdot)} > 1$. So, we get

$$J(u) \geq \min \left(\frac{\alpha}{p^+}; \frac{\sigma_0}{p^+} \right) \|u\|_{1,p(\cdot)}^{p^-} - \delta_0 \bar{\beta} M.$$

Since $\min \left(\frac{\alpha}{p^+}; \frac{\sigma_0}{p^+} \right) > 0$, we deduct that J is coercive.

It is easy to see that

$$J(u) \geq \min \left(\frac{\alpha}{p^+}; \frac{\sigma_0}{p^+} \right) \varrho_{1,p(\cdot)}(u) - \delta_0 \bar{\beta} M \geq -\delta_0 \bar{\beta} M.$$

Namely J is bounded from below □

Since J is weakly lower semi-continuous, bounded from below and coercive on H , using the relation between critical points of J and problem (4.1), we deduce that J has a minimizer which is a weak solution of problem (4.1).

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