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Dynamic equation on time scale with almost periodic coefficients

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Abstract: In this paper, we discuss a nonautonomous dynamical equation on time scale in a Banach space. The nonautonomous case is particularly important and needs to be studied because it is frequently met in the mathematical models of evolutionary processes. We give sufficient condition for equation to have an exponentially stable almost periodic solution in terms of the accretiveness of an operator. At the end, examples are given to illustrate the analytical findings.

Keywords: Time scale, Nonlinear equations, Evolution semigroup, Accretive operator, Almost periodic function

MSC: 47D06, 42A75, 34N05

1 Introduction

Time scale calculus was first introduced by Stefan Hilger [1] in order to unify the theory of continuous and discrete calculus. Apart from these two calculus, the theory unifies several other calculus including quantum calculus and equation defined over Cantor set etc. One advantage to work on differential equation on time scale is that the results are more general and contains several results as a particular case. Also, it is useful in several situations when we need simultaneous modelling of both continuous and discrete calculus. For example, the insect "Paroha cicada" lives as larva for 17 years and then as adults for 7 days. In this case the time domain is the union of closed subsets of the real line. For the given species, we need the following time-scale:

$$\mathbb{D} = \bigcup_{k=0}^{\infty} [k(17Y + 7D), k(17Y + 7D) + 7D],$$

where $D := \text{Days}$ and $Y := \text{Years}$. Recent decades have seen tremendous interest in the field of differential equations defined on time scale. It covers differential equations, difference equations and several other kinds of evolutionary processes. For more details, we refer to [2–9] and references therein.

Almost periodicity is very important property of a dynamical system. It was introduced by Bohr [10] as a generalization of periodicity. The other definition which in terms of sequence is given by Bochner [11]. There are several work on the almost periodic solutions of differential equations, we refer to a nice monograph [12]. This concept is more natural especially in mathematical modeling as the growth rates may not be exactly periodic but periodic with certain error. Moreover, it may capture the dynamics which may not be possible using periodic functions.

In this work, we consider the following dynamical equation on time scale

$$x^\Delta = f(t, x),$$

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where $t \in \mathbb{T}$ and x belongs to a Banach space X . The set $\mathbb{T}$ denote the time scale which is any nonempty closed subset of real line. Here, we consider the problem over unbounded time scale only. The function $f : \mathbb{T} \times X \rightarrow X$ and time scale $\mathbb{T}$ are assumed to be almost periodic in t . If f does not depend on t , then it is well known that the equation $x^\Delta = f(x)$ generates a dynamical system on time scale under some additional hypothesis. These kind of systems are called autonomous systems. The theory for such systems is very rich and lots of analysis can be performed. But if f depends explicitly on t , which is nonautonomous case, the above mentioned fact is not true. As it is evident that most of the rates appearing in the mathematical models are time dependent, hence it is very important to study nonautonomous dynamical systems. There are several work when $T = \mathbb{R}$, we refer to [13–16] and references therein, but there are not much work in this direction for general time scale $\mathbb{T}$.

Several authors have developed methods to study nonautonomous equations, for more details, we refer to [14, 15, 17, 18] and the references therein. Another direction is to employ theory of semigroups of linear as well as nonlinear operators to study such systems. Some works in this direction are by [19, 20] for linear equations, and [21–23] for nonlinear equations. In the current work, we adopt the approach given in [13], which is continuation of the approach studied in [20]. Of course, our equation is on time scale, so introduction of new term are required. We establish that the evolution semigroup associated with dynamical equation on time scale leaves the function space of almost periodic functions from $\mathbb{T}$ to X invariant. So, the smaller the space of "test functions" is the better the results are. We derive the sufficient condition which ensure the existence of an exponentially stable almost periodic solution in terms of the accretiveness of the operator $L = -\frac{d^\Delta}{dt^\Delta} + f(t, \cdot)$ which acts on the set of almost periodic functions.

2 Preliminaries

In this section, we give important definitions and results for time scales which are required for further work. A time scale, $\mathbb{T}$, is subset of real line which is non empty and closed. We denote $T_+ = \mathbb{T} \cap \mathbb{R}^{\geq 0}$. Some important operators are backward, forward and graininess operators. The backward, forward and graininess operators are define by

$$\begin{aligned}\rho(t) &= \sup\{s \in \mathbb{T} : s < t\}, \\ \sigma(t) &= \inf\{s \in \mathbb{T} : s > t\}, \\ \mu(t) &= \sigma(t) - t, \quad \forall t \in \mathbb{T},\end{aligned}$$

respectively.

Definition 2.1. A point t is a left dense point and left scattered point when $\rho(t) = t$ and $\rho(t) < t$ respectively with $t > \inf \mathbb{T}$. Also, t is right scattered point and right dense when $\sigma(t) > t$ and $\sigma(t) = t$ respectively with $t < \sup \mathbb{T}$.

An interval in time scale is defined by $[c, d]_{\mathbb{T}} = \{t \in \mathbb{T} : c \leq t \leq d\}$, $c, d \in \mathbb{T}$.

Definition 2.2. If $\Lambda : \mathbb{T} \rightarrow \mathbb{R}$ is a function and at left dense points, its left-side limits exist and continuous at right dense points of $\mathbb{T}$ then it is known as rd-continuous. The collection of all rd-continuous functions $\Lambda : \mathbb{T} \rightarrow \mathbb{R}$ will be mean by $C_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 2.3. If $f : \mathbb{T} \times X \rightarrow X$ is a function and at left dense points, its left-side limits exist and continuous at right dense points of $\mathbb{T}$ for fixed $x \in X$, then it is known as rd-continuous. The collection of all rd-continuous functions $f : \mathbb{T} \times X \rightarrow X$ will be mean by $C_{rd}(\mathbb{T} \times X, X)$.

Definition 2.4. [7] A function $q : \mathbb{T} \rightarrow \mathbb{R}$ is said to be regressive (positive regressive) if $1 + \mu(t)q(t) \neq 0 (> 0)$, $\forall t \in \mathbb{T}$. The collection of regressive (positive regressive) functions is represented by $\mathcal{R}(\mathbb{R}^+)$.

Definition 2.5. [7] Let $\Lambda : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}$. Δ -derivative, $\Lambda^\Delta(t)$ is the number if exist, such that given any $\varepsilon > 0$, $\exists$ a neighbourhood U of t such that

$$\left| [\Lambda(\sigma(t)) - \Lambda(r)] - \Lambda^\Delta(t)[\sigma(t) - r] \right| \leq \varepsilon |\sigma(t) - r|, \quad \forall r \in U.$$

Let Λ is rd-continuous; if $\Lambda_\star^\Delta(t) = \Lambda(t)$, the delta integral is defined by

$$\int_r^t \Lambda(t) \Delta t = \Lambda_\star(t) - \Lambda_\star(r), \quad r, t \in \mathbb{T}.$$

Definition 2.6. The exp function on $\mathbb{T}$ is defined as

$$e_q(r, t) = \exp \left(\int_r^t \xi_{\mu(t)}(q(t)) \Delta t \right), \quad t, r \in \mathbb{T}, q \in \mathcal{R}.$$

For $w > 0$,

$$\xi_w(Z) = \frac{1}{w} \log(1 + Zw).$$

For $w = 0$, $\xi_0(Z) = Z$.

Note that the function $e_q(t, s)$ is solution of $x^\Delta(t) = q(t)x(t)$, $t \in T$, $x(s) = 1$, where q is a regressive function.

Definition 2.7. [8] Let $q, p \in \mathcal{R}$, the following operations are defined as

$$\ominus q = \frac{-q}{1 + \mu q}, \quad q \oplus p = q + p + \mu q p, \quad q \ominus p = q \oplus (\ominus p).$$

Lemma 2.8. [8] Let us suppose that $p, q \in \mathcal{R}$, then

1. $e_0(t, r) = 1, \quad e_p(t, t) = 1;$
2. $e_p(\sigma(t), r) = (1 + \mu(t)p)e_p(t, r);$
3. $e_p(t, r) = 1/e_p(r, t) = e_{\ominus p}(r, t);$
4. $e_p(t, r)e_p(r, s) = e_p(t, s);$
5. $e_p(t, r)e_q(t, r) = e_{p \oplus q}(t, r);$
6. $(1/e_p(t, r))^\Delta = -p(t)/e_p(\sigma(t), r).$

One can see that for $\lambda > 0$, the following claim holds

$$e_{\ominus \lambda}(t, s) \leq e^{-\frac{\lambda(t-s)}{(1+\lambda\|\mu\|_\infty)}}, \quad t \geq s \in \mathbb{T}$$

with the condition that μ is bounded. Moreover, for $\lambda > 0$ we also have

$$0 < e_{\ominus \lambda}(t, s) \leq \frac{1}{1 + \lambda(t-s)}.$$

Lemma 2.9. [8] Let $q \in \mathcal{R}$ and $c, d, a \in \mathbb{T}$, then

$$\int_c^d q(\zeta) e_q(a, \sigma(\zeta)) \Delta \zeta = e_q(a, c) - e_q(a, d).$$

Definition 2.10. The time scale $\mathbb{T}$ is called periodic time scale if

$$\Pi := \{w \in \mathbb{R} : t \pm w \in \mathbb{T}, \text{ for each } t \in \mathbb{T}\} \neq \{0\}.$$

Here $\mathbb{R}$ denotes the set of real numbers.

Now, we give the definition of almost periodic function in the sense of Bochner.

Definition 2.11. A function $h : \mathbb{T} \rightarrow X$ is said to be almost periodic if for every given sequence $\{t_n\}$ there exists a subsequence $\{t_{n_k}\}$ such that the sequence of functions $h(\cdot + t_{n_k})$ is uniformly convergent on $\mathbb{T}$ as $k \rightarrow \infty$.

Definition 2.12. A function $h : \mathbb{T} \times X \rightarrow X$ is said to be almost periodic in t and uniformly for all x in any bounded subset $K \subset X$ if for every given sequence $\{t_n\}$ there exists a subsequence $\{t_{n_k}\}$ such that the sequence of functions $h(\cdot + t_{n_k}, x)$ is uniformly convergent on $\mathbb{T}$ for every $x \in K$ as $k \rightarrow \infty$.

The equation is said to have almost periodic coefficients if for every fixed $x \in X$ the function $f(\cdot, x)$ is almost periodic. The notation $AP(X)$ is the collection of all almost periodic functions from $\mathbb{T}$ to X . One can note that this space is a Banach space under the supremum norm.

Definition 2.13. A mapping $f : \mathbb{T} \times X \rightarrow X$ is said to be admissible if it satisfies the following conditions:

1. $f(t, x)$ is rd-continuous with respect to $(t, x) \in \mathbb{T} \times X$,
2. $f(t, x)$ is Lipschitz continuous with respect to x with Lipschitz constant L_f independent (dependent) on t ,
3. there exists a positive constant N such that

$$\|f(t, x)\| \leq N(1 + \|x\|)$$

for all $t \in \mathbb{T}$ and $x \in X$.

We may replace the growth condition by

$$\|f(t, x)\| \leq N(1 + \mu(t) + \|x\|),$$

the similar analysis can be performed without much change.

Definition 2.14. A mapping $f : \mathbb{T} \times X \rightarrow X$ is said to satisfy condition H if for every $v \in AP(X)$ the function $f(\cdot, v(\cdot)) \in AP(X)$.

We have the following easy proposition.

Lemma 2.15. An admissible function f satisfies condition H if and only if for every fixed $x \in X$ the function $f(\cdot, x)$ is almost periodic.

Let us denote by $X(t, s)x$ the solution of the Cauchy problem

$$x^\Delta = f(t, x) \quad t \in \mathbb{T}, \quad x(s) = x.$$

The following definition is given in [6].

Definition 2.16. Let S be a C_0 semigroup, the linear operator A is generator of S if

$$\lim_{s \rightarrow 0} \frac{S(\mu(t))x - S(s)x}{\mu(t) - s}$$

exists and equal to Ax for $x \in D(A)$.

The domain $D(A)$ of A is the set of all $x \in X$ for which the above limit exists uniformly in t .

Definition 2.17. The zero solution is called exponentially stable if there exists a positive constant d , a constant $C \in \mathbb{R}^+$, and an $M > 0$ such that for any solution $x(t)$, we have $\|x(t)\| \leq C(e_{\ominus M}(t, t_0))^d$, $t \in \mathbb{T}$.

We discuss now several versions of the Gronwall lemma, which is very important to prove several results [7].

Lemma 2.18. Let y, f be rd-continuous functions and $p \in R^+$, then

$$y(t) \leq f(t) + \int_{t_0}^t y(s)p(s)\Delta s,$$

implies

$$y(t) \leq f(t) + \int_{t_0}^t e_p(t, \sigma(s))f(s)p(s)\Delta s, \quad t_0 \in \mathbb{T}.$$

Lemma 2.19. Let y, f be rd-continuous functions and $p \in R^+$, then

$$y(t) \leq \alpha + \int_{t_0}^t y(s)p(s)\Delta s,$$

implies

$$y(t) \leq \alpha e_p(t, t_0).$$

Another version which we use is the following.

Lemma 2.20. Let y be rd-continuous functions and $a, b, c \in R$, with $c > 0$, then

$$y(t) \leq a + b(t - t_0) + c \int_{t_0}^t y(s)\Delta s,$$

implies

$$y(t) \leq \left(a + \frac{b}{c}\right)e_c(t, t_0) - \frac{b}{c}.$$

We make use of all the above versions in order to prove our results.

3 Nonlinear semigroup and almost periodicity

Let us consider the following differential equation on a Banach space X :

$$x^\Delta = f(t, x), \quad t \in \mathbb{T}, \quad (3.1)$$

with initial data $x(s) = x \in X$. Let $X(t, s)$ is the associate Cauchy operator. Now define the following operator $(S^h v)(t) = X(t, t-h)v(t-h)$, for $h \in \mathbb{T}$. We first check that the $\{S^h : h \in \mathbb{T}\}$ is evolution group. We need to just show that $\{S^h : h \in T_+\}$ is an evolution semigroup.

Lemma 3.1. Assume that the function f is admissible and satisfies condition H . Then for every $h \in T_+$, the operator S^h acts on $AP(X)$ as a Lipschitz continuous operator. In addition, the semigroup $\{S^h, h \in T_+\}$ is strongly continuous with infinitesimal generator $L = -\frac{d^\Delta}{dt^\Delta} + f(t, \cdot)$, with $D(L)$ is the set of almost periodic functions with derivative $\frac{d^\Delta u}{dt^\Delta} = u^\Delta$ is almost periodic as well.

Proof: Let $u \in AP(X)$. It is easy to see that $X(t, t-h)x$ is Lipschitz in x . Hence, for fixed h, x , the operator $X(\cdot, \cdot - h)x$ is almost periodic. Let us define

$$(S^* u)(t) = x + \int_{t-h}^t f(\xi, u(\xi))\Delta \xi.$$

Computing norm of S^* , we obtain

$$\|(S^*u)(t)\| \leq \|x\| + \int_{t-h}^t \|f(\xi, u(\xi))\| \Delta \xi \quad (3.2)$$

$$\leq \|x\| + \int_{t-h}^t N(1 + \|u(\xi)\|) \Delta \xi \quad (3.3)$$

$$\leq \|x\| + \int_{t-h}^t Nh(1 + \|u\|) \Delta \xi < \infty. \quad (3.4)$$

Now we show that $(S^*u)(t)$ is almost periodic. We use Bochner criteria to establish this result. Let $\{t_n\}_{n \in \mathbb{N}}$ be any sequence in $\mathbb{T}$, then there exists a subsequence $\{t_{n_k}\}_{k \in \mathbb{N}}$ such that $\{f(t + t_{n_k}, u(t + t_{n_k}))\}_{k \in \mathbb{N}}$ converges uniformly in t . It is due to the fact that $f(t, u(t))$ is almost periodic since $u(t)$ is almost periodic. Hence, we obtain

$$= \|(S^*u)(t + t_{n_k}) - (S^*u)(t + t_{n_p})\| \quad (3.5)$$

$$= \left\| \int_{t+t_{n_k}-h}^{t+t_{n_k}} f(\xi, u(\xi)) \Delta \xi - \int_{t+t_{n_p}-h}^{t+t_{n_p}} f(\xi, u(\xi)) \Delta \xi \right\| \quad (3.6)$$

$$\leq \int_{t-h}^t \|f(\xi + t_{n_k}, u(\xi + t_{n_k})) - f(\xi + t_{n_p}, u(\xi + t_{n_p}))\| \Delta \xi, \quad (3.7)$$

which implies that $(S^*u)(t)$ is almost periodic in t . Moreover, we can see that

$$\|S^*u - S^*v\| \leq L_f h \|u - v\|,$$

for each $u, v \in AP(X)$. Hence if h is such that $hL_f < 1$, the operator S^* has a fixed point in $AP(X)$. In addition, it coincides with $X(t, t-h)x$, which implies that $X(t, t-h)$ is almost periodic. Moreover, since the equation has almost periodic coefficients, one can see

$$\lim_{\mu(t) \rightarrow 0} \sup_t \left\| \frac{1}{\mu(t)} \int_{t-\mu(t)}^t f(\xi, X(\xi, t-\mu(t))v(t-\mu(t))) \Delta \xi - f(t, v(t)) \Delta \xi \right\| = 0.$$

Let us now denote by A the infinitesimal generator of the semigroup $\{S^h, h \in T_+\}$. By definition then v belongs to $D(A)$ if and only if the limit

$$\frac{(S^{\mu(t)}v - S^s v)}{\mu(t) - s}$$

exists as $s \rightarrow 0^+$. So, $v \in D(A)$ if and only if

$$\lim_{s \rightarrow 0^+} \sup_t \left\| \frac{X(t, t-\mu(t))v(t-\mu(t)) - X(t, t-s)v(t-s)}{\mu(t) - s} - (Av)(t) \right\| = 0$$

as $s \rightarrow 0^+$. The above equation can be written as

$$\begin{aligned} & \lim_{s \rightarrow 0^+} \sup_t \left\| \frac{X(t, t-\mu(t))v(t-\mu(t)) - X(t, t-s)v(t-s)}{\mu(t) - s} - (Av)(t) \right\| \\ &= \lim_{s \rightarrow 0^+} \sup_t \left\| \left[\frac{X(t, t-\mu(t))v(t-\mu(t)) - X(t+\mu(t), (t+\mu(t)-s)v(t+\mu(t)-s))}{\mu(t) - s} \right. \right. \\ & \quad \left. \left. - f(t, v(t)) \right] \right. \\ &+ \left. \left[f(t, v(t)) + \frac{X(t+\mu(t), (t+\mu(t)-s)v(t+\mu(t)-s)) - X(t, t-s)v(t-s)}{\mu(t) - s} \right. \right. \\ & \quad \left. \left. - (Av)(t) \right] \right\| = 0. \end{aligned}$$

Now after taking limit $s \rightarrow 0$, the above relation implies

$$\lim_{s \rightarrow 0} \sup_t \left\| f(t, v(t)) + \frac{X(t, t - (t + \mu(t)))v(t + \mu(t)) - X(t, t - s)v(t - s)}{\mu(t) - s} - (Av)(t) \right\| = 0.$$

The functions f satisfies the required condition and $f(\cdot, v(\cdot))$ is almost periodic. Hence, we can infer that the $D(A)$ consists of all functions v which is almost periodic with the property that

$$\begin{aligned} \lim_{s \rightarrow 0} \frac{X(t + \mu(t), (t + \mu(t) - s))v(t + \mu(t) - s) - X(t, t - s)v(t - s)}{\mu(t) - s} &= \lim_{s \rightarrow 0} \frac{v(t + \mu(t) - s) - v(t - s)}{\mu(t) - s} \\ &= \frac{v(t + \mu(t)) - v(t)}{\mu(t)} = \frac{v(\sigma(t)) - v(t)}{\sigma(t) - t} = v^\Delta(t). \end{aligned} \quad (3.8)$$

The above calculation is valid for $\sigma(t) > t$. If $\sigma(t) = t$, then similar analysis will work by taking limit when $t \rightarrow 0$. The later case covers $\mathbb{R}$ and any other set which is dense in $\mathbb{R}$. This implies the delta derivative v^Δ is almost periodic. Hence $D(A) = AP(X)$. Moreover, we have established that $A = -\frac{d^\Delta}{dt^\Delta} + f(t, \cdot)$.

We can also deduce from the above analysis that if the function $f(t, u)$ is uniform with respect to $u \in X$, then $m(X(\cdot, -h)u) \subset m(f)$. Here the symbol $m(u)$ denotes the module (see [12]) of an almost periodic function or a family of almost periodic functions.

Now, we prove our next result.

Theorem 3.2. *Let f satisfies condition H , let us consider $L = -\frac{d^\Delta}{dt^\Delta} + f(t, \cdot)$. The operator $(I^\sigma - \lambda L)^{-1}$ exists for sufficiently small $\lambda > 0$, where $D((I^\sigma - \lambda L)^{-1}) = AP(X)$, and $I^\sigma x = x^\sigma$.*

Proof: In order to prove the above result, we need to show that for every almost periodic g the following differential equation

$$x^\Delta = -\frac{1}{\lambda} x^\sigma + f(t, x) + \frac{1}{\lambda} g(t)$$

has a unique solution in the set of almost periodic functions $AP(X)$. We may consider time dependent t and get a more general result. So, let us consider the modified equation

$$x^\Delta = -\frac{1}{\lambda(t)} x^\sigma + f(t, x) + \frac{1}{\lambda(t)} g(t),$$

where $p(t) := \frac{1}{\lambda(t)}$ is regressive function. Hence $1 + p(t)\mu(t) \neq 0$, which is equivalent to the condition $\lambda(t) + \mu(t) \neq 0$. The functions $F_1(t, x) = -\frac{1}{\lambda} x^\sigma + f(t, x) + \frac{1}{\lambda} g(t)$ satisfies condition H as f satisfies condition H . Let us assume $P^h : h \in T_+$ is the semigroup associate with the above equation. Also, let $Y(t, s)x$ denotes the solution of above equation with the condition that $Y(s, s) = I$. Similar to last lemma, we can conclude that P^h acts on the set of almost periodic functions for every $h \in T_+$. Let us denote $F(t, x) = f(t, x) + \frac{1}{\lambda} g(t)$. Since $Y(t, s)$ is a solution, we have

$$Y(t, s)x = e_{\ominus p}(t, s)x + \int_s^t e_{\ominus p}(t, \xi) F(\xi, Y(\xi, s)x) \Delta \xi.$$

Now, for $x, y \in X$, we obtain

$$\begin{aligned} \|Y(t, s)x - Y(t, s)y\| &\leq e_{\ominus p}(t, s)\|x - y\| + \int_s^t e_{\ominus p}(t, \xi) \|F(\xi, Y(\xi, s)x) - F(\xi, Y(\xi, s)y)\| \Delta \xi \\ &\leq e_{\ominus p}(t, s)\|x - y\| + \int_s^t e_{\ominus p}(t, \xi) \|f(\xi, Y(\xi, s)x) - f(\xi, Y(\xi, s)y)\| \Delta \xi \\ &\leq e_{\ominus p}(t, s)\|x - y\| + L_f \int_s^t e_{\ominus p}(t, \xi) \|Y(\xi, s)x - Y(\xi, s)y\| \Delta \xi \\ &\leq e^{-\frac{p(t-s)}{1+\|\mu\|_\infty}} \|x - y\| + L_f \int_s^t e^{-\frac{p(t-\xi)}{1+\|\mu\|_\infty}} \|Y(\xi, s)x - Y(\xi, s)y\| \Delta \xi. \end{aligned}$$

Multiplying both side by $e^{\frac{pt}{1+\|\mu\|_\infty}}$, we obtain

$$e^{\frac{pt}{1+\|\mu\|_\infty}} \|Y(t, s)x - Y(t, s)y\| \leq e^{\frac{ps}{1+\|\mu\|_\infty}} \|x - y\| + L_f \int_s^t e^{\frac{p\xi}{1+\|\mu\|_\infty}} \|Y(\xi, s)x - Y(\xi, s)y\| \Delta\xi.$$

Applying the Gronwall's lemma 2.19 for time scale, we obtain

$$e^{\frac{pt}{1+\|\mu\|_\infty}} \|Y(t, s)x - Y(t, s)y\| \leq e^{\frac{ps}{1+\|\mu\|_\infty}} e_{L_f}(t, s) \|x - y\|.$$

Here, we may assume that L_f is time dependent. Rearranging the terms, we obtain

$$\|Y(t, s)x - Y(t, s)y\| \leq e^{-\frac{p(t-s)}{1+\|\mu\|_\infty}} e_{L_f}(t, s) \|x - y\|.$$

We compute the following

$$\begin{aligned} \|P^h v - P^h w\| &= \sup_t \|(P^h v)(t) - (P^h w)(t)\| \\ &= \sup_t \|Y(t, t-h)v(t-h) - Y(t, t-h)w(t-h)\| \\ &\leq e^{-\frac{ph}{1+\|\mu\|_\infty}} e_{L_f}(t, t-h) \|v(t-h) - w(t-h)\| \\ &\leq e^{-\frac{ph}{1+\|\mu\|_\infty}} e_{L_f}(t, t-h) \|v - w\|. \end{aligned} \quad (3.9)$$

Thus for $0 < e^{-\frac{ph}{1+\|\mu\|_\infty}} e_{L_f}(t, t-h) < 1$, the operator P^h is a contraction. Hence, for every fixed $h \in T_+ - \{0\}$, employing Banach contraction theorem, there exists a unique fixed point v_h of P^h . Moreover, it is evident that this fixed point is almost periodic. We can see that $P^h P^1 v_h = P^{h+1} v_h = P^1 P^h v_h = P^1 v_h$, which implies $P^1 v_h = v_h$. Hence, $v_1 = v_h$ for all $h \in T_+$. Moreover, $v_1 \in D(L)$ and $Lv_1 = 0$, which implies that it is almost periodic. So if $w(t)$ is any other almost periodic solution, then $w(t)$ is a fixed point of P^h . Thus, we conclude that $w(t) = v_1$. Hence the differential equation has at most one almost periodic solution.

Theorem 3.3. *Let f satisfies condition H and the operator $\alpha I - L$ is accretive for some α . Then there exists a unique almost periodic solution of equation (3.1) which is globally exponentially stable.*

Proof: We show that the operator $L = -\frac{d^A}{dt^A} + f(t, \cdot)$ is closed. Let $\{x_n\}_{n \in \mathbb{N}}$ is a sequence in the space of almost periodic functions and $x_n \rightarrow x$, $Lx_n \rightarrow y$. Now, we have

$$\|f(t, x_n(t)) - f(t, x(t))\| \leq L_f \|u_n(t) - u(t)\|$$

for $t \in R$. Moreover, $\frac{d^A}{dt^A}$ is closed in $AP(X)$. (easy to check), which implies that the operator L is closed. If we pick $y \in AP^1(X)$, then $S^h y$ is also in $AP^1(X)$ due to fact that

$$\begin{aligned} (S^h y)(t) &= X(t, t-h)y(t-h) \\ &= y(t-h) + \int_{t-h}^t f(\xi, X(\xi, t-h)y(t-h)) \Delta\xi. \end{aligned}$$

Hence $\frac{d^A}{dt^A} S^t y = L S^t y$ for all $y \in AP^1(X)$., Using Brezis-Pazy theorem [24], we obtain (i) $\lim_{n \rightarrow \infty} (I - \frac{h}{n} L)^{-n} u = S^h u$ for u almost periodic and (ii) The inequality $\|S^h u - S^h v\| \leq e_\alpha(h, 0) \|u - v\|$. The mapping S^h is a contraction if $\alpha < 0$. (may depend on t .) So, one can infer the existence of unique almost periodic solution. Let us call this solution $x^*(t)$. Furthermore, for $y \in X$, let $u(t) = y$, we obtain

$$(S^h u)(h) - (S^h x^*)(h) \leq e_\alpha(h, 0) \sup_t \|y - x^*(t)\|.$$

The above relation implies that x^* is globally exponentially stability.

We can also observe that $m(x^*) \subset m(f)$. Here the symbol $m(u)$ denotes the module (see [12]) of an almost periodic function or a family of almost periodic functions

Let us consider the space $AP_f(X)$ which consists of all almost periodic function v such that $m(v) \subset m(f)$. Define the following operator

$$(Qy)(t) = v(t-h) + \int_{t-h}^t f(\xi, y(\xi)) \Delta \xi.$$

The operator Q is contraction with fixed point given by $(S^h v)(t)$. Hence $x^* \in AP_f(X)$.

Next theorem is about perturbation of the equation (1).

Theorem 3.4. *If all condition of Theorem (3.3) are satisfied and let g satisfies condition H with Lipschitz constant L_g such that $\beta = \alpha + L_g < 0$, then the operator $\gamma I - A$ is accretive. The notation $A = -\frac{d^\Delta}{dt^\Delta} + f(t, \cdot) + g(t, \cdot)$ and the corresponding perturbed equation*

$$x^\Delta = f(t, x) + g(t, x)$$

has a unique globally exponentially stable almost periodic solution.

Proof: It is enough to show that the operator $\gamma I - A$ is accretive and then we can apply Theorem (3.3). For every $u, v \in AP(X)$ and $\lambda > 0$, computing the norm, we obtain

$$\begin{aligned} & \| (I + \lambda(\gamma I - L - g))u - (I + \lambda(\gamma I - L - g))v \| \\ &= \| ((1 + \lambda L_g)I + \lambda(\alpha I - L))u - ((1 + \lambda L_g)I + \lambda(\alpha I - L))v + \lambda g u - \lambda g v \| \\ &\geq (1 + \lambda L_g) \| u - v \| - \lambda L_g \| u - v \| \\ &= \| u - v \|. \end{aligned}$$

Hence, we have our desired result.

Next, we prove results regarding perturbed equation. Let us assume all the assumptions of previous theorem. may also note that, one can choose time dependent N and δ which are bounded. The proof will still work. Then, we have the following result.

Theorem 3.5. *Let $x_0(t)$ is the unique almost periodic solution which is exponentially stable of the unperturbed equation. The unique exponentially stable almost periodic solution $x_p(t)$ of perturbed equation depends continuously on the perturbation.*

Proof: Let us assume that S_*^h is the evolution operator and $X_*(t, s)$ is the Cauchy operator associated with the perturbed equation. We can observe the following

$$\begin{aligned} \| X(t, s)x - X_*(t, s)x \| &\leq \int_s^t \| f(\xi, X(\xi, s)x) - f(\xi, X_*(\xi, s)x) - g(\xi, X_*(\xi, s)x) \| \Delta \xi \\ &\leq \int_s^t L_f \| X(\xi, s)x - X_*(\xi, s)x \| \Delta \xi + L_g \int_s^t \| X_*(\xi, s)x \| \Delta \xi + \int_s^t \| g(\xi, 0) \| \Delta \xi. \end{aligned}$$

Denote $\delta_1 = \sup_\xi \| g(\xi, 0) \|$, $\delta = \max\{\delta_1, L_g\}$. We can compute

$$\| f(t, x) + g(t, x) \| \leq (N + \delta)(1 + \mu(t) + \| x \|).$$

Thus, we obtain

$$\begin{aligned} \| X_*(t, s)x \| &\leq \| x \| + \int_s^t (N + \delta)(1 + X_*(\xi, s)x) \Delta \xi \\ &= (\| x \| + (N + \delta)(t - s) + \int_s^t (N + \delta) X_*(\xi, s)x \Delta \xi). \end{aligned}$$

Using the another version of Gronwall 2.20, we have

$$\|X_*(t, s)x\| \leq (\|x\| + 1)e_{N+\delta}(t, s) - 1.$$

Substituting the above relation, we obtain

$$\begin{aligned} \|X(t, s)x - X_*(t, s)x\| &\leq \int_s^t L_f \|X(\xi, s)x - X_*(\xi, s)x\| \Delta \xi \\ &\quad + L_g \int_s^t [(\|x\| + 1)e_{N+\delta}(\xi, s) - 1] \Delta \xi + \delta h \\ &\leq \int_s^t L_f \|X(\xi, s)x - X_*(\xi, s)x\| \Delta \xi \\ &\quad + L_g [(\|x\| + 1)[e_{N+\delta}(t, s) - 1] - (t - s)] + \delta h \end{aligned}$$

for $|t - s| \leq h$. Again applying Gronwall 2.18, we obtain

$$\begin{aligned} \|X(t, s)x - X_*(t, s)x\| &\leq [(\|x\| + 1)\frac{1}{N+\delta}[e_{N+\delta}(t, s) - 1] - (t - s)] + \delta h \\ &\quad + \int_s^t \{[(\|x\| + 1)[e_{N+\delta}(\xi, s) - 1] - (\xi - s)] + \delta h\} e_{L_g}(t, \sigma(\xi)) \Delta \xi \\ &\leq [(\|x\| + 1)[(N + \delta)h - 1] - h] + \delta h \\ &\quad + \{[(\|x\| + 1)[\frac{1}{N+\delta}(e_{N+\delta}(t, s) - 1) - (t - s)] - \frac{1}{2}(t - s)^2] \\ &\quad + (t - s)\delta h\} \frac{1}{L_g} [1 - e_{L_g}(t, s)] \\ &\leq [(\|x\| + 1)[(N + \delta)h] - h] + \delta h \\ &\quad + \{[(\|x\| + 1)[\frac{1}{N+\delta}(N + \delta h)] + h\delta h\} \frac{1}{L_g}, \end{aligned}$$

for $t \in [s, s + h]_T$, we have also used the positivity of the exponential function since $N + \delta$ is positive. As we know

$$\|x_0 - x_p\| = \|S^1 x_0 - S_*^1 x_p\| \leq \|S^1 x_0 - S_*^1 x_0\| + \|S_*^1 x_0 - S_*^1 x_p\|.$$

Hence we need to compute

$$\begin{aligned} \|S^1 x_0 - S_*^1 x_0\| &= \sup_t \|X(t, t-1)x_0(t-1) - X_*(t, t-1)x_0(t-1)\| \\ &\leq \sup_t \{[(\|x_0\| + 1)[(N + \delta)]] + \delta + \{[(\|x_0\| + 1)[\frac{1}{N+\delta}(N + \delta)] + \delta\} \frac{1}{L_g}\} \\ &\leq \sup_t \{[(\|x_0\| + 1)[(N + \delta)]] + \delta + [(\|x_0\| + 1) + \delta] \frac{1}{L_g}\}. \end{aligned}$$

Thus, we obtain

$$\|x_0 - x_p\| \leq 2\delta + [(1 + \frac{1}{L_g})(\|x_0(\cdot)\| + 1)[(N + \delta)]] + e_{\alpha+L_g}(\cdot, \cdot-1)\|x_0 - x_p\|.$$

So, we have

$$\|x_0 - x_p\| \leq \frac{2\delta}{e_{\alpha+L_g}(\cdot, \cdot-1)} \{1 + (1 + \frac{1}{L_g})(N + \delta)(\|x_0(\cdot)\| + 1)\}.$$

Since $\alpha + L_g < 0$, we have

$$\|x_0 - x_p\| \leq 2\delta \{1 + (1 + \frac{1}{L_g})(N + \delta)(\|x_0(\cdot)\| + 1)\}.$$

Hence, we have the desired result.

4 Example

Let us consider the following equation

$$x^\Delta = A(t)x + b(t), \quad t \in \mathbb{T}, x \in X, \quad (4.10)$$

where $A, b : \mathbb{T} \rightarrow \mathbb{R}$ is rd-continuous and almost periodic functions. It is easy to see that $f(t, x) = A(t)x + b(t)$, computing the norm, we get $\|f(t, x)\| \leq \|A(t)\| \|x\| + \|b(t)\|$. Under the assumption that A and b are bounded, we obtain $\|f(t, x)\| \leq \max\{\|A\|, \|b\|\}(1 + \|x\|)$. Hence, we can apply our results to ensure the existence of almost periodic solutions. One may consider the function $b(t) = \sin \frac{1}{2 + \cos t + \cos \sqrt{2}t}$, $t \in \mathbb{T}$. Another example, we may consider

$$\frac{\Delta}{\Delta t} x(t, z) = c(t, z) \frac{\Delta^2}{\Delta z^2} x(t, z) + f(t, x(t, z)), \quad t \in \mathbb{T}, z \in [0, 1]_{\mathbb{T}} \quad (4.11)$$

with $x(0, z) = x_0$, and $x(t, 0) = x(t, 1) = 0$. Let $Y = L^2[0, 1]_{\mathbb{T}}$ and $c(t, z)$ is a continuous almost periodic function. The notations $\frac{\Delta}{\Delta t}$, $\frac{\Delta^2}{\Delta z^2}$ denote the delta derivative with respect to t and double delta derivative with respect to z respectively. Let $A(t)X = c(t, z) \frac{\Delta^2}{\Delta z^2} X$ with $X \in D(A) = \{X \in H_0^1[0, 1]_{\mathbb{T}} \cap H^2[0, 1]_{\mathbb{T}}\}$. The operator $A(t)$ generates an analytic semigroup with $U(t, s) \leq K_0 e_{\ominus \nu}(t, s)$, for each $t \geq s \in \mathbb{T}$.

Using the above formulations, the considered equation can be rewritten in the following abstract form

$$X^\Delta X(t) = A(t)X(t) + F(t, X(t)), \quad X(0) = X_0 \quad t \in \mathbb{T}, \quad (4.12)$$

in $Y = L^2[0, 1]_{\mathbb{T}}$ and $X(t) = x(t, \cdot)$ which means $X(t)z = x(t, z)$. Hence under the condition that F is almost periodic and satisfies the conditions-A, the existence of almost periodic solution is ensured.

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