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Pseudo almost automorphic solutions of class r in α -norm under the light of measure theory

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Abstract: Using the spectral decomposition of the phase space developed in Adimy and co-authors, we present a new approach to study weighted pseudo almost automorphic functions in the α -norm using the measure theory.

Keywords: Measure theory; ergodicity; (μ, ν) -pseudo almost automorphic function; evolution equations; reaction diffusion system; partial functional differential equations.

AMS Subject Classification: 34K14; 34K30; 35B15; 35K57; 44A35; 47D06.

1 Introduction

In this work, present a new approach to study weighted pseudo almost automorphic functions in α -norm for the following partial functional differential equation

$$u'(t) = -Au(t) + L(u_t) + f(t) \text{ for } t \in \mathbb{R}, \quad (1)$$

where $-A : D(A) \rightarrow X$ is the infinitesimal generator of compact analytic semigroup of uniformly bounded linear operators on a Banach space X , $C_\alpha = C([-r, 0], D(A^\alpha))$, $0 < \alpha < 1$, denotes the space of continuous functions from $[-r, 0]$ into $D(A^\alpha)$, A^α is the fractional α -power of A . This operator $(A^\alpha, D(A^\alpha))$ will be describe later and

$$\|\varphi\|_{C_\alpha} = \|A^\alpha \varphi\|_{C([-r, 0], X)}.$$

For $t \geq 0$, and $u \in C([-r, a], D(A^\alpha))$, $r > 0$ and u_t denotes the history function of C_α defined by

$$u_t(\theta) = u(t + \theta) \text{ for } -r \leq \theta \leq 0.$$

L is a bounded linear operator from C_α into X and $f : \mathbb{R} \rightarrow X$ is a continuous function.

Some recent contributions concerning pseudo almost periodic solutions for abstract differential equations similar to equation (1) have been made.

In [3], the authors present new approach to study weighted pseudo almost automorphic functions using the measure theory. They present a new concept of weighted ergodic functions which is more general than the classical one. Then they establish many interesting results on the functional space of such functions like completeness and composition theorems. Our aim of this work is to generalize the results obtain in [5].

This work is organised as follow, in section 2 we recall some preliminary results about analytic semi-groups and fractional power associated to its generator will be used throughout this work. In section 3, we

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recall some preliminary results on spectral decomposition. In section 4, we recall some preliminary results on $\alpha - (\mu, \nu)$ -pseudo almost automorphic functions and neutral partial functional differential equations that will be used in this work. In section 5, we give some properties of $\alpha - (\mu, \nu)$ -pseudo almost automorphic functions of class r . In section 6, we discuss the main result of this paper. Using the strict contraction principle we show the existence and uniqueness of $\alpha - (\mu, \nu)$ -pseudo almost automorphic solution of class r for equation (1). Last section is devoted to some applications arising in population dynamics.

2 Analytic semigroup

Let $(X, \|\cdot\|)$ be a Banach space and α be a constant such that $0 < \alpha < 1$ and $-A$ be the infinitesimal generator of a bounded analytic semigroup of linear operator $(T(t))_{t \geq 0}$ on X . We assume without loss of generality that $0 \in \rho(A)$. Note that if the assumption $0 \in \rho(A)$ is not satisfied, one can substitute the operator A by the operator $(A - \sigma I)$ with σ large enough such that $0 \in \rho(A - \sigma I)$. This allows us to define the fractional power A^α for $0 < \alpha < 1$, as a closed linear invertible operator with domain $D(A^\alpha)$ dense in X . The closeness of A^α implies that $D(A^\alpha)$, endowed with the graph norm of A^α , $|x| = \|x\| + \|A^\alpha x\|$, is a Banach space. Since A^α is invertible, its graph norm $|\cdot|$ is equivalent to the norm $|x|_\alpha = \|A^\alpha x\|$. Thus, $D(A^\alpha)$ equipped with the norm $|\cdot|_\alpha$, is a Banach space, which we denote by X_α . For $0 < \beta \leq \alpha < 1$, the imbedding $X_\alpha \hookrightarrow X_\beta$ is compact if the resolvent operator of A is compact. Also, the following properties are well known.

Proposition 2.1. [8] Let $0 < \alpha < 1$. Assume that the operator $-A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t \geq 0}$ on the Banach space X satisfying $0 \in \rho(A)$. Then we have

i) $T(t) : X \rightarrow D(A^\alpha)$ for every $t > 0$.

ii) $T(t)A^\alpha x = A^\alpha T(t)x$ for every $x \in D(A^\alpha)$ and $t \geq 0$.

iii) for every $t > 0$, $A^\alpha T(t)$ is bounded on X and there exist $M_\alpha > 0$ and $\omega > 0$ such that

$$\|A^\alpha T(t)\| \leq M_\alpha e^{-\omega t} t^{-\alpha} \text{ for } t > 0.$$

iv) If $0 < \alpha \leq \beta < 1$, $D(A^\beta) \hookrightarrow D(A^\alpha)$.

v) There exists $N_\alpha > 0$ such that

$$\|(T(t) - I)A^{-\alpha}\| \leq N_\alpha t^\alpha \text{ for } t > 0.$$

Recall that $A^{-\alpha}$ is given by the following formula

$$A^{-\alpha} = \frac{1}{\Gamma(\delta)} \int_0^{+\infty} t^{\alpha-1} T(t) dt,$$

where the integral converges in the uniform operator topology for every $\alpha > 0$.

Consequently, if $T(t)$ is compact for each $t > 0$, then $A^{-\alpha}$ is compact.

3 Spectral decomposition

To equation (1), we associate the following initial value problem

$$\begin{cases} \frac{d}{dt} u(t) = -Au(t) + L(u_t) + f(t) \text{ for } t \geq 0 \\ u_0 = \varphi \in C_\alpha, \end{cases} \quad (2)$$

where $f : \mathbb{R}^+ \rightarrow X$ is a continuous function.

For each $t \geq 0$, we define the linear operator $\mathcal{U}(t)$ on C_α by

$$\mathcal{U}(t)\varphi = v_t(\cdot, \varphi)$$

where $v(\cdot, \varphi)$ is the solution of the following homogeneous equation

$$\begin{cases} \frac{d}{dt}v(t) = -Av(t) + L(v_t) \text{ for } t \geq 0 \\ v_0 = \varphi \in C_\alpha. \end{cases}$$

Proposition 3.1. [1] Let $\mathcal{A}_U$ defined on C_α by

$$\begin{cases} D(\mathcal{A}_U) = \left\{ \varphi \in C_\alpha, \varphi' \in C_\alpha, \varphi(0) \in D(A), \varphi(0)' \in \overline{D(A)} \text{ and } \varphi(0)' = -A\varphi(0) + L(\varphi) \right\} \\ \mathcal{A}_U \varphi = \varphi' \text{ for } \varphi \in D(\mathcal{A}_U). \end{cases}$$

Then $\mathcal{A}_U$ is the infinitesimal generator of the semigroup $(\mathcal{U}(t))_{t \geq 0}$ on C_α .

Let $\langle X_0 \rangle$ be the space defined by

$$\langle X_0 \rangle = \{X_0 c : c \in X\}$$

where the function $X_0 c$ is defined by

$$(X_0 c)(\theta) = \begin{cases} 0 & \text{if } \theta \in [-r, 0[, \\ c & \text{if } \theta = 0. \end{cases}$$

Consider the extension $\mathcal{A}_U$ defined on $C_\alpha \oplus \langle X_0 \rangle$ by

$$\begin{cases} D(\widetilde{\mathcal{A}_U}) = \left\{ \varphi \in C^1([-r, 0]; X_\alpha) : \varphi(0) \in D(A) \text{ and } \varphi(0)' \in \overline{D(A)} \right\} \\ \widetilde{\mathcal{A}_U} \varphi = \varphi' + X_0(A\varphi(0) + L(\varphi) - \varphi(0)'). \end{cases}$$

We make the following assertion:

(H₀) The operator $-A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t \geq 0}$ on the Banach space X and satisfies $0 \in \rho(A)$.

Lemma 3.2. [2] Assume that **(H₀)** holds. Then, $\widetilde{\mathcal{A}_U}$ satisfies the Hille-Yosida condition on $C_\alpha \oplus \langle X_0 \rangle$ there exist $\tilde{M} \geq 0$, $\tilde{\omega} \in \mathbb{R}$ such that $]\tilde{\omega}, +\infty[\subset \rho(\widetilde{\mathcal{A}_U})$ and

$$|(\lambda I - \widetilde{\mathcal{A}_U})^{-n}| \leq \frac{\tilde{M}}{(\lambda - \tilde{\omega})^n} \text{ for } n \in \mathbb{N} \text{ and } \lambda > \tilde{\omega}.$$

Now, we can state the variation of constants formula associated to equation (2).

Theorem 3.3. [1] Assume that **(H₀)** holds. Then for all $\varphi \in C_\alpha$, the solution u of equation (2) is given by the following variation of constants formula

$$u_t = \mathcal{U}(t)\varphi + \lim_{\lambda \rightarrow +\infty} \int_0^t \mathcal{U}(t-s) \tilde{B}_\lambda(X_0 f(s)) ds \text{ for } t \geq 0,$$

where $\tilde{B}_\lambda = \lambda(\lambda I - \widetilde{\mathcal{A}_U})^{-1}$.

Definition 3.4. We say that a semigroup $(\mathcal{U}(t))_{t \geq 0}$ is hyperbolic if

$$\sigma(\mathcal{A}_U) \cap i\mathbb{R} = \emptyset.$$

For the sequel, we make the following assumption:

(H₁) $T(t)$ is compact on $\overline{D(A)}$ for every $t > 0$.

We get the following result on the spectral decomposition of the phase space C_α .

Proposition 3.5. [1] Assume that **(H₀)** and **(H₁)** hold. If the semigroup $(\mathcal{U}(t))_{t \geq 0}$ is hyperbolic, then the space C_α is decomposed as a direct sum

$$C_\alpha = S \oplus U$$

of two $\mathcal{U}(t)$ invariant closed subspaces S and U such that the restriction of $(\mathcal{U}(t))_{t \geq 0}$ on U is a group and there exist positive constants $\overline{M}$ and ω such that

$$|\mathcal{U}(t)\varphi|_{C_\alpha} \leq \overline{M}e^{-\omega t}|\varphi|_{C_\alpha} \text{ for } t \geq 0 \text{ and } \varphi \in S$$

$$|\mathcal{U}(t)\varphi|_{C_\alpha} \leq \overline{M}e^{\omega t}|\varphi|_{C_\alpha} \text{ for } t \leq 0 \text{ and } \varphi \in U,$$

where S and U are called respectively the stable and unstable space, Π^s and Π^u denote respectively the projection operator on S and U .

4 (μ, ν) -Pseudo almost automorphic functions

In this section, we recall some properties about pseudo almost automorphic functions. Let $BC(\mathbb{R}, X)$ be the space of all bounded and continuous function from $\mathbb{R}$ to X equipped with the uniform topology norm.

Definition 4.1. A bounded continuous function $\phi : \mathbb{R} \rightarrow X$ is called almost automorphic if for each real sequence (s_m) , there exists a subsequence (s_n) such that

$$g(t) = \lim_{n \rightarrow +\infty} \phi(t + s_n)$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n \rightarrow +\infty} g(t - s_n) = \phi(t)$$

for each $t \in \mathbb{R}$.

We denote by $AA(\mathbb{R}, X)$, the space of all such functions.

Proposition 4.2. [7] $AA(\mathbb{R}, X)$ equipped with the sup norm is a Banach space.

Definition 4.3. Let X_1 and X_2 be two Banach spaces. A bounded continuous function $\phi : \mathbb{R} \times X_1 \rightarrow X_2$ is called almost automorphic in $t \in \mathbb{R}$ uniformly for each x in X_1 if for every real sequence (s_m) , there exists a subsequence (s_n) such that

$$g(t, x) = \lim_{n \rightarrow +\infty} \phi(t + s_n, x) \text{ in } X_2$$

is well defined for each $t \in \mathbb{R}$ and each $x \in X_1$ and

$$\lim_{n \rightarrow +\infty} g(t - s_n, x) = \phi(t, x) \text{ in } X_2$$

for each $t \in \mathbb{R}$ and for every $x \in X_1$.

Denote by $AA(\mathbb{R} \times X_1; X_2)$ the space of all such functions.

Definition 4.4. A bounded continuous function $\phi : \mathbb{R} \rightarrow X$ is called compact almost automorphic if for each real sequence (s_m) , there exists a subsequence (s_n) such that

$$g(t) = \lim_{n \rightarrow +\infty} \phi(t + s_n) \text{ and } \lim_{n \rightarrow +\infty} g(t - s_n) = \phi(t)$$

uniformly on compact subsets of $\mathbb{R}$.

We denote by $AA_c(\mathbb{R}; X)$, the space of all such functions.

It is well known that $AA_c(\mathbb{R}; X)$ is closed subsets of $(BC(\mathbb{R}, X), \|\cdot\|_\infty)$.

In view of the above, the proof of the next lemma is straightforward.

Lemma 4.5. [7] $AA_c(\mathbb{R}; X)$ equipped with the sup norm is a Banach space.

Definition 4.6. Let X_1 and X_2 be two Banach spaces. A continuous function $\phi : \mathbb{R} \times X_1 \rightarrow X_2$ is called compact almost automorphic in $t \in \mathbb{R}$ if every real sequence (s_m) , there exists a subsequence (s_n) such that

$$g(t, x) = \lim_{n \rightarrow +\infty} \phi(t + s_n, x) \text{ and } \lim_{n \rightarrow +\infty} g(t - s_n, x) = \phi(t, x) \text{ in } X_2$$

where the limits are uniform on compact subsets of $\mathbb{R}$ for each $x \in X_1$.

Denote by $AA_c(\mathbb{R} \times X_1; X_2)$ the space of all such functions.

In the sequel, we recall some preliminary results concerning the $\alpha - (\mu, \nu)$ -pseudo almost automorphic functions.

$\mathcal{E}(\mathbb{R}; X_\alpha, \mu, \nu)$ stands for the space of functions

$$\mathcal{E}(\mathbb{R}; X_\alpha, \mu, \nu) = \left\{ u \in BC(\mathbb{R}; X_\alpha) : \lim_{\tau \rightarrow +\infty} \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{+\tau} |u(t)|_\alpha d\mu(t) = 0 \right\}.$$

To study delayed differential equations for which the history belongs to $C([-r, 0]; X_\alpha)$, we need to introduce the space

$$\mathcal{E}(\mathbb{R}; X_\alpha, \mu, \nu, r) = \left\{ u \in BC(\mathbb{R}; X_\alpha) : \lim_{\tau \rightarrow +\infty} \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{+\tau} \left(\sup_{\theta \in [t-r, t]} |u(\theta)|_\alpha \right) d\mu(t) = 0 \right\}.$$

In addition to above-mentioned spaces, we consider the following spaces

$$\mathcal{E}(\mathbb{R} \times X_\alpha, \mu, \nu) = \left\{ u \in BC(\mathbb{R} \times X_\alpha; X_\alpha) : \lim_{\tau \rightarrow +\infty} \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{+\tau} |u(t, x)|_\alpha d\mu(t) = 0 \right\},$$

$$\mathcal{E}(\mathbb{R} \times X_\alpha, \mu, \nu, r) = \left\{ u \in BC(\mathbb{R} \times X_\alpha; X_\alpha) : \lim_{\tau \rightarrow +\infty} \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{+\tau} \left(\sup_{\theta \in [t-r, t]} |u(\theta, x)|_\alpha \right) d\mu(t) = 0 \right\},$$

where in both cases the limit (as $\tau \rightarrow +\infty$) is uniform in compact subset of X_α .

In view of previous definitions, it is clear that the space $\mathcal{E}(\mathbb{R}; X_\alpha, \mu, \nu, r)$ is continuously embedded in $\mathcal{E}(\mathbb{R}; X_\alpha, \mu, \nu)$.

On the other hand, one can observe that a ρ -weighted pseudo almost automorphic functions is μ -pseudo almost automorphic, where the measure μ is absolutely continuous with respect to the Lebesgue measure and its Radon-Nikodym derivative is ρ :

$$d\mu(t) = \rho(t)dt$$

and ν is the usual Lebesgue measure on $\mathbb{R}$, i.e $\nu([- \tau, \tau]) = 2\tau$ for all $\tau \geq 0$.

Example 4.7. [3] Let ρ be a nonnegative $\mathcal{B}$ -measurable function. Denote by μ the positive measure defined by

$$\mu(A) = \int_A \rho(t) dt, \quad \text{for } A \in \mathcal{B}, \quad (3)$$

where dt denotes the Lebesgue measure on $\mathbb{R}$. The function ρ which occurs in equation (3) is called the Radon-Nikodym derivative of μ with respect to the Lebesgue measure on $\mathbb{R}$.

Definition 4.8. A bounded continuous function $\phi : \mathbb{R} \rightarrow X$ is called $\alpha - (\mu, \nu)$ -pseudo almost automorphic if $\phi = \phi_1 + \phi_2$ where $\phi_1 \in AA(\mathbb{R}; X_\alpha)$ and $\phi_2 \in \mathcal{E}(\mathbb{R}; X_\alpha, \mu, \nu)$.

We denote by $PAA(\mathbb{R}; X_\alpha, \mu, \nu)$, the space of all such functions.

Definition 4.9. A bounded continuous function $\phi : \mathbb{R} \times X_\alpha \rightarrow X_\alpha$ is called uniformly $\alpha - (\mu, \nu)$ -pseudo almost automorphic if $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AA(\mathbb{R} \times X_\alpha; X_\alpha)$ and $\phi_2 \in \mathcal{E}(\mathbb{R} \times X_\alpha; X_\alpha, \mu, \nu)$.

We denote by $PAA(\mathbb{R} \times X_\alpha; X_\alpha, \mu, \nu)$, the space of all such functions.

We now introduce some new spaces used in the sequel.

Definition 4.10. A bounded continuous function $\phi : \mathbb{R} \rightarrow X$ is called compact $\alpha - (\mu, \nu)$ -pseudo almost automorphic if $\phi = \phi_1 + \phi_2$ where $\phi_1 \in AA_c(\mathbb{R}; X_\alpha)$ and $\phi_2 \in \mathcal{E}(\mathbb{R}; X_\alpha, \mu, \nu)$.

We denote by $PAA_c(\mathbb{R}; X_\alpha, \mu, \nu)$, the space of all such functions.

Definition 4.11. A bounded continuous function $\phi : \mathbb{R} \times X_\alpha \rightarrow X_\alpha$ is called uniformly compact $\alpha - (\mu, \nu)$ -pseudo almost automorphic if $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AA_c(\mathbb{R} \times X_\alpha; X_\alpha)$ and $\phi_2 \in \mathcal{E}(\mathbb{R} \times X_\alpha; X_\alpha, \mu, \nu)$.

We denote by $PAA_c(\mathbb{R} \times X_\alpha; X_\alpha, \mu, \nu)$, the space of all such functions.

Definition 4.12. A bounded continuous function $\phi : \mathbb{R} \rightarrow X$ is called $\alpha - (\mu, \nu)$ -pseudo almost automorphic of class r (respectively compact $\alpha - (\mu, \nu)$ -pseudo almost automorphic of class r) if $\phi = \phi_1 + \phi_2$ where $\phi_1 \in AA(\mathbb{R}; X_\alpha)$ and $\phi_2 \in \mathcal{E}(\mathbb{R}; X_\alpha, \mu, \nu, r)$ (respectively if $\phi = \phi_1 + \phi_2$ where $\phi_1 \in AA_c(\mathbb{R}; X_\alpha)$ and $\phi_2 \in \mathcal{E}(\mathbb{R}; X_\alpha, \mu, \nu, r)$).

We denote by $PAA(\mathbb{R}; X_\alpha, \mu, \nu, r)$ (respectively $PAA_c(\mathbb{R}; X_\alpha, \mu, \nu, r)$) the space of all such functions.

Definition 4.13. A bounded continuous function $\phi : \mathbb{R} \times X_\alpha \rightarrow X_\alpha$ is called uniformly pseudo almost automorphic of class r (respectively uniformly pseudo compact almost automorphic of class r) if $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AA(\mathbb{R} \times X_\alpha; X_\alpha)$ and $\phi_2 \in \mathcal{E}(\mathbb{R} \times X_\alpha; X_\alpha, \mu, \nu, r)$ (respectively if $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AA_c(\mathbb{R} \times X_\alpha; X_\alpha)$ and $\phi_2 \in \mathcal{E}(\mathbb{R} \times X_\alpha; X_\alpha, \mu, \nu, r)$).

We denote by $PAA(\mathbb{R} \times X_\alpha; X_\alpha, \mu, \nu, r)$ (respectively $PAA_c(\mathbb{R} \times X_\alpha; X_\alpha, \mu, \nu, r)$) the space of all such functions.

5 Properties of (μ, ν) -pseudo almost automorphic functions of class r

From $\mu, \nu \in \mathcal{M}$, we formulate the following hypothesis.

(H₂) Let $\mu, \nu \in \mathcal{M}$ be such that $\limsup_{\tau \rightarrow +\infty} \frac{\mu([- \tau, \tau])}{\nu([- \tau, \tau])} = \delta < \infty$.

We have the following result.

Lemma 5.1. Assume (H_2) holds and let $f \in BC(\mathbb{R}; X_\alpha)$. Then $f \in \mathcal{E}(\mathbb{R}; X_\alpha, \mu, \nu)$ if and only if for any $\varepsilon > 0$,

$$\lim_{\tau \rightarrow +\infty} \frac{\mu(M_{\tau, \varepsilon}(f))}{\nu([- \tau, \tau])} = 0$$

where

$$M_{\tau, \varepsilon}(f) = \{t \in [-\tau, \tau] : |f(t)|_\alpha \geq \varepsilon\}.$$

Proof. Suppose that $f \in \mathcal{E}(\mathbb{R}; X_\alpha, \mu, \nu)$. Then

$$\begin{aligned} \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{+\tau} |f(t)|_\alpha d\mu(t) &= \frac{1}{\nu([- \tau, \tau])} \int_{M_{\tau, \varepsilon}(f)} |f(t)|_\alpha d\mu(t) + \frac{1}{\nu([- \tau, \tau])} \int_{[- \tau, \tau] \setminus M_{\tau, \varepsilon}(f)} |f(t)|_\alpha d\mu(t) \\ &\geq \frac{1}{\nu([- \tau, \tau])} \int_{M_{\tau, \varepsilon}(f)} |f(t)|_\alpha d\mu(t) \\ &\geq \frac{\varepsilon \mu(M_{\tau, \varepsilon}(f))}{\nu([- \tau, \tau])}. \end{aligned}$$

Consequently

$$\lim_{\tau \rightarrow +\infty} \frac{\mu(M_{\tau, \varepsilon}(f))}{\nu([- \tau, \tau])} = 0.$$

Suppose that $f \in BC(\mathbb{R}; X_\alpha)$ such that for any $\varepsilon > 0$,

$$\lim_{\tau \rightarrow +\infty} \frac{\mu(M_{\tau, \varepsilon}(f))}{\nu([- \tau, \tau])} = 0.$$

We can assume $|f(t)|_\alpha \leq N$ for all $t \in \mathbb{R}$. Using (H_2) , we have

$$\begin{aligned} \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{+\tau} |f(t)|_\alpha d\mu(t) &= \frac{1}{\nu([- \tau, \tau])} \int_{M_{\tau, \varepsilon}(f)} |f(t)|_\alpha d\mu(t) + \frac{1}{\nu([- \tau, \tau])} \int_{[- \tau, \tau] \setminus M_{\tau, \varepsilon}(f)} |f(t)|_\alpha d\mu(t) \\ &\leq \frac{N}{\nu([- \tau, \tau])} \int_{M_{\tau, \varepsilon}(f)} d\mu(t) + \frac{1}{\nu([- \tau, \tau])} \int_{[- \tau, \tau] \setminus M_{\tau, \varepsilon}(f)} |f(t)|_\alpha d\mu(t) \\ &\leq \frac{N}{\nu([- \tau, \tau])} \int_{M_{\tau, \varepsilon}(f)} d\mu(t) + \frac{\varepsilon}{\nu([- \tau, \tau])} \int_{[- \tau, \tau]} d\mu(t) \\ &\leq \frac{N}{\nu([- \tau, \tau])} \mu(M_{\tau, \varepsilon}(f)) + \frac{\varepsilon \mu([- \tau, \tau])}{\nu([- \tau, \tau])}. \end{aligned}$$

Which implies that

$$\lim_{\tau \rightarrow +\infty} \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{+\tau} |f(t)|_\alpha d\mu(t) \leq \delta \varepsilon \text{ for any } \varepsilon > 0.$$

Therefore $f \in \mathcal{E}(\mathbb{R}; X_\alpha, \mu, \nu)$. ■

Lemma 5.2. Assume that (H_2) holds. The space $\mathcal{E}(\mathbb{R}; X_\alpha, \mu, \nu, r)$ endowed with the uniform topology norm is a Banach space.

Proof. We can see that $\mathcal{E}(\mathbb{R}; X_\alpha, \mu, \nu, r)$ is a vector subspace of $BC(\mathbb{R}; X_\alpha)$. To complete the proof, it is enough to prove that $\mathcal{E}(\mathbb{R}; X_\alpha, \mu, \nu, r)$ is closed in $BC(\mathbb{R}; X_\alpha)$. Let $(z_n)_n$ be a sequence in $\mathcal{E}(\mathbb{R}; X_\alpha, \mu, \nu, r)$ such that $\lim_{n \rightarrow +\infty} z_n = z$ uniformly in $\mathbb{R}$. From $\nu(\mathbb{R}) = +\infty$, it follows $\nu([- \tau, \tau]) > 0$ for τ sufficiently large. Let $\|z\|_{\infty, \alpha} = \sup_{t \in \mathbb{R}} |z(t)|_\alpha$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\|z_n - z\|_{\infty, \alpha} < \varepsilon$. Let $n \geq n_0$, then we have

$$\begin{aligned} \frac{1}{\nu([- \tau, \tau])} \int_{- \tau}^{+ \tau} \left(\sup_{\theta \in [t-r, t]} |z(\theta)|_\alpha \right) d\mu(t) &\leq \frac{1}{\nu([- \tau, \tau])} \int_{- \tau}^{+ \tau} \left(\sup_{\theta \in [t-r, t]} |z_n(\theta) - z(\theta)|_\alpha \right) d\mu(t) \\ &\quad + \frac{1}{\nu([- \tau, \tau])} \int_{- \tau}^{+ \tau} \left(\sup_{\theta \in [t-r, t]} |z_n(\theta)|_\alpha \right) d\mu(t) \\ &\leq \frac{1}{\nu([- \tau, \tau])} \int_{- \tau}^{+ \tau} \left(\sup_{t \in \mathbb{R}} |z_n(t) - z(t)|_\alpha \right) d\mu(t) \\ &\quad + \frac{1}{\nu([- \tau, \tau])} \int_{- \tau}^{+ \tau} \left(\sup_{\theta \in [t-r, t]} |z_n(\theta)|_\alpha \right) d\mu(t) \\ &\leq \|z_n - z\|_{\infty, \alpha} \times \frac{\mu([- \tau, \tau])}{\nu([- \tau, \tau])} + \frac{1}{\nu([- \tau, \tau])} \int_{- \tau}^{+ \tau} \left(\sup_{\theta \in [t-r, t]} |z_n(\theta)|_\alpha \right) d\mu(t). \end{aligned}$$

We deduce that

$$\limsup_{\tau \rightarrow +\infty} \frac{1}{\nu([- \tau, \tau])} \int_{- \tau}^{+ \tau} \left(\sup_{\theta \in [t-r, t]} |z(\theta)|_\alpha \right) d\mu(t) \leq \delta \varepsilon \text{ for any } \varepsilon > 0. \blacksquare$$

From the definition of $PAA(\mathbb{R}; X_\alpha, \mu, \nu, r)$, we deduce the following result.

Proposition 5.3. Assume that (H_2) holds and let $\mu, \nu \in \mathcal{M}$. The space $PAA(\mathbb{R}; X_\alpha, \mu, \nu, r)$ endowed with the uniform topology norm is a Banach space.

Next result is a characterization of $\alpha - (\mu, \nu)$ -ergodic functions of class r .

Theorem 5.4. Assume that (H_2) holds and let $\mu, \nu \in \mathcal{M}$ and I be a bounded interval (eventually $I = \cdot$). Assume that $f \in BC(\mathbb{R}, X_\alpha)$. Then the following assertions are equivalent:

i) $f \in \mathcal{E}(\mathbb{R}, X_\alpha, \mu, \nu, r)$.

ii) $\lim_{\tau \rightarrow +\infty} \frac{1}{\nu([- \tau, \tau] \setminus I)} \int_{[- \tau, \tau] \setminus I} \left(\sup_{\theta \in [t-r, t]} |f(\theta)|_\alpha \right) d\mu(t) = 0$.

iii) For any $\varepsilon > 0$, $\lim_{\tau \rightarrow +\infty} \frac{\mu\left(\left\{t \in [- \tau, \tau] \setminus I : \sup_{\theta \in [t-r, t]} |f(\theta)|_\alpha > \varepsilon\right\}\right)}{\nu([- \tau, \tau] \setminus I)} = 0$.

Proof. i) $\Leftrightarrow$ ii) Denote by $A = \nu(I)$, $B = \int_I \left(\sup_{\theta \in [t-r, t]} |f(\theta)|_\alpha \right) d\mu(t)$. We have A and $B \in \mathbb{R}$, since the interval I is bounded and the function f is bounded and continuous. For $\tau > 0$ such that $I \subset [- \tau, \tau]$ and $\nu([- \tau, \tau] \setminus I) > 0$, we have

$$\begin{aligned} &\frac{1}{\nu([- \tau, \tau] \setminus I)} \int_{[- \tau, \tau] \setminus I} \left(\sup_{\theta \in [t-r, t]} |f(\theta)|_\alpha \right) d\mu(t) = \frac{1}{\nu([- \tau, \tau]) - A} \left[\int_{[- \tau, \tau]} \left(\sup_{\theta \in [t-r, t]} |f(\theta)|_\alpha \right) d\mu(t) - B \right] \\ &= \frac{\nu([- \tau, \tau])}{\nu([- \tau, \tau]) - A} \left[\frac{1}{\nu([- \tau, \tau])} \int_{[- \tau, \tau]} \left(\sup_{\theta \in [t-r, t]} |f(\theta)|_\alpha \right) d\mu(t) - \frac{B}{\nu([- \tau, \tau])} \right]. \end{aligned}$$

From above equalities and the fact that $\nu(\mathbb{R}) = +\infty$, we deduce that $ii)$ is equivalent to

$$\lim_{\tau \rightarrow +\infty} \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{+\tau} \left(\sup_{\theta \in [t-r, t]} |f(\theta)|_{\alpha} \right) d\mu(t) = 0,$$

that is $i)$.

$iii) \Rightarrow ii)$ Denote by A_{τ}^{ε} and B_{τ}^{ε} the following sets

$$A_{\tau}^{\varepsilon} = \left\{ t \in [-\tau, \tau] \setminus I : \sup_{\theta \in [t-r, t]} |f(\theta)|_{\alpha} > \varepsilon \right\} \text{ and } B_{\tau}^{\varepsilon} = \left\{ t \in [-\tau, \tau] \setminus I : \sup_{\theta \in [t-r, t]} |f(\theta)|_{\alpha} \leq \varepsilon \right\}.$$

Assume that $iii)$ holds, that is

$$\lim_{\tau \rightarrow +\infty} \frac{\mu(A_{\tau}^{\varepsilon})}{\nu([- \tau, \tau] \setminus I)} = 0. \quad (4)$$

From the equality

$$\int_{[-\tau, \tau] \setminus I} \left(\sup_{\theta \in [t-r, t]} |f(\theta)|_{\alpha} \right) d\mu(t) = \int_{A_{\tau}^{\varepsilon}} \left(\sup_{\theta \in [t-r, t]} |f(\theta)|_{\alpha} \right) d\mu(t) + \int_{B_{\tau}^{\varepsilon}} \left(\sup_{\theta \in [t-r, t]} |f(\theta)|_{\alpha} \right) d\mu(t),$$

we deduce that for τ sufficiently large

$$\frac{1}{\nu([- \tau, \tau] \setminus I)} \int_{[-\tau, \tau] \setminus I} \left(\sup_{\theta \in [t-r, t]} |f(\theta)|_{\alpha} \right) d\mu(t) \leq \|f\|_{\infty, \alpha} \frac{\mu(A_{\tau}^{\varepsilon})}{\nu([- \tau, \tau] \setminus I)} + \varepsilon \frac{\mu(B_{\tau}^{\varepsilon})}{\nu([- \tau, \tau] \setminus I)}.$$

By using (H_2) , it follows that

$$\lim_{\tau \rightarrow +\infty} \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{+\tau} \left(\sup_{\theta \in [t-r, t]} |f(\theta)|_{\alpha} \right) d\mu(t) \leq \delta \varepsilon, \text{ for any } \varepsilon > 0,$$

consequently $ii)$ holds.

$ii) \Rightarrow iii)$ Assume that $ii)$ holds. From the following inequality

$$\begin{aligned} \int_{[-\tau, \tau] \setminus I} \left(\sup_{\theta \in [t-r, t]} |f(\theta)|_{\alpha} \right) d\mu(t) &\geq \int_{A_{\tau}^{\varepsilon}} \left(\sup_{\theta \in [t-r, t]} |f(\theta)|_{\alpha} \right) d\mu(t) \\ \frac{1}{\nu([- \tau, \tau] \setminus I)} \int_{[-\tau, \tau] \setminus I} \left(\sup_{\theta \in [t-r, t]} |f(\theta)|_{\alpha} \right) d\mu(t) &\geq \varepsilon \frac{\mu(A_{\tau}^{\varepsilon})}{\nu([- \tau, \tau] \setminus I)} \\ \frac{1}{\varepsilon \nu([- \tau, \tau] \setminus I)} \int_{[-\tau, \tau] \setminus I} \left(\sup_{\theta \in [t-r, t]} |f(\theta)|_{\alpha} \right) d\mu(t) &\geq \frac{\mu(A_{\tau}^{\varepsilon})}{\nu([- \tau, \tau] \setminus I)}, \end{aligned}$$

for τ sufficiently large, we obtain equation (4), that is $iii)$. ■

From $\mu \in \mathcal{M}$, we formulate the following hypotheses.

(H_3) For all a, b and $c \in \mathbb{R}$, such that $0 \leq a < b \leq c$, there exist δ_0 and $\alpha_0 > 0$ such that

$$|\delta| \geq \delta_0 \Rightarrow \mu(a + \delta, b + \delta) \geq \alpha_0 \mu(\delta, c + \delta).$$

(H_4) For all $\tau \in \mathbb{R}$, there exist $\beta > 0$ and a bounded interval I such that

$$\mu(\{a + \tau : a \in A\}) \leq \beta \mu(A) \text{ when } A \in \mathcal{B} \text{ satisfies } A \cap I = \emptyset.$$

We have the following results due to [3]

Lemma 5.5. [3] Hypothesis (H_4) implies (H_3) .

Proposition 5.6. [3, 4] $\mu, \nu \in \mathcal{M}$ satisfy (H_3) and $f \in PAA(\mathbb{R}; X, \mu, \nu)$ be such that

$$f = g + h$$

where $g \in AA(\mathbb{R}, X)$ and $h \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$. Then

$$\{g(t), t \in \mathbb{R}\} \subset \overline{\{f(t), t \in \mathbb{R}\}} \text{ (the closure of the range of } f\text{)}.$$

Corollary 5.7. [4] Assume that (H_3) holds. Then the decomposition of a (μ, ν) -pseudo almost automorphic function in the form $f = g + \phi$ where $g \in AA(\mathbb{R}; X)$ and $\phi \in \mathcal{E}(\mathbb{R}; X, \mu, \nu)$, is unique.

We deduce following result.

Proposition 5.8. Let $\mu, \nu \in \mathcal{M}$. Assume (H_3) holds. Then the decomposition of a $\alpha - (\mu, \nu)$ -pseudo-almost periodic function $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AA(\mathbb{R}; X_\alpha)$ and $\phi_2 \in \mathcal{E}(\mathbb{R}; X_\alpha, \mu, \nu, r)$, is unique.

Proof. In fact, since as a consequence of Corollary 5.7, the decomposition of a (μ, ν) -pseudo-almost periodic function $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AA(\mathbb{R}; X_\alpha)$ and $\phi_2 \in \mathcal{E}(\mathbb{R}; X_\alpha, \mu, \nu)$, is unique. Since $PAA(\mathbb{R}; X_\alpha, \mu, \nu) \subset PAA(\mathbb{R}; X, \mu, \nu)$ and $PAA(\mathbb{R}; X_\alpha, \mu, \nu, r) \subset PAA(\mathbb{R}; X_\alpha, \mu, \nu)$, we get the desired result. ■

Definition 5.9. Let $\mu_1, \mu_2 \in \mathcal{M}$. We say that μ_1 is equivalent to μ_2 , denoting this as $\mu_1 \sim \mu_2$ if there exist constants α and $\beta > 0$ and a bounded interval I (eventually $I = \mathbb{R}$) such that

$$\alpha\mu_1(A) \leq \mu_2(A) \leq \beta\mu_1(A), \text{ when } A \in \mathcal{B} \text{ satisfies } A \cap I = \emptyset.$$

From [3] $\sim$ is a binary equivalence relation on $\mathcal{M}$. The equivalence class of a given measure $\mu \in \mathcal{M}$ will then be denoted by

$$cl(\mu) = \{\varpi \in \mathcal{M} : \mu \sim \varpi\}.$$

Theorem 5.10. Let $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathcal{M}$. If $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$, then $PAA(\mathbb{R}; X_\alpha, \mu_1, \nu_1, r) = PAA(\mathbb{R}; X_\alpha, \mu_2, \nu_2, r)$.

Proof. Since $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$ there exist some constants $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ and a bounded interval I (eventually $I = \mathbb{R}$) such that $\alpha_1\mu_1(A) \leq \mu_2(A) \leq \beta_1\mu_1(A)$ and $\alpha_2\nu_1(A) \leq \nu_2(A) \leq \beta_2\nu_1(A)$ for each $A \in \mathcal{B}$ satisfies $A \cap I = \emptyset$ i.e

$$\frac{1}{\beta_2\nu_1(A)} \leq \frac{1}{\nu_2(A)} \leq \frac{1}{\alpha_2\nu_1(A)}.$$

Since $\mu_1 \sim \mu_2$ and $\mathcal{B}$ is the Lebesgue σ -field, we obtain for τ sufficiently large that

$$\begin{aligned} \frac{\alpha_1\mu_1\left(\left\{t \in [-\tau, \tau] \setminus I : \sup_{\theta \in [t-r, t]} |f(\theta)|_\alpha > \varepsilon\right\}\right)}{\beta_2\nu_1([-\tau, \tau] \setminus I)} &\leq \frac{\mu_2\left(\left\{t \in [-\tau, \tau] \setminus I : \sup_{\theta \in [t-r, t]} |f(\theta)|_\alpha > \varepsilon\right\}\right)}{\nu_2([-\tau, \tau] \setminus I)} \\ &\leq \frac{\beta_1\mu_1\left(\left\{t \in [-\tau, \tau] \setminus I : \sup_{\theta \in [t-r, t]} |f(\theta)|_\alpha > \varepsilon\right\}\right)}{\alpha_2\nu_1([-\tau, \tau] \setminus I)}. \end{aligned}$$

By using Theorem 5.4 we deduce that $\mathcal{E}(\mathbb{R}, X_\alpha, \mu_1, \nu_1, r) = \mathcal{E}(\mathbb{R}, X_\alpha, \mu_2, \nu_2, r)$. From the definition of a (μ, ν) -pseudo almost automorphic function, we deduce that $PAA(\mathbb{R}; X_\alpha, \mu_1, \nu_1, r) = PAA(\mathbb{R}; X_\alpha, \mu_2, \nu_2, r)$. ■

Let $\mu, \nu \in \mathcal{M}$. We denote by

$$cl(\mu, \nu) = \{\varpi_1, \varpi_2 \in \mathcal{M} : \mu \sim \varpi_1 \text{ and } \nu \sim \varpi_2\}.$$

In what follows, we prove some preliminary results concerning the composition of (μ, ν) -pseudo almost automorphic functions of class r .

Theorem 5.11. Let $\mu, \nu \in \mathcal{M}$, $\phi \in PAA(\mathbb{R} \times X_\alpha, \mu, \nu, r)$ and $h \in PAA(\mathbb{R}; X_\alpha, \mu, \nu, r)$. Assume that there exists a function $L_\phi : \mathbb{R} \rightarrow [0, +\infty[$ such that

$$|\phi(t, x_1) - \phi(t, x_2)| \leq L_\phi(t)|x_1 - x_2|_\alpha \text{ for } t \in \mathbb{R} \text{ and for } x_1, x_2 \in X_\alpha. \quad (5)$$

If

$$\limsup_{\tau \rightarrow +\infty} \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \left(\sup_{\theta \in [t-r, t]} L_\phi(\theta) \right) d\mu(t) < \infty \text{ and } \lim_{\tau \rightarrow +\infty} \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{+\tau} \left(\sup_{\theta \in [t-r, t]} L_\phi(\theta) \right) \xi(t) d\mu(t) = 0 \quad (6)$$

for each $\xi \in \mathcal{E}(\mathbb{R}, \mathbb{R}, \mu, \nu)$, then the function $t \rightarrow \phi(t, h(t))$ belongs to $PAA(\mathbb{R}; X_\alpha, \mu, \nu, r)$.

Proof. Assume that $\phi = \phi_1 + \phi_2$, $h = h_1 + h_2$ where $\phi_1 \in AA(\mathbb{R} \times X_\alpha; X_\alpha)$, $\phi_2 \in \mathcal{E}(\mathbb{R} \times X_\alpha, \mu, \nu, r)$ and $h_1 \in AA(\mathbb{R}; X_\alpha)$, $h_2 \in \mathcal{E}(\mathbb{R}; X_\alpha, \mu, \nu, r)$. Consider the following decomposition

$$\phi(t, h(t)) = \phi_1(t, h_1(t)) + [\phi(t, h(t)) - \phi(t, h_1(t))] + \phi_2(t, h_1(t)).$$

Since $\phi_1 \in AA(\mathbb{R} \times X_\alpha; X_\alpha)$, then $\phi_1(t, x)$ is uniformly continuous on any bounded subset uniformly for $t \in \mathbb{R}$. Define the function g on $\mathbb{R}$ by $g(t) = \phi_1(t, h_1(t))$, then $g \in AA(\mathbb{R}; X_\alpha)$. In fact since $h_1 \in AA(\mathbb{R}; X_\alpha)$ then for a given sequence $(s_m)_{m \in \mathbb{N}}$ of real numbers, we can extract a subsequence $(s_n)_{n \in \mathbb{N}}$ and $\nu \in BC(\mathbb{R}; X_\alpha)$ such that

$$\nu(t) = \lim_{n \rightarrow +\infty} h_1(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n \rightarrow +\infty} \nu(t - s_n) = h_1(t)$$

for each $t \in \mathbb{R}$. Since $\phi_1 \in AA(\mathbb{R} \times X_\alpha; X_\alpha)$, then

$$u(t, \nu(t)) = \lim_{n \rightarrow +\infty} \phi_1(t + s_n, \nu(t)) = \lim_{n \rightarrow +\infty} \phi_1(t + s_n, h_1(t + s_n)) = g(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n \rightarrow +\infty} u(t - s_n, \nu(t - s_n)) = \phi_1\left(t, \lim_{n \rightarrow +\infty} \nu(t - s_n)\right) = \phi_1(t, h_1(t)) = g(t)$$

for each $t \in \mathbb{R}$. We conclude that $g \in AA(\mathbb{R}; X_\alpha)$.

It remains to prove that both $\phi(\cdot, h(\cdot)) - \phi(\cdot, h_1(\cdot))$ and $\phi_2(\cdot, h_1(\cdot))$ belong to $\mathcal{E}(\mathbb{R}; X_\alpha, \mu, \nu, r)$. Consequently, using inequality (5), it follows that

$$\begin{aligned} \frac{\mu\left(\left\{t \in [-\tau, \tau] : \sup_{\theta \in [t-r, t]} |\phi(\theta, h(\theta)) - \phi(\theta, h_1(\theta))|_\alpha > \varepsilon\right\}\right)}{\nu([- \tau, \tau])} &\leq \frac{\mu\left(\left\{t \in [-\tau, \tau] : \sup_{\theta \in [t-r, t]} (L_\phi(\theta)|h_2(\theta)|_\alpha) > \varepsilon\right\}\right)}{\nu([- \tau, \tau])} \\ &\leq \frac{\mu\left(\left\{t \in [-\tau, \tau] : \left(\sup_{\theta \in [t-r, t]} L_\phi(\theta)\right) \left(\sup_{\theta \in [t-r, t]} |h_2(\theta)|_\alpha\right) > \varepsilon\right\}\right)}{\nu([- \tau, \tau])}. \end{aligned}$$

Since h_2 is (μ, ν) -ergodic of class r , Theorem 5.4 and equation (6) yield that for the above-mentioned ε , we have

$$\lim_{\tau \rightarrow +\infty} \frac{\mu\left(\left\{t \in [-\tau, \tau] : \left(\sup_{\theta \in [t-r, t]} L_\phi(\theta)\right) \left(\sup_{\theta \in [t-r, t]} |h_2(\theta)|_\alpha\right) > \varepsilon\right\}\right)}{\nu([- \tau, \tau])} = 0,$$

and then we obtain

$$\lim_{\tau \rightarrow +\infty} \frac{\mu\left(\left\{t \in [-\tau, \tau] : \sup_{\theta \in [t-r, t]} |\phi(\theta, h(\theta)) - \phi(\theta, h_1(\theta))|_\alpha > \varepsilon\right\}\right)}{\nu([-\tau, \tau])} = 0. \quad (7)$$

By Theorem 5.4, equation (7) shows that $t \mapsto \phi(t, h(t)) - \phi(t, h_1(t))$ is (μ, ν) -ergodic of class r .

Now to complete the proof, it is enough to prove that $t \mapsto \phi_2(t, h(t))$ is (μ, ν) -ergodic of class r . Since ϕ_2 is uniformly continuous on the compact set $K = \{h_1(t) : t \in \mathbb{R}\}$ with respect to the second variable x , we deduce that for given $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $t \in \mathbb{R}$, ξ_1 and $\xi_2 \in K$, one has

$$|\xi_1 - \xi_2| \leq \delta \Rightarrow |\phi_2(t, \xi_1) - \phi_2(t, \xi_2)|_\alpha \leq \varepsilon.$$

Therefore, there exist $n(\varepsilon)$ and $\{z_i\}_{i=1}^{n(\varepsilon)} \subset K$, such that

$$K \subset \bigcup_{i=1}^{n(\varepsilon)} B_\delta(z_i, \delta)$$

and then

$$\|\phi_2(t, h_1(t))\| \leq \varepsilon + \sum_{i=1}^{n(\varepsilon)} \|\phi_2(t, z_i)\|$$

Since

$$\forall i \in \{1, \dots, n(\varepsilon)\}, \quad \lim_{\tau \rightarrow +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \left(\sup_{\theta \in [t-r, t]} |\phi_2(\theta, z_i)|_\alpha \right) d\mu(t) = 0,$$

we deduce that

$$\forall \varepsilon > 0, \quad \limsup_{\tau \rightarrow +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \left(\sup_{\theta \in [t-r, t]} |\phi_2(\theta, h_1(t))|_\alpha \right) d\mu(t) \leq \varepsilon \delta,$$

that implies

$$\lim_{\tau \rightarrow +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \left(\sup_{\theta \in [t-r, t]} |\phi_2(\theta, h_1(\theta))|_\alpha \right) d\mu(t) = 0.$$

Consequently $t \mapsto \phi_2(t, h(t))$ is (μ, ν) -ergodic of class r . ■

For $\mu \in \mathcal{M}$ and $\delta \in \mathbb{R}$, we denote μ_δ the positive measure on $(\mathbb{R}, \mathcal{B})$ defined by

$$\mu_\delta(A) = \mu([a + \delta : a \in A]) \quad (8)$$

Lemma 5.12. [3] Let $\mu \in \mathcal{M}$ satisfy (H_4) . Then the measures μ and μ_δ are equivalent for all $\delta \in \mathbb{R}$.

Lemma 5.13. [3] (H_4) implies

$$\text{for all } \sigma > 0 \quad \limsup_{\tau \rightarrow +\infty} \frac{\mu([-\tau - \sigma, \tau + \sigma])}{\mu([-\tau, \tau])} < +\infty.$$

We have the following result.

Theorem 5.14. Assume that (H_4) holds. Let $\mu, \nu \in \mathcal{M}$ and $\phi \in PAA_c(\mathbb{R}; X_\alpha, \mu, \nu, r)$, then the function $t \rightarrow \phi_t$ belongs to $PAA_c(C_\alpha, \mu, \nu, r)$.

Proof. Assume that $u = g + h$ where $g \in AA(\mathbb{R}; X_\alpha)$ and $h \in \mathcal{E}(\mathbb{R}; X_\alpha, \mu, \nu, r)$. We can see that $u_t = g_t + h_t$. We want to show that $g_t \in AA(\mathbb{R}; X_\alpha)$ and $h_t \in \mathcal{E}(\mathbb{R}; X_\alpha, \mu, \nu, r)$.

Firstly for a given sequence $(s_m)_{m \in \mathbb{N}}$ of real numbers, fix a subsequence $(s_n)_{n \in \mathbb{N}}$ and $\nu \in BC(\mathbb{R}; X_\alpha)$ such that

$g(s + s_n) \rightarrow v(s)$ uniformly on compact subsets of $\mathbb{R}$. Let $K \subset \mathbb{R}$ be an arbitrary compact and $L > 0$ such that $K \subset [-L, L]$. For $\varepsilon > 0$, fix $N_{\varepsilon, L} \in \mathbb{N}$ such that $|g(s + s_n) - v(s)|_\alpha \leq \varepsilon$ for $s \in [-L, L]$ whenever $n \geq N_{\varepsilon, L}$. For $t \in K$ and $n \geq N_{\varepsilon, L}$, we have

$$\begin{aligned} |g_{t+s_n} - v_t|_\alpha &\leq \sup_{\theta \in [-L, L]} |g(\theta + s_n) - v(\theta)|_\alpha \\ &\leq \varepsilon. \end{aligned}$$

In view of above, g_{t+s_n} converges to v_t uniformly on K . Similarly, one can prove that v_{t-s_n} converges to u_t uniformly on K . Thus, the function $s \rightarrow g_s$ belongs to $AA_c(\mathbb{R}; X_\alpha)$.

Let us denote by

$$M_\delta(\tau) = \frac{1}{v_\delta([- \tau, \tau])} \int_{-\tau}^{+\tau} \left(\sup_{\theta \in [t-r, t]} |h(\theta)|_\alpha \right) d\mu_\delta(t),$$

where μ_δ and v_δ are the positive measures defined by equation (8). By using Lemma 5.12, it follows that μ_δ and μ are equivalent and v_δ and v are also equivalent. Then by using Theorem 5.10 we have $\mathcal{E}(\mathbb{R}; X_\alpha, \mu_\delta, v_\delta, r) = \mathcal{E}(\mathbb{R}; X_\alpha, \mu, v, r)$, therefore $h \in \mathcal{E}(\mathbb{R}; X, \mu_\delta, v_\delta, r)$, that is

$$\lim_{\tau \rightarrow +\infty} M_\delta(\tau) = 0, \text{ for all } \delta \in \mathbb{R}.$$

On the other hand, for $r > 0$ we have

$$\begin{aligned} &\frac{1}{v([- \tau, \tau])} \int_{-\tau}^{+\tau} \left(\sup_{\theta \in [t-r, t]} \left[\sup_{\xi \in [-r, 0]} |h(\theta + \xi)|_\alpha \right] \right) d\mu(t) \leq \frac{1}{v([- \tau, \tau])} \int_{-\tau}^{+\tau} \left(\sup_{\theta \in [t-2r, t]} |h(\theta)|_\alpha \right) d\mu(t) \\ &\leq \frac{1}{v([- \tau, \tau])} \int_{-\tau}^{+\tau} \left(\sup_{\theta \in [t-2r, t-r]} |h(\theta)|_\alpha + \sup_{\theta \in [t-r, t]} |h(\theta)|_\alpha \right) d\mu(t) \\ &\leq \frac{1}{v([- \tau, \tau])} \int_{-\tau-r}^{+\tau-r} \left(\sup_{\theta \in [t-r, t]} |h(\theta)|_\alpha \right) d\mu(t+r) + \frac{1}{v([- \tau, \tau])} \int_{-\tau}^{+\tau} \left(\sup_{\theta \in [t-r, t]} |h(\theta)|_\alpha \right) d\mu(t) \\ &\leq \frac{1}{v([- \tau, \tau])} \int_{-\tau-r}^{+\tau+r} \left(\sup_{\theta \in [t-r, t]} |h(\theta)|_\alpha \right) d\mu(t+r) + \frac{1}{v([- \tau, \tau])} \int_{-\tau}^{+\tau} \left(\sup_{\theta \in [t-r, t]} |h(\theta)|_\alpha \right) d\mu(t) \\ &\leq \left[\frac{v([- \tau-r, \tau+r])}{v([- \tau, \tau])} \right] \times \frac{1}{v([- \tau-r, \tau+r])} \int_{-\tau-r}^{+\tau+r} \left(\sup_{\theta \in [t-r, t]} |h(\theta)|_\alpha \right) d\mu(t+r) \\ &\quad + \frac{1}{v([- \tau, \tau])} \int_{-\tau}^{+\tau} \left(\sup_{\theta \in [t-r, t]} |h(\theta)|_\alpha \right) d\mu(t). \end{aligned}$$

Consequently

$$\begin{aligned} \frac{1}{v([- \tau, \tau])} \int_{-\tau}^{+\tau} \left(\sup_{\theta \in [t-r, t]} \left[\sup_{\xi \in [-r, 0]} |h(\theta + \xi)|_\alpha \right] \right) d\mu(t) &\leq \left[\frac{v([- \tau-r, \tau+r])}{v([- \tau, \tau])} \right] \times M_r(\tau+r) \\ &\quad + \frac{1}{v([- \tau, \tau])} \int_{-\tau}^{+\tau} \left(\sup_{\theta \in [t-r, t]} |h(\theta)|_\alpha \right) d\mu(t), \end{aligned}$$

which shows using Lemma 5.12 and Lemma 5.13 that ϕ_t belongs to $PAA_c(C_\alpha, \mu, v, r)$. Thus, we obtain the desired result. ■

Lemma 5.15. [3] Let $\mu, v \in \mathcal{M}$ satisfy (H_4) . Then $PAA(\mathbb{R}, X, \mu, v)$ is invariant by translation, that is $f \in PAA(\mathbb{R}, X, \mu, v)$ implies $f_\gamma \in PAA(\mathbb{R}, X, \mu, v)$ for all $\gamma \in \mathbb{R}$.

Corollary 5.16. Let $\mu, v \in \mathcal{M}$ satisfy (H_4) . Then $PAA(\mathbb{R}, X_\alpha, \mu, v, r)$ is invariant by translation, that is $f \in PAA(\mathbb{R}, X_\alpha, \mu, v, r)$ implies $f_\gamma \in PAA(\mathbb{R}, X_\alpha, \mu, v, r)$ for all $\gamma \in \mathbb{R}$.

Proof. It suffices to prove that $\mathcal{E}(\mathbb{R}, X, \mu, \nu, r)$ is invariant by translation. Let $f \in \mathcal{E}(\mathbb{R}, X, \mu, \nu, \infty)$ and $F^t(\theta) = \sup_{\theta \in [t-r, t]} |f(\theta)|$. Then $F^t \in \mathcal{E}(\mathbb{R}, \mathbb{R}, \mu, \nu)$, but since $\mathcal{E}(\mathbb{R}, \mathbb{R}, \mu, \nu)$ is invariant by translation, it follows that

$$\lim_{\tau \rightarrow +\infty} \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} F^t(\theta + \gamma) d\mu(t) = \lim_{\tau \rightarrow +\infty} \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} |f(\theta + \gamma)| d\mu(t) = 0,$$

which implies that $f(\cdot + \gamma) \in PAA(\mathbb{R}, X, \mu, \nu, r)$. ■

6 (μ, ν) -pseudo almost automorphic solutions of class r

In what follows, we will be looking at the existence of bounded integral solutions of class r of equation (1).

Proposition 6.1. [1] Assume that (H_0) and (H_1) hold and the semigroup $(\mathcal{U}(t))_{t \geq 0}$ is hyperbolic. If f is bounded on $\mathbb{R}$, then there exists a unique bounded solution u of equation (1) on $\mathbb{R}$, given by

$$u_t = \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 f(s)) ds + \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^t \mathcal{U}^u(t-s) \Pi^u(\tilde{B}_\lambda X_0 f(s)) ds \quad \text{for } t \in \mathbb{R},$$

where $\tilde{B}_\lambda = \lambda(\lambda I - \mathcal{A}_\mathcal{U})^{-1}$ for $\lambda > \tilde{\omega}$, Π^s and Π^u are the projections of C_α onto the stable and unstable subspaces, respectively.

Proposition 6.2. Let $h \in AA_c(\mathbb{R}; X)$ and Γ be the mapping defined for $t \in \mathbb{R}$ by

$$\Gamma h(t) = \left[\lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 h(s)) ds + \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^t \mathcal{U}^u(t-s) \Pi^u(\tilde{B}_\lambda X_0 h(s)) ds \right](0).$$

Then $\Gamma h \in AA_c(\mathbb{R}, X_\alpha)$.

Proof. We can see that $\Gamma h \in BC(\mathbb{R}; X_\alpha)$. In fact

$$\begin{aligned} |\Gamma h(t)|_\alpha &\leq \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t |\mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 h(s))|_\alpha ds + \lim_{\lambda \rightarrow +\infty} \int_t^{+\infty} |\mathcal{U}^u(t-s) \Pi^u(\tilde{B}_\lambda X_0 h(s))|_\alpha ds \Big| (0) \\ &\leq \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \|\mathcal{A}_\mathcal{U}^\alpha \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 h(s))\| ds + \lim_{\lambda \rightarrow +\infty} \int_t^{+\infty} \|\mathcal{A}_\mathcal{U}^\alpha \mathcal{U}^u(t-s) \Pi^u(\tilde{B}_\lambda X_0 h(s))\| ds \Big| (0) \\ &\leq \overline{M} \tilde{M} \int_{-\infty}^t \frac{e^{-\omega(t-s)}}{(t-s)^\alpha} |\Pi^s| \|h(s)\| ds + \overline{M} \tilde{M} \int_t^{+\infty} \frac{e^{\omega(t-s)}}{(s-t)^\alpha} |\Pi^u| \|h(s)\| ds \\ &\leq \overline{M} \tilde{M} \int_{-\infty}^t \frac{e^{-\omega(t-s)}}{(t-s)^\alpha} |\Pi^s| \|h(s)\| ds + \overline{M} \tilde{M} \int_t^{+\infty} \frac{e^{\omega(t-s)}}{(s-t)^\alpha} |\Pi^u| \|h(s)\| ds \\ &\leq \overline{M} \tilde{M} \int_{-\infty}^t \frac{e^{-\omega(t-s)}}{(t-s)^\alpha} |\Pi^s| \|h(s)\| ds + \overline{M} \tilde{M} \int_t^{+\infty} \frac{e^{-\omega(s-t)}}{(s-t)^\alpha} |\Pi^u| \|h(s)\| ds \\ &\leq \frac{2K \|h\|_\infty}{\omega^{1-\alpha}} \int_0^{+\infty} e^{-s} s^{-\alpha} ds = \frac{2K \|h\|_\infty \Gamma(1-\alpha)}{\omega^{1-\alpha}} < \infty, \end{aligned} \tag{9}$$

where $K = \max(\overline{M} \tilde{M} |\Pi^s|, \overline{M} \tilde{M} |\Pi^u|)$.

For a given sequence $(s_m)_{m \in \mathbb{N}}$ of real numbers, fix a subsequence $(s_n)_{n \in \mathbb{N}}$ and $v \in BC(\mathbb{R}; X_\alpha)$ such that $h(t + s_n)$ converges to $v(t)$ and $v(t - s_n)$ converges to $h(t)$ uniformly on compact subsets of $\mathbb{R}$. Then, if

$$w(t + s_n) = \lim_{\lambda \rightarrow -\infty} \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 h(s + s_n)) ds + \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^t \mathcal{U}^u(t-s) \Pi^u(\tilde{B}_\lambda X_0 h(s + s_n)) ds \text{ for } t \in \mathbb{R}, n \in \mathbb{N},$$

then by equation(9) and the Lebesgue Dominated Convergence Theorem, it follows that $w(t + s_n)$ converges to

$$z(t) = \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 v(s)) ds + \lim_{\lambda \rightarrow +\infty} \int_{+\infty}^t \mathcal{U}^u(t-s) \Pi^u(\tilde{B}_\lambda X_0 v(s)) ds.$$

It remains to prove that the convergence is uniform on all compact subset of $\mathbb{R}$. Let $K \subset \mathbb{R}$ be an arbitrary compact and let $\varepsilon > 0$. Fix $L > 0$ and $N_\varepsilon \in \mathbb{N}$ such that $K \subset [-\frac{L}{2}, \frac{L}{2}]$ with

$$\int_{\frac{L}{2}}^{+\infty} e^{-s} s^{-\alpha} < \varepsilon$$

$$\|h(s + s_n) - v(s)\| \leq \varepsilon \text{ for } n \geq N_\varepsilon \text{ and } s \in [-L, L].$$

Then, for each $t \in K$, ones has

$$\begin{aligned} |w(t + s_n) - z(t)| &\leq \overline{MM} |\Pi^s| \int_{-\infty}^t \frac{e^{-\omega(t-s)}}{(t-s)^\alpha} \|h(s + s_n) - v(s)\| ds \\ &\quad + \overline{MM} |\Pi^u| \int_t^{+\infty} \frac{e^{-\omega(s-t)}}{(s-t)^\alpha} \|h(s + s_n) - v(s)\| ds \\ &\leq \overline{MM} |\Pi^s| \int_{-\infty}^{-L} \frac{e^{-\omega(t-s)}}{(t-s)^\alpha} \|h(s + s_n) - v(s)\| ds \\ &\quad + \overline{MM} |\Pi^s| \int_{-L}^t \frac{e^{-\omega(t-s)}}{(t-s)^\alpha} \|h(s + s_n) - v(s)\| ds \\ &\quad + \overline{MM} |\Pi^u| \int_t^{+\infty} \frac{e^{-\omega(s-t)}}{(s-t)^\alpha} \|h(s + s_n) - v(s)\| ds \\ &\leq \frac{2\overline{MM} |\Pi^s| \|h\|_\infty}{\omega^{1-\alpha}} \int_{t+L}^{\infty} e^{-s} s^{-\alpha} ds + \frac{\overline{MM} \varepsilon (|\Pi^s| + |\Pi^u|)}{\omega^{1-\alpha}} \int_0^{\infty} e^{-s} s^{-\alpha} ds \\ &\leq \frac{2\overline{MM} |\Pi^s| \|h\|_\infty}{\omega^{1-\alpha}} \int_{\frac{L}{2}}^{\infty} e^{-s} s^{-\alpha} ds + \frac{\overline{MM} \varepsilon (|\Pi^s| + |\Pi^u|)}{\omega^{1-\alpha}} \int_0^{\infty} e^{-\omega s} ds \\ &\leq \frac{2\overline{MM}}{\omega^{1-\alpha}} \left(2|\Pi^s| \|h\|_\infty + (|\Pi^s| + |\Pi^u|) \int_0^{\infty} e^{-\omega s} ds \right) \varepsilon, \end{aligned}$$

which proves that the convergence is uniform on K , by the fact that the last estimate is independent of $t \in K$. Proceeding as previously, one can similarly prove that $z(t - s_n)$ converges to w uniformly on compact subsets in $\mathbb{R}$. This completes the proof. ■

Theorem 6.3. Let $\mu, \nu \in \mathcal{M}$ satisfy (H_4) and $g \in \mathcal{E}(\mathbb{R}; X, \mu, \nu, r)$. Then $\Gamma g \in \mathcal{E}(\mathbb{R}; X_\alpha, \mu, \nu, r)$.

Proof. In fact, for $\tau > 0$ we get

$$\begin{aligned}
 \int_{-\tau}^{\tau} \left(\sup_{\theta \in [t-r, t]} |\Gamma g(\theta)|_\alpha \right) d\mu(t) &\leq \int_{-\tau}^{\tau} \left(\sup_{\theta \in [t-r, t]} \left[\lim_{\lambda \rightarrow +\infty} \int_{-\infty}^{\theta} |\mathcal{U}^s(\theta - s) \Pi^s(\tilde{B}_\lambda X_0 g(s))|_\alpha ds \right. \right. \\
 &\quad \left. \left. + \lim_{\lambda \rightarrow +\infty} \int_{\theta}^{+\infty} |\mathcal{U}^u(\theta - s) \Pi^u(\tilde{B}_\lambda X_0 g(s))|_\alpha ds \right] (0) \right) d\mu(t) \\
 &\leq \int_{-\tau}^{\tau} \left(\sup_{\theta \in [t-r, t]} \left[\lim_{\lambda \rightarrow +\infty} \int_{-\infty}^{\theta} \|\mathcal{A}_\mathcal{U}^\alpha \mathcal{U}^s(\theta - s) \Pi^s(\tilde{B}_\lambda X_0 g(s))\| ds \right. \right. \\
 &\quad \left. \left. + \lim_{\lambda \rightarrow +\infty} \int_{\theta}^{+\infty} \|\mathcal{A}_\mathcal{U}^\alpha \mathcal{U}^u(\theta - s) \Pi^u(\tilde{B}_\lambda X_0 g(s))\| ds \right] (0) \right) d\mu(t) \\
 &\leq \overline{MM} \int_{-\tau}^{\tau} \left(\sup_{\theta \in [t-r, t]} \int_{-\infty}^{\theta} \frac{e^{-\omega(\theta-s)}}{(\theta-s)^\alpha} |\Pi^s| |g(s)| ds \right) d\mu(t) \\
 &\quad + \overline{MM} \int_{-\tau}^{\tau} \left(\sup_{\theta \in [t-r, t]} \int_{\theta}^{+\infty} \frac{e^{\omega(\theta-s)}}{(s-\theta)^\alpha} |\Pi^u| |g(s)| ds \right) d\mu(t) \\
 &\leq K \left[\int_{-\tau}^{\tau} \left(\sup_{\theta \in [t-r, t]} \int_{-\infty}^{\theta} \frac{e^{-\omega(\theta-s)}}{(\theta-s)^\alpha} |g(s)| ds \right) d\mu(t) + \int_{-\tau}^{\tau} \left(\sup_{\theta \in [t-r, t]} \int_{\theta}^{+\infty} \frac{e^{\omega(\theta-s)}}{(s-\theta)^\alpha} |g(s)| ds \right) d\mu(t) \right],
 \end{aligned}$$

where $K = \max(\overline{MM} |\Pi^s|, \overline{MM} |\Pi^u|)$. On the one hand using Fubini's theorem, we have

$$\begin{aligned}
 \int_{-\tau}^{\tau} \left(\sup_{\theta \in [t-r, t]} \int_{-\infty}^{\theta} \frac{e^{-\omega(\theta-s)}}{(\theta-s)^\alpha} |g(s)| ds \right) d\mu(t) &\leq \int_{-\tau}^{\tau} \left(\sup_{\theta \in [t-r, t]} \int_0^{+\infty} \frac{e^{-\omega s}}{s^\alpha} |g(\theta - s)| ds \right) d\mu(t) \\
 &\leq \int_0^{+\infty} \frac{e^{-\omega s}}{s^\alpha} \left(\sup_{\theta \in [t-r, t]} \int_{-\tau}^{\tau} |g(\theta - s)| d\mu(t) \right) ds.
 \end{aligned}$$

By the Lebesgue dominated convergence theorem and by using Corollary 5.16, it follows that

$$\lim_{\tau \rightarrow +\infty} \int_0^{+\infty} \frac{e^{-\omega s}}{s^\alpha} \frac{1}{\nu([- \tau, \tau])} \left(\sup_{\theta \in [t-r, t]} \int_{-\tau}^{\tau} |g(\theta - s)| d\mu(t) \right) ds = 0.$$

On the other hand by Fubini's theorem, we also have

$$\begin{aligned}
 \int_{-\tau}^{\tau} \left(\sup_{\theta \in [t-r, t]} \int_{\theta}^{+\infty} \frac{e^{\omega(\theta-s)}}{(s-\theta)^\alpha} |g(s)| ds \right) d\mu(t) &\leq \int_{-\tau}^{\tau} \left(\sup_{\theta \in [t-r, t]} \int_0^{+\infty} \frac{e^{-\omega s}}{s^\alpha} |g(s + \theta)| ds \right) d\mu(t) \\
 &\leq \int_0^{+\infty} \frac{e^{-\omega s}}{s^\alpha} \left(\sup_{\theta \in [t-r, t]} \int_{-\tau}^{\tau} |g(s + \theta)| d\mu(t) \right) ds.
 \end{aligned}$$

Reasoning like above, it follows that

$$\lim_{\tau \rightarrow +\infty} \int_0^{+\infty} \frac{e^{-\omega s}}{s^\alpha} \left(\sup_{\theta \in [t-r, t]} \int_{-\tau}^{\tau} |g(s + \theta)| d\mu(t) \right) ds = 0.$$

Consequently

$$\lim_{\tau \rightarrow +\infty} \frac{1}{\nu([- \tau, \tau])} \int_{-\tau}^{\tau} \left(\sup_{\theta \in [t-r, t]} |(\Gamma g)(\theta)|_\alpha \right) d\mu(t) = 0.$$

Thus, we obtain the desired result. ■

For the existence of (μ, ν) -pseudo almost automorphic solution of class r , we make the following assumption.

(H₅) $f : \mathbb{R} \rightarrow X$ is compact $cl(\mu, \nu)$ -pseudo almost automorphic of class r .

Proposition 6.4. Assume **(H₀)**, **(H₁)**, **(H₃)** and **(H₅)** hold. Then equation (1) has a unique compact $\alpha - cl(\mu, \nu)$ -pseudo almost automorphic solution of class r .

Proof. Since f is a (μ, ν) -pseudo almost automorphic function, f has a decomposition $f = f_1 + f_2$ where $f_1 \in AA_c(\mathbb{R}; X)$ and $f_2 \in \mathcal{E}(\mathbb{R}; X, \mu, \nu, r)$. Using Proposition 6.1, Proposition 6.2 and Theorem 6.3, we get the desired result. ■

Our next objective is to show the existence of (μ, ν) -pseudo almost automorphic solutions of class r for the following problem

$$u'(t) = -Au(t) + L(u_t) + f(t, u_t) \text{ for } t \in \mathbb{R} \quad (10)$$

where $f : \mathbb{R} \times C_\alpha \rightarrow X$ is continuous.

For the sequel, we make the following assumptions.

(H₆) The instable space $U \equiv \{0\}$.

(H₇) Let $\mu, \nu \in \mathcal{M}$ and $f : \mathbb{R} \times C_\alpha \rightarrow X$ $cl(\mu, \nu)$ -pseudo almost automorphic of class r such that there exists a positive constant L_f such that

$$\|f(t, \varphi_1) - f(t, \varphi_2)\| \leq L_f \|\varphi_1 - \varphi_2\|_{C_\alpha} \text{ for all } t \in \mathbb{R} \text{ and } \varphi_1, \varphi_2 \in C_\alpha$$

and L_f satisfies (6).

Theorem 6.5. Assume **(H₀)**, **(H₁)**, **(H₂)**, **(H₃)**, **(H₆)** and **(H₇)** hold. If

$$\frac{\overline{MM}|\Pi^s|L_f\Gamma(1-\alpha)}{\omega^{1-\alpha}} < 1,$$

then equation (10) has a unique compact $\alpha - cl(\mu, \nu)$ -pseudo almost automorphic solution of class r .

Proof. Let x be a function in $PAA_c(\mathbb{R}; X, \mu, \nu, r)$, from Theorem 5.14 the function $t \rightarrow x_t$ belongs to $PAA_c(C_\alpha, \mu, r)$. Hence Theorem 5.11 implies that the function $g(\cdot) := f(\cdot, x_\cdot)$ is in $PAA(\mathbb{R}; X, \mu, r)$. Since the unstable space $U \equiv \{0\}$, then $\Pi^u \equiv 0$. Consider the mapping

$$\mathcal{H} : PAA_c(\mathbb{R}; X_\alpha, \mu, \nu, r) \rightarrow PAA_c(\mathbb{R}; X_\alpha, \mu, \nu, r)$$

defined for $t \in \mathbb{R}$ by

$$(\mathcal{H}x)(t) = \left[\lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0 f(s, x_s)) ds \right](0).$$

From Proposition 6.1, Proposition 6.2 and taking into account Theorem 6.3, it suffices now to show that the operator $\mathcal{H}$ has a unique fixed point in $PAA(\mathbb{R}; X_\alpha, \mu, r)$.

Let $x_1, x_2 \in PAA_c(\mathbb{R}; X_\alpha, r)$. Then we have

$$\begin{aligned}
 |\mathcal{H}x_1(t) - \mathcal{H}x_2(t)|_\alpha &\leq \left| \lim_{\lambda \rightarrow +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s) \Pi^s(\tilde{B}_\lambda X_0(f(s, x_{1s}) - f(s, x_{2s}))) ds \right|_\alpha \\
 &\leq \overline{M} \tilde{M} |\Pi^s| L_f \int_{-\infty}^t \frac{e^{-\omega(t-s)}}{(t-s)^\alpha} |x_{1s} - x_{2s}| ds \\
 &\leq \overline{M} \tilde{M} |\Pi^s| |x_1 - x_2| L_f \int_{-\infty}^t \frac{e^{-\omega(t-s)}}{(t-s)^\alpha} ds \\
 &\leq \frac{\overline{M} \tilde{M} |\Pi^s| L_f}{\omega^{1-\alpha}} \left(\int_0^{+\infty} e^{-s} s^{-\alpha} ds \right) |x_1 - x_2| \\
 &\leq \frac{\overline{M} \tilde{M} |\Pi^s| L_f \Gamma(1-\alpha)}{\omega^{1-\alpha}} |x_1 - x_2|.
 \end{aligned}$$

This means that $\mathcal{H}$ is a strict contraction. Thus by Banach's fixed point theorem, $\mathcal{H}$ has a unique fixed point u in $PAA_c(\mathbb{R}; X_\alpha, \mu, \nu, r)$. We conclude that equation (10), has one and only one $cl(\mu, \nu)$ -pseudo almost automorphic solution of class r . ■

7 Application

For illustration, we propose to study the existence of solutions for the following model

$$\begin{cases} \frac{\partial z(t, x)}{\partial t} = -\frac{\partial^2 z(t, x)}{\partial x^2} + \int_{-r}^0 G(\theta) z(t + \theta, x) d\theta + \int_0^t \sin\left(\frac{1}{2 + \cos s + \cos \sqrt{2}s}\right) ds \\ \quad + \arctan(t) + h\left(t, \frac{\partial}{\partial x} z(t + \theta, x)\right) \text{ for } t \in \mathbb{R} \text{ and } x \in [0, \pi] \\ z(t, 0) = z(t, \pi) = 0 \text{ for } t \in \mathbb{R}, \end{cases} \quad (11)$$

where $G : [-r, 0] \rightarrow \mathbb{R}$ is a continuous function and $h : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz continuous with respect to the second argument. To rewrite equation (11) in the abstract form, we introduce the space $X = L^2([0, \pi]; \mathbb{R})$ vanishing at 0 and π , equipped with the L^2 norm that is to say for all $x \in X$,

$$\|x\|_{L^2} = \left(\int_0^\pi |x(s)|^2 ds \right)^{\frac{1}{2}}.$$

Let $A : X \rightarrow X$ be defined by

$$\begin{cases} D(A) = H^2(0, \pi) \cap H_0^1(0, \pi) \\ Ay = y''. \end{cases}$$

Then the spectrum $\sigma(A)$ of A equals to the point spectrum $\sigma_p(A)$ and is given by

$$\sigma(A) = \sigma_p(A) = \{-n^2 : n \geq 1\}$$

and the associated eigenfunctions $(e_n)_{n \geq 1}$ are given by

$$e_n(s) = \sqrt{\frac{2}{\pi}} \sin(ns), \quad s \in [0, \pi].$$

Then the operator is computed by

$$Ay = \sum_{n=1}^{+\infty} n^2(y, e_n)e_n, \quad y \in D(A).$$

For each $y \in D(A^{\frac{1}{2}}) = \{y \in X : \sum_{n=1}^{+\infty} n(y, e_n)e_n \in X\}$, the operator $A^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}}y = \sum_{n=1}^{+\infty} n(y, e_n)e_n, \quad y \in D(A).$$

Lemma 7.1. [9] If $y \in D(A^{\frac{1}{2}})$, then y is absolutely continuous, $y' \in X$ and $|y'| = |A^{\frac{1}{2}}y|$.

It is well known that $-A$ is the generator of a compact analytic semigroup $(T(t))_{t \geq 0}$ on X which is given by

$$T(t)x = \sum_{n=1}^{+\infty} e^{-n^2 t}(x, e_n)e_n, \quad x \in X.$$

Then (H_0) and (H_1) are satisfied. Here we choose $\alpha = \frac{1}{2}$.

We define $f : \mathbb{R} \times C_{\frac{1}{2}} \rightarrow X$ and $L : C_{\frac{1}{2}} \rightarrow X$ as follows

$$f(t, \varphi)(x) = \int_0^t \sin\left(\frac{1}{2 + \cos s + \cos \sqrt{2}s}\right) ds + \arctan(t) + h\left(t, \frac{\partial}{\partial x} \varphi(\theta, x)\right) \text{ for } x \in [0, \pi] \text{ and } t \in \mathbb{R},$$

$$L(\varphi)(x) = \int_{-r}^0 G(\theta) \varphi(\theta, x) d\theta \text{ for } -r \leq \theta \leq 0 \text{ and } x \in [0, \pi].$$

Let us pose $v(t) = z(t, x)$. Then equation (11) takes the following abstract form

$$v'(t) = -Av(t) + L(v_t) + f(t, v_t) \text{ for } t \in \mathbb{R}. \quad (12)$$

Consider the measures μ and ν where its Radon-Nikodym derivative are respectively $\rho_1, \rho_2 : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\rho_1(t) = \begin{cases} 1 & \text{for } t > 0 \\ e^t & \text{for } t \leq 0. \end{cases}$$

and

$$\rho_2(t) = |t| \text{ for } t \in \mathbb{R}$$

i.e $d\mu(t) = \rho_1(t)dt$ and $d\nu(t) = \rho_2(t)dt$ where dt denotes the Lebesgue measure on $\mathbb{R}$ and

$$\mu(A) = \int_A \rho_1(t)dt \text{ for } \nu(A) = \int_A \rho_2(t)dt \text{ for } A \in \mathcal{B}.$$

From [?] $\mu, \nu \in \mathcal{M}$, μ, ν satisfy Hypothesis (H_4) .

We have

$$\limsup_{\tau \rightarrow +\infty} \frac{\mu([- \tau, \tau])}{\nu([- \tau, \tau])} = \limsup_{\tau \rightarrow +\infty} \frac{\int_{-\tau}^0 e^t dt + \int_0^{\tau} dt}{2 \int_0^{\tau} t dt} = \limsup_{\tau \rightarrow +\infty} \frac{1 - e^{-\tau} + \tau}{\tau^2} = 0 < \infty,$$

which implies that (H_2) is satisfied.

Since $A^{\frac{1}{2}}\left(\int_0^t \sin\left(\frac{1}{2+\cos s+\cos\sqrt{2}s}\right)ds\right) = \sin\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right)$ and $t \rightarrow \sin\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right)$ belongs to $AA_c(\mathbb{R}, X)$, it follows that $t \rightarrow \int_0^t \sin\left(\frac{1}{2+\cos s+\cos\sqrt{2}s}\right)ds$ belongs to $AA_c(\mathbb{R}, X_{\frac{1}{2}})$. On the other hand, we have the following:

$$\begin{aligned} \frac{1}{v([- \tau, \tau])} \int_{-\tau}^{+\tau} \sup_{\theta \in [t-r, t]} |\arctan(\theta)|_{\frac{1}{2}} dt &= \frac{1}{v([- \tau, \tau])} \int_{-\tau}^{+\tau} \sup_{\theta \in [t-r, t]} |A^{\frac{1}{2}} \arctan(\theta)| dt \\ &= \frac{1}{v([- \tau, \tau])} \int_{-\tau}^{+\tau} \sup_{\theta \in [t-r, t]} \left(\frac{1}{1+\theta^2}\right) dt \\ &\leq \frac{\mu([- \tau, \tau])}{v([- \tau, \tau])} \rightarrow 0 \text{ as } \tau \rightarrow +\infty. \end{aligned}$$

It follows that $t \mapsto \arctan t$ is (μ, v) -ergodic of class r , consequently, f is uniformly compact (μ, v) -pseudo almost automorphic of class r . Moreover, L is a bounded linear operator from $C_{\frac{1}{2}}$ to X .

Let k be the lipschitz constant of h , then for every $\varphi_1, \varphi_2 \in C_{\frac{1}{2}}$ and $t \geq 0$, we have

$$\begin{aligned} \|f(t, \varphi_1)(x) - f(t, \varphi_2)(x)\| &= \left(\int_0^\pi \left[h\left(\theta, \frac{\partial}{\partial x} \varphi_1(\theta, x)\right) - h\left(\theta, \frac{\partial}{\partial x} \varphi_2(\theta, x)\right) \right]^2 dx \right)^{\frac{1}{2}} \\ &\leq L_h \left[\int_0^\pi \left(\frac{\partial}{\partial x} \varphi_1(\theta, x) - \frac{\partial}{\partial x} \varphi_2(\theta, x) \right)^2 dx \right]^{\frac{1}{2}} \\ &\leq L_h \sup_{-r \leq \theta \leq 0} \left[\int_0^\pi \left(\frac{\partial}{\partial x} \varphi_1(\theta, x) - \frac{\partial}{\partial x} \varphi_2(\theta, x) \right)^2 dx \right]^{\frac{1}{2}} \\ &\leq L_h \|\varphi_1 - \varphi_2\|_{C_a} \end{aligned}$$

Consequently, we conclude that f is Lipschitz continuous and $cl(\mu, v)$ -pseudo almost automorphic of class r .

Lemma 7.2. [6] If $\int_{-r}^0 |G(\theta)| d\theta < 1$, then the semigroup $(\mathcal{U}(t))_{t \geq 0}$ is hyperbolic.

For example, let us pose $G(\theta) = \frac{\theta^2 - 1}{(\theta^2 + 1)^2}$ for $\theta \in [-r, 0]$. Then we can see that

$$\int_{-r}^0 |G(\theta)| d\theta = \int_{-r}^0 \left| \frac{\theta^2 - 1}{(\theta^2 + 1)^2} \right| d\theta = \left[\frac{\theta}{\theta^2 + 1} \right]_{-r}^0 = \frac{r}{r^2 + 1} < 1 \text{ if } r < 1$$

and

$$\int_{-r}^0 |G(\theta)| d\theta = \int_{-r}^0 \left| \frac{\theta^2 - 1}{(\theta^2 + 1)^2} \right| d\theta = \int_{-r}^{-1} \frac{\theta^2 - 1}{(\theta^2 + 1)^2} d\theta + \int_{-1}^0 \frac{-\theta^2 + 1}{(\theta^2 + 1)^2} d\theta = 1 - \frac{r}{r^2 + 1} < 1 \text{ if } r \geq 1,$$

By Proposition 6.5 we deduce the following result.

Theorem 7.3. Under the above assumptions, if $Lip(h)$ is small enough, then equation (12) has a unique compact $\alpha - cl(\mu, v)$ -pseudo almost automorphic solution v of class r .

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