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Mehdi Pourbarat*

On some kinds of dense orbits in general nonautonomous dynamical systems

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Abstract: We study the theory of universality for the nonautonomous dynamical systems from topological point of view related to hypercyclicity. The conditions are provided in a way that Birkhoff transitivity theorem can be extended. In the context of generalized linear nonautonomous systems, we show that either one of the topological transitivity or hypercyclicity give sensitive dependence on initial conditions. Meanwhile, some examples are presented for topological transitivity, hypercyclicity and topological conjugacy.

Keywords: Birkhoff transitivity theorem, Hypercyclicity, Topological conjugacy

MSC: 37D45, 37B55, 57R18

1 Introduction

Nonautonomous dynamical systems provide a successful framework to describe a large variety of phenomena in biology, informatics, economy, medicine, quantum mechanics and etc [2, 8, 9, 17], in the case that they can not be modelled by autonomous systems. A knowledge of the topological and geometrical structure of nonautonomous discrete system is useful both in developing theory and in applications [6, 10]. In the last decade chaotic behaviour of nonautonomous discrete systems attracted many researchers [1, 3, 14, 16, 18]. Chaos theory deals with complex and unpredictable behaviour of phenomena over time. Even in the context of dynamical systems, different definitions of chaos has been posed which some of them are classified by Forti [4]. In most of the definitions, topological transitivity plays a central role. In this context, Birkhoff transitivity theorem establishes a straight relationship between topological transitivity and density of orbits on a separable complete metric space without isolated points [5, 13].

The present paper focuses on the generalization of nonautonomous dynamical systems and challenges some topological and dynamical characteristics like topological transitivity, sensitivity, hypercyclicity and conjugacy. Under suitable assumptions, some new results have been achieved that Birkhoff transitivity theorem is one of them. Suppose that X and X_0 are two nonempty topological spaces and $f_n : X_0 \rightarrow X$, $n \in \mathbb{N}$ are maps. The orbit of $x_0 \in X_0$ is defined as $orb(x_0) := \{f_n(x_0) | n \in \mathbb{N}\}$. When $orb(x_0)$ is a dense subset of X , the point x_0 is called universal [5]. We call the sequence $\{f_n\}_{n=1}^{\infty}$ a generalized nonautonomous discrete system (for short GNDS) on X_0 , that determines an equation in the form

$$x_{n+1} = f_n(x_0).$$

The definition of a GNDS on X_0 is more general than the definition of a NDS introduced in [7, 15] which is as follows: assume that X is a metric space and for each $n \in \mathbb{N} \cup \{0\}$, X_n is its subspace. Also, let $g_n : X_n \rightarrow X_{n+1}$ be an arbitrary map, then they call the family $\{g_n\}_{n=0}^{\infty}$ a NDS. Indeed, such a family generates a NDS via time-dependent difference equations $x_{n+1} = g_n(x_n)$. Here, for a given element $x_0 \in X_0$, a sequence $\{x_n\}_{n=0}^{\infty}$ is called the orbit of x_0 (or an orbit of $\{g_n\}_{n=0}^{\infty}$ in the iterative way) and is defined as

$$x_0, x_1 = g_0(x_0), x_2 = g_1(x_1), \dots, x_{n+1} = g_n(x_n), \dots$$

*Corresponding Author: Mehdi Pourbarat: Dep. Math., Shahid Beheshti University, Tehran, Iran

Consider $f_n := g_{n-1} \circ g_{n-2} \cdots \circ g_0$, we observe that f_n is a map from X_0 to X , and therefore our definition is a generalization (see Example 1.(h) and the explanation after that).

Throughout the paper, we always assume that X_0 is a subspace of X . Hence, for given $x \in X_0$ one may expect two main types of density by restricting to the subspace X_0 . In the other word, for the underlying GNDS we always deal with the subsets

$$O_1 := \{x \in X_0 \mid \overline{\text{orb}(x)} \cap X_0 = X_0\}, \quad O_2 := \{x \in X_0 \mid \overline{\text{orb}(x)}^X \cap X_0 = X_0\}.$$

Investigating topological structure of these subspaces leads us to achieve some results that are organized as follows:

In Section 2, some properties of subspaces O_1 and O_2 are discussed, and then we challenge these subspaces with topological transitivity. In fact we prove that if $\overline{O_1} = X_0$, the system is topologically transitive. Also, under certain conditions for underlying spaces and maps f_n (see Theorem 2), the topological transitivity yields $\overline{O_2} = X_0$.

In Section 3, the concepts of topological h -conjugacy and topological $\{h_n\}$ -conjugacy are defined for GNDSs on X_0 . We show that these two definitions fit exactly to Definition 3.6 in [16] and Definition 3.1 in [15], respectively. Then considering the hypothesis h_n some classical dynamical properties are preserved under two $\{h_n\}$ -conjugated equations.

In Section 4, assuming underlying space X as a Frechet space we focus on the generalized linear nonautonomous discrete systems (in short GLNDSs). Indeed we prove that either topological transitivity or hypercyclicity always implies sensitive dependence on initial conditions. Also, in this section two examples are posed, the first system is topologically transitive while $\overline{O_1} \neq X_0$, and for the second one (that satisfies hypothesis of Theorem 2) $\overline{O_2} = X_0$ holds while it is not topologically transitive.

In the study of dynamical systems, we may deal with results that less hypothesis for their proofs can be written. That is why we choose the underling system in the general form, and then we provide the conditions for the system in a way that some classical results obtained. As we mentioned in above, these results are related to chaos, topological conjugacy and linear systems. To continue this, it seems that we can find more results around expansiveness, sensitivity and topological entropy. Another extension is to interpret a kind of closeness for two general nonautonomous dynamical systems, and then to achieve the results in some contexts like symbolic dynamic, bifurcation theory and etc.

2 Hypercyclic Orbits and Topological Transitivity

We begin the section by posing the definition of a GNDS.

Definition 1. Let X be a topological space and X_0 be a subspace of X . A sequence $\{f_n\}_{n=1}^\infty$ of maps $f_n : X_0 \rightarrow X$ is called a GNDS on X_0 and the orbit of $x_0 \in X_0$ is defined as $\text{orb}(x_0) := \{f_n(x_0) \mid n \in \mathbb{N}\}$.

We write $f_{1,\infty}$ instead of $\{f_n\}_{n=1}^\infty$. Obviously, a dynamical system $g : X \rightarrow X$ can be considered as a GNDS on X_0 by setting $f_n := g^n|_{X_0}$. Many other examples are presented here that may be useful in the study of the theory of GNDSs.

Example 1. (a) Let Y be a topological space and $F : X_0 \times Y \rightarrow X$ be an arbitrary map. Also, let $\{a_n\}_{n=1}^\infty$ be a sequence in Y . Then $f_{1,\infty}$ given by $f_n(x) = F(x, a_n)$ is a GNDS on X_0 .

(b) Let M be a smooth manifold and $f \in \text{Diff}(M)$. Also, let p be a hyperbolic point and $X_0 = W_{loc}^u(p)$. Then $f_{1,\infty}$ given by $f_n(x) = f^n(x)$ is a GNDS on $W_{loc}^u(p)$. The similar situation can be written for $W_{loc}^s(p)$.

(c) Let g_1 and g_2 be two discrete dynamical systems on X and A be any subset of natural numbers. Then $f_{1,\infty}$ given by

$$f_n(x) = \begin{cases} g_1^n(x) & n \in A \\ g_2^n(x) & n \notin A \end{cases}, \quad x \in X_0$$

is a GNDS on X_0 .

(d) Let $\{\alpha_n\}_{n=1}^\infty$ be a sequence of real numbers. Then $f_{1,\infty}$ given by $f_n(x) = \alpha_n x(1-x)$ is a GNDS on $[0, 1]$. We call it Logistic GNDS corresponding to the sequence $\{\alpha_n\}_{n=1}^\infty$.

(e) Let $\mathbb{H}(\mathbb{C})$ be a set of all holomorphic functions from $\mathbb{C}$ into itself and $\{g_n\}_{n=0}^\infty$ be a sequence in $\mathbb{H}(\mathbb{C})$. Also, let D be the differentiable operator and maps f_n defined by

$$f_n(g) = \sum_{k=0}^n \frac{D^k g(0)}{k!} g_k.$$

Then $f_{1,\infty}$ is a GLNDS on any given subspace X_0 of $\mathbb{H}(\mathbb{C})$. Notice that in the case $g_n(x) := x^n$, $f_n(g)$ is the map that associates with g its Taylor polynomial of degree n at 0.

(f) Let $\{\alpha_n\}_{n=1}^\infty$ be a sequence of real numbers and X be the space of all continuous functions on $\mathbb{R}$ endowed with the sup norm. Also, let $f_n(g) = \tilde{g}$ where $\tilde{g}$ is Birkhoff's translation $\tilde{g}(x) := g(x + \alpha_n)$. Then $f_{1,\infty}$ is a GLNDS on the subspace X_0 .

(g) Let $\{\alpha_n\}_{n=1}^\infty$ be a sequence of real numbers. For each $1 \leq p < \infty$, let $X := l^p$ be the space of all p -summable sequences. Then $f_{1,\infty}$ given by

$$f_n((x_1, x_2, \dots, x_n, \dots)) = \alpha_n(x_n, x_{n+1}, \dots)$$

is a linear GNDS on any subspace X_0 of l^p .

(h) Let $f_n : [0, 1) \rightarrow [0, 1)$ be given by $f_n(x) = nx \bmod 1$. Then $f_{1,\infty}$ is a GNDS on $[0, 1)$.

In general a GNDS can not be stated in the iterative way, but if all maps f_n are bijective, by getting $g_0 := f_1$ and $g_n := f_{n+1} \circ f_n^{-1}$ as the maps from $X_n := f_n(X_0)$ to $X_{n+1} := f_{n+1}(X_0)$ then this GNDS is in the iterative way.

In the case that the maps f_n (even one of them) are not injective, maybe GNDS can not be stated in the iterative way. For instance, in Example 1.(h) there is no function $g_n : X_n \rightarrow X_{n+1}$ with $X_n = X_{n+1} = [0, 1)$ such that $g_n \circ f_n = f_{n+1}$.

To achieve this, let $A_n := \{a \mid f_n(a) = \frac{1}{2}\}$. Thus, $A_n = \{\frac{2k+1}{2n} \mid 0 \leq k < n, k \in \mathbb{N} \cup \{0\}\}$. Hence, for every $0 \leq k < n$ one follows:

$$g_n \circ f_n(\frac{2k+1}{2n}) = g_n(\frac{1}{2}), \quad f_{n+1}(\frac{2k+1}{2n}) = \frac{n+1}{2n}(2k+1) \bmod 1.$$

Consequently, $g_n(\frac{1}{2}) = \frac{n+1}{2n}(2k+1) \bmod 1$, which is in contradiction with the well-definitive of the function g_n .

The subset $A \subset X_0$ is said to be *invariant* if $f_n(A) \subset A$ for every $n \in \mathbb{N}$. The point x_0 is a *fixed* point if $\{x_0\}$ be invariant. It is called *weakly k -periodic* if $f_{mk}(x_0) = x_0$, for every $m \in \mathbb{N}$. The smallest such k is called the *period* of x_0 and the set of weakly periodic points is denoted by $WP(f_n)$. Also, weakly k -periodic point x_0 is called *k -periodic* if $f_{mk+n}(x_0) = f_n(x_0)$, for every $m, n \in \mathbb{N}$. The set of periodic points is denoted by $P(f_n)$, and so $P(f_n) \subset WP(f_n)$. The point x_0 is called a *finite orbit* point if the set $orb(x_0)$ is finite. The set of finite orbit points is denoted by $Fin(f_n)$. Notice that, $P(f_n) \subset Fin(f_n)$ while the following example demonstrates that in general the relation $WP(f_n) \subset Fin(f_n)$ does not hold.

Example 2. Consider $f_n(x) = \begin{cases} \frac{x}{2}(n+1+(-1)^n(1-n)) & x > 0 \\ -x & x \leq 0 \end{cases}$. The point 0 is the only fixed point for GNDS $f_{1,\infty}$. It is easy to verify that $P(f_n) = \{0\}$, $WP(f_n) = \mathbb{R}^+ \cup \{0\}$ and $Fin(f_n) = \mathbb{R}^- \cup \{0\}$.

Now we are going to generalize some topological concepts and results that is stated in [11], for GNDSs on X_0 and discuss the complicated behaviour that appears in the universality process. There are two kinds of densities of orbits for GNDSs on X_0 as follows:

- GNDS $f_{1,\infty}$ is called *hypercyclic or universal* on X_0 , if there is $x_0 \in X_0$ such that $\text{orb}(x_0) \cap X_0$ is dense in X_0 . In such a case, x_0 is called a hypercyclic point and the set of hypercyclic points is denoted by

$$O_1 := \{x \in X_0 \mid \overline{\text{orb}(x) \cap X_0}^{X_0} = X_0\}.$$

- GNDS $f_{1,\infty}$ is called *weakly hypercyclic or weakly universal* on X_0 , if there exists a point whose orbit's closure contains X_0 . Such a point is called a weakly hypercyclic point and the set of weakly hypercyclic points is denoted by

$$O_2 := \{x \in X_0 \mid \overline{\text{orb}(x)}^X \cap X_0 = X_0\}.$$

Thus, in dealing with the terminology “density of an orbit” we must specify which one of the above subspaces is intended. When these sets are large subspaces from topological point of view, we will expect a rich dynamic around X_0 . Before going any further, it is important to know the relationship between O_1 and O_2 .

Proposition 1. *With notations as above, we have:*

- $O_1 = \{x \in X_0 \mid \overline{\text{orb}(x) \cap X_0}^X \supseteq X_0\},$
- $O_1 \subseteq O_2,$
- if X_0 is an open set in X , then $O_1 = O_2$.

Proof. The same proof of Theorem 1 in [11]. □

Thus, every hypercyclic point is a weak hypercyclic point since $O_1 \subset O_2$. The sets O_1 and O_2 emerged in the following example are non trivial subsets of X_0 and $O_1 \neq O_2$.

Example 3. Let $X = \mathbb{R}^4$ and $X_0 = \mathbb{R}^3 \times \{0\}$. Let $\{(r_n, s_n, t_n)\}_{n=1}^\infty$ be a dense subsequence in $\mathbb{R}^3$ and $a_1, a_2, \dots, a_{10}$ ten arbitrary constants. Consider the GLNDS $T_{1,\infty}$ given by $T_n(x) = A_n x$, where

$$A_n = \begin{bmatrix} r_n & a_1 & a_2 & a_3 \\ s_n & a_4 & a_5 & a_6 \\ t_n & a_7 & a_8 & a_9 \\ 0 & \frac{1}{n} & n & a_{10} \end{bmatrix}.$$

Then $O_1 = \mathbb{R}^* \times \{(0, 0, 0)\}$ and $O_2 = \mathbb{R}^* \times \mathbb{R} \times \{(0, 0)\}$. Indeed, the assertions follow immediately from equalities

$$A_n \begin{bmatrix} x \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} r_n x \\ s_n x \\ t_n x \\ 0 \end{bmatrix}, \quad A_n \begin{bmatrix} x \\ y \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} r_n x + a_1 y \\ s_n x + a_4 y \\ t_n x + a_7 y \\ \frac{y}{n} \end{bmatrix}, \quad A_n \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix} = \begin{bmatrix} r_n x + a_1 y + a_2 z \\ s_n x + a_4 y + a_5 z \\ t_n x + a_7 y + a_8 z \\ \frac{y}{n} + n z \end{bmatrix},$$

where $x, y, z \in \mathbb{R}$. Notice that, for all the points $(x, y, z, 0) \in X_0$ with $z \in \mathbb{R}^*$, we always have $\lim_{n \rightarrow \infty} |T_n(x, y, z, 0)| = \infty$.

In the above example, if all the matrixs A_n are invertible, then by taking $g_n := A_{n+1} A_n^{-1}$ as the functions from $A_n X_0$ to $A_{n+1} X_0$, this GNDS is stated in the iterative way (see the explanation after Example 1).

Topological transitivity can be extended as below (see Definition 2.3 in [16]).

Definition 2. GNDS $f_{1,\infty}$ is called *topologically transitive* on X_0 if for any two nonempty open sets U_0 and V_0 of X_0 , there exists some $n \in \mathbb{N}$ such that $f_n(U_0) \cap V_0 \neq \emptyset$.

Now we are going to state the relation between density and topological transitivity.

Theorem 1. If $\overline{O_1} = X_0$, then GNDS $f_{1,\infty}$ is topologically transitive on X_0 .

Proof. Let U_0 and V_0 be non-empty open subsets of X_0 . Since O_1 is dense in X_0 , there exists $x \in U_0 \cap O_1$ such that $\text{orb}(x) \cap X_0$ is dense in X_0 . This implies the existence of $n \in \mathbb{N}$ such that $f_n(x) \in V_0$. Hence, $f_n(U_0) \cap V_0 \neq \emptyset$ which completes the proof. $\square$

The GLNDS appeared in Example 3 is hypercyclic but not topologically transitive. Indeed, Theorem 1 expresses that to have topological transitivity property, we need all hypercyclic points to be dense in X_0 . Unfortunately, topological transitivity does not imply even density of O_2 in general. The next theorem expresses that under suitable assumptions, if a GNDS is topologically transitive, then O_2 forms a large subspace from topological point of view.

Theorem 2. *Let X_0 be second countable and Baire subspace of X , and X be first countable at any point of X_0 . Also, suppose that all f_n s are continuous. If GNDS $f_{1,\infty}$ is topologically transitive on X_0 , then $\overline{O_2} = X_0$ and further that O_2 is a G_δ -set.*

Proof. Since X_0 is a second countable subspace and X is the first countable at any point of X_0 , it is easy to show the existence of a collection $\{U_m\}_{m \in \mathbb{N}}$ of open sets in X such that

- i) $U_m \cap X_0 \neq \emptyset$,
- ii) the family $\{U_m \cap X_0\}_{m \in \mathbb{N}}$ is a basis for X_0 ,
- iii) for each $x_0 \in X_0$, the family $\{U_m\}_{m \in \mathbb{N}}$ is a local basis for x_0 in X .

We claim that

$$O_2 = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} f_n^{-1}(U_m).$$

To prove the claim, let $O^* := \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} f_n^{-1}(U_m)$. First, we show that $O_2 \subseteq O^*$. Suppose otherwise, there is $x \in O_2$ such that $x \notin O^*$. So if $x \notin O^*$, there exists $m \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ we have $f_n(x) \notin U_m$. Hence, $\text{orb}(x) \cap U_m = \emptyset$. Since $U_m \cap X_0 \neq \emptyset$, there exists an element $z \in U_m \cap X_0$, such that $z \notin \overline{\text{orb}(x)}^X$. But $z \in X_0$ and so $\overline{\text{orb}(x)}^X \cap X_0 \neq X_0$. Therefore $x \notin O_2$ which contradicts the choice of x . Now, we show $O^* \subseteq O_2$. Let $x \in O^*$ but $x \notin O_2$. So as $x \in O^*$, for each $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $f_n(x) \in U_m$. Thus, $\text{orb}(x) \cap U_m \neq \emptyset$. Moreover, the relation $x \notin O_2$ indicates that there exists $z \in X_0$ such that $z \notin \overline{\text{orb}(x)}^X$. Consequently, there exists U_k containing z , such that $U_k \cap \text{orb}(x) = \emptyset$. This contradicts the fact that $\text{orb}(x) \cap U_m \neq \emptyset$, for each $m \in \mathbb{N}$, that proves our claim. By continuity of f_n , each set $\bigcup_{n=1}^{\infty} f_n^{-1}(U_m)$ is open and because of transitivity, these open sets are dense in X_0 . Thus, $O_2 = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} f_n^{-1}(U_m)$ is a dense G_δ -set in X_0 , since X_0 is a Baire space. This also tells us that $\overline{O_2} = X_0$ and the proof of Theorem is complete. $\square$

Theorems 1 and 2 enable us to state a generalization of Birkhoff transitivity theorem.

Theorem 3. *Suppose that X_0 is a second countable and Baire subspace of topological space X . Also, suppose that X is first countable at any point of X_0 and all the maps f_n are continuous. Then,*

- density of hypercyclic points implies topological transitivity of GNDS $f_{1,\infty}$ on X_0 ,
- topological transitivity of GNDS $f_{1,\infty}$ on X_0 implies density of a G_δ -set of all weakly hypercyclic points.

In particular, when $X_0 = X$ we can drop “weakly” from the statement of theorem.

3 Topological Conjugacy

The conjugacy is an important feature that worths to mention in the study of GNDSs. We begin by an extension of Definition 3.6 in [16] in the usual way.

Definition 3. Let X and Y be two topological spaces and $X_0 \subset X$, $Y_0 \subset Y$. Let $f_{1,\infty}$ and $g_{1,\infty}$ be two GNDSs on X_0 and Y_0 , respectively. If there exists a homeomorphism $h : X \rightarrow Y$ such that $h \circ f_n = g_n \circ h$ on X_0 , for each $n \in \mathbb{N}$, then $f_{1,\infty}$ is said to be topologically h -conjugate or simply conjugate to $g_{1,\infty}$.

The conjugacy establishes an equivalence relation on GNDSs. It is easy to show that orbits, invariant sets, weakly k -periodic points, k -periodic points and finite orbit points are preserved by the conjugated GNDSs. Next theorem asserts that hypercyclic points, weakly hypercyclic points and transitivity are preserved under conjugacy too.

Theorem 4. Let $h : X \rightarrow Y$ be a homeomorphism and GNDS $g_{1,\infty}$ be topologically h -conjugate to GNDS $f_{1,\infty}$. Then,

- i) if x_0 is a hypercyclic point (resp. weakly hypercyclic point) for $f_{1,\infty}$, then $h(x_0)$ is a hypercyclic point (resp. weakly hypercyclic point) for $g_{1,\infty}$,
- ii) if $f_{1,\infty}$ is topologically transitive on X_0 , then $g_{1,\infty}$ is topologically transitive on Y_0 .

Proof. (i) As $x_0 \in O_1$ (resp. $x_0 \in O_2$) in $f_{1,\infty}$, thus $\overline{\text{orb}(x)} \cap X_0^{X_0} = X_0$ (resp. $\overline{\text{orb}(x_0)}^X \cap X_0 = X_0$). Since h is a homeomorphism, we have $\overline{h(\text{orb}(x))} \cap Y_0^{Y_0} = Y_0$ (resp. $\overline{h(\text{orb}(x_0))}^Y \cap Y_0 = Y_0$). Thus, $h(x_0) \in O_1$ (resp. $h(x_0) \in O_2$) in $g_{1,\infty}$, since $h(\text{orb}(x_0)) = \text{orb}(h(x_0))$. This completes assertion (i).

(ii) Suppose that U_0 and V_0 are two nonempty open subsets of Y_0 . Since, GNDS $f_{1,\infty}$ is topologically transitive on X_0 , there exists a number $n \in \mathbb{N}$ such that $f_n(h^{-1}(U_0)) \cap h^{-1}(V_0) \neq \emptyset$. Since h is a homeomorphism and GNDS $g_{1,\infty}$ is h -conjugate to $f_{1,\infty}$, then

$$g_n(U_0) \cap V_0 = h \circ f_n \circ h^{-1}(U_0) \cap V_0 \neq \emptyset.$$

Thus, GNDS $g_{1,\infty}$ is topologically transitive on Y_0 that implies the proof for (ii). $\square$

Consequently, if GNDS $g_{1,\infty}$ is conjugate to GNDS $f_{1,\infty}$, then for $i = 1, 2$, $\overline{O_i}^{X_0} = X_0$, if and only if $\overline{O_i}^{Y_0} = Y_0$.

In the context of metric spaces, Tian and Chen proved that two uniformly topologically conjugate NDSs share the sensitive dependence on initial conditions (see Definition 5) too. But similar conclusion does not necessarily hold for two conjugate NDSs [16]. As we observed, homeomorphism h plays a key role in transferring the classical dynamical properties of GNDS $f_{1,\infty}$ to GNDS $\{g_n\}$. Since the maps emerged in GNDSs may be selected randomly, one do not expect such homeomorphism always exists, see Example 4. For the moment, let $X^* := \bigcap_{n=0}^{\infty} X_n$ and $Y^* := \bigcap_{n=0}^{\infty} Y_n$. In the lack of a conjugacy between GNDS $\{f_n\}$ and GNDS $\{g_n\}$, the existence of a conjugacy between GNDS $\{F_n\}$ and GNDS $\{G_n\}$ is possible, where $F_n := f_n|_{X^*}$ and $G_n := g_n|_{Y^*}$. In such cases, most of the dynamical properties are permanent for GNDSs $\{f_n\}$ and $\{g_n\}$, see Theorem 5. This leads us to another type of conjugacy as below, which completely fits into other Definitions. Two conjugated GNDSs in the sense of Definition 4 may share a given dynamical property under special additional assumptions. For the sake of simplicity, let $X_n := f_n(X_0)$ and $Y_n := g_n(Y_0)$.

Definition 4. Let X and Y be two topological spaces and $X_0 \subset X$, $Y_0 \subset Y$. Let $f_{1,\infty}$ and $g_{1,\infty}$ be two GNDSs on X_0 and Y_0 , respectively. If there exist homeomorphisms $h_0 : X_0 \rightarrow Y_0$ and $h_n : X_n \rightarrow Y_n$ such that $h_n \circ f_n = g_n \circ h_0$, for each $n \in \mathbb{N}$. Then GNDS $f_{1,\infty}$ is said to be topologically $\{h_n\}$ -conjugate or simply $\{h_n\}$ -conjugate to GNDS $g_{1,\infty}$.

Obviously, the $\{h_n\}$ -conjugacy establishes an equivalent relation on GNDSs. Also, if GNDS $\{f_n\}$ is topologically h -conjugate to GNDS $\{g_n\}$, then by taking $h_0 := h|_{X_0}$ and $h_n := h|_{X_n}$, they are $\{h_n\}$ -conjugate. The condition $h_n \circ f_n = g_n \circ h_0$ indicates that if $\{x_n\}_{n=0}^{\infty}$ is an orbit of the $\{f_n\}$, then $\{h_n(x_n)\}_{n=0}^{\infty}$ is an orbit of GNDS $\{g_n\}$ that starts from $h_0(x_0)$, since

$$h_n(x_n) = h_n(f_n(x_0)) = g_n(h_0(x_0)).$$

Indeed, in the case that all maps f_n and g_n are bijective, $\{h_n\}$ -conjugacy is equivalent to $\{x_n\}_{n=0}^{\infty}$ is an orbit of GNDS $\{f_n\}$ if and only if, $\{h_n(x_n)\}_{n=0}^{\infty}$ is an orbit of GNDS $\{g_n\}$ that starts from $h_0(x_0)$. This fact, for instance,

says that Definition 4 fits into the situation that any two GNDSs are conjugated by the conjugacy notion in [10]. Example 5.1.1 in the book expresses that every difference equation generating a 2-parameter group is conjugated to the trivial identical equation. Precisely, if $(\hat{\varphi}_k)$ and $(\hat{\psi}_k)$ are generators related with difference equation and identical equation, respectively, then they are conjugated via the homeomorphism T_k in the sense of Definition 4. For GNDSs stated in the iterative way, $\{h_n\}$ -conjugacy is defined in the following.

Assume that X and Y are two metric spaces and for each $n \in \mathbb{N} \cup \{0\}$, X_n and Y_n are subspaces of X and Y , respectively. Assume that $g_n : X_n \rightarrow X_{n+1}$ and $\tilde{g}_n : Y_n \rightarrow Y_{n+1}$ are arbitrary maps. Also, assume that homeomorphisms $h_n : X_n \rightarrow Y_n$ satisfying $h_{n+1} \circ g_n = \tilde{g}_n \circ h_n$, for each n . Then NDS $\{g_n\}_{n=0}^\infty$ is said to be *topologically $\{h_n\}$ -conjugate* to NDS $\{\tilde{g}_n\}_{n=0}^\infty$, see Definition 3.1 in [15]. As we explained in Section 1, by taking $f_n := g_{n-1} \circ g_{n-2} \cdots \circ g_0$ and $\tilde{f}_n := \tilde{g}_{n-1} \circ \tilde{g}_{n-2} \cdots \circ \tilde{g}_0$, $f_{1,\infty}$ and $\tilde{f}_{1,\infty}$ are two GNDSs on X_0 and Y_0 , respectively that are stated in the successive way.

Proposition 2. *With above notation, $h_{n+1} \circ g_n = \tilde{g}_n \circ h_n$ for each $n \in \mathbb{N} \cup \{0\}$, if and only if $h_n \circ f_n = \tilde{f}_n \circ h_0$ for each $n \in \mathbb{N}$.*

Proof. Firstly, assume that $h_{n+1} \circ g_n = \tilde{g}_n \circ h_n$ for each $n \in \mathbb{N} \cup \{0\}$. Thus,

$$\begin{aligned} h_n \circ f_n &= h_n \circ g_{n-1} \circ g_{n-2} \circ \cdots \circ g_0 \\ &= \tilde{g}_{n-1} \circ h_{n-1} \circ g_{n-2} \circ \cdots \circ g_0 = \cdots \\ &= \tilde{g}_{n-1} \circ \tilde{g}_{n-2} \circ \cdots \circ \tilde{g}_0 \circ h_0 = \tilde{f}_n \circ h_0. \end{aligned}$$

To prove the inverse, let $x \in X_n$. Then there is $x_0 \in X_0$ such that $f_n(x_0) = x$. Thus,

$$\begin{aligned} h_{n+1} \circ g_n(x) &= h_{n+1} \circ f_{n+1}(x_0) \\ &= \tilde{f}_{n+1} \circ h_0(x_0) \\ &= \tilde{g}_n \circ \tilde{f}_n \circ h_0(x_0) \\ &= \tilde{g}_n \circ h_n \circ f_n(x_0) = \tilde{g}_n \circ h_n(x). \end{aligned}$$

This completes the proof of the proposition. $\square$

Thus, Definition 4 is an extension of Definition 3.1 in [15] (as mentioned above). Roughly speaking, if $\{g_n\}_{n=0}^\infty$ and $\{\tilde{g}_n\}_{n=0}^\infty$ are two NDSs on X_0 and Y_0 corresponding to $f_{1,\infty}$ and $\tilde{f}_{1,\infty}$, respectively, then by proposition 2, $\{g_n\}_{n=0}^\infty$ is topologically $\{h_n\}$ -conjugate to $\{\tilde{g}_n\}_{n=0}^\infty$ if and only if, $f_{1,\infty}$ is topologically $\{h_n\}$ -conjugate to $\tilde{f}_{1,\infty}$. The following theorem expresses that if homeomorphisms h_0 and h_n agree on the intersection of their domains, then h_0 transfers elementary properties such as invariant sets, weakly k -periodic points, the set O_1 and transitivity.

Theorem 5. *Let GNDS $f_{1,\infty}$ be $\{h_n\}$ -conjugate to GNDSs $g_{1,\infty}$ and $h_n = h_0$ on $X_0 \cap X_n$. Then,*

- i) *if $A \subset X_0$ be an invariant subset for $\{f_n\}$, then $h_0(A) \subset Y_0$ is an invariant subset for $\{g_n\}$,*
- ii) *if x_0 is a weakly k -periodic point for $\{f_n\}$, then $h(x_0)$ is a weakly k -periodic point for $\{g_n\}$,*
- iii) *if x_0 is a hypercyclic point for $\{f_n\}$, then $h(x_0)$ is a hypercyclic point for $\{g_n\}$,*
- iv) *if $f_{1,\infty}$ is topologically transitive on X_0 , then $g_{1,\infty}$ is topologically transitive on Y_0 .*

Proof. (i) Because, $A \subset X_0$ is an invariant subset for $\{f_n\}$. Thus,

$$g_n(h_0(A)) = h_n(f_n(A)) \subset h_n(A) = h_0(A),$$

for any $n \in \mathbb{N}$. This means that $h_0(A) \subset Y_0$ is an invariant subset for $\{g_n\}$.

(ii) Because, x_0 is a weakly k -periodic point for $\{f_n\}$, then $f_{mk}(x_0) = x_0$, for every $m \in \mathbb{N}$. Thus,

$$g_{mk}(h_0(x_0)) = h_{mk}(f_{mk}(x_0)) = h_{mk}(x_0) = h_0(x_0),$$

for every $m \in \mathbb{N}$. This means $h_0(x_0)$ is a weakly k -periodic point for $\{g_n\}$.

(iii) Let $y_0 := h_0(x_0)$ and $U \in \tau_{Y_0}$. Then $h_0^{-1}(U) \in \tau_{X_0}$ and so

$$\text{orb}(x_0) \cap X_0 \cap h_0^{-1}(U) \neq \emptyset,$$

since $x_0 \in O_1$ in $\{f_n\}$. This guarantees the existence of $x_n \in \text{orb}(x_0)$ such that $x_n \in X_0 \cap h_0^{-1}(U)$. Due to the fact $x_n \in X_0$ and by the assumption of the theorem, we obtain $h_n(x_n) = h_0(x_n) \in Y_0 \cap U$. On the other hand, the sequence $\{y_n\}_{n=0}^\infty$ with $y_n = h_n(x_n)$ forms the same orbit of $g_{1,\infty}$ which starts from $y_0 = h_0(x_0)$. Hence, $\text{orb}(y_0) \cap Y_0 \cap U \neq \emptyset$ since $y_n \in Y_0 \cap U$. This implies that $y_0 \in O_1$ in $g_{1,\infty}$ and completes the assertion.

(iv) Suppose that U_0 and V_0 are two nonempty open subsets of Y_0 . Since, $f_{1,\infty}$ is topologically transitive on X_0 , there exists some $n \in \mathbb{N}$ such that $\emptyset \neq f_n(h_0^{-1}(U_0)) \cap h_0^{-1}(V_0) \in X_n \cap X_0$. Since, h_0 is a homeomorphism, by the assumption of the theorem, we have

$$g_n(U_0) \cap V_0 = h_n \circ f_n \circ h_0^{-1}(U_0) \cap V_0 \neq \emptyset.$$

Thus, $g_{1,\infty}$ is topologically transitive on Y_0 that implies the proof for (ii). $\square$

By the hypothesis of Theorem 5, if invariant subset A is dense in X_0 , $h_0(A)$ is an invariant subset and dense in Y_0 . If $WP(f_n)$ (resp. O_1 in GNDS $f_{1,\infty}$) is dense in X_0 , then $WP(g_n)$ (resp. O_1 in GNDS $g_{1,\infty}$) is dense in Y_0 . The following example says that periodic points, finite orbit points and O_2 may not remain preserved under a $\{h_n\}$ -conjugacy, even for the linear GNDSs stated in the iterative way with the assumption that $h_n = h_0$ on $X_0 \cap X_n$ and $h_n^{-1} = h_0^{-1}$ on $Y_0 \cap Y_n$.

Example 4. Let $\{\alpha_n\}_{n=0}^\infty$ be a dense sequences of non zero real numbers. Let $X = Y = \mathbb{R}^2$ and $X_0 = Y_0 = \mathbb{R} \times \{0\}$. For each $n \in \mathbb{N}$, consider the linear maps

$$f_n(x, 0) = (x, \frac{1 - (-1)^n}{2}x), \quad g_n(x, 0) = \begin{cases} (\alpha_{\frac{n-1}{2}}x, \frac{x}{n}) & n \text{ is odd} \\ (x, 0) & n \text{ is even} \end{cases}.$$

There is no homeomorphism h such that the linear GNDS $\{f_n\}$ is h -conjugate to the linear GNDS $\{g_n\}$. Likewise, $h \circ f_n(x, 0) = g_n \circ h(x, 0)$, for all $x \in \mathbb{R}$, so sending n to ∞ makes a contradiction. While GNDS $\{f_{2n}\}$ is $\{h_n\}$ -conjugate to GNDS $\{g_n\}$ with $h_0 = \text{ID}$ and $h_n(x, y) = g_n(x, 0)$. Note that all maps are one to one and this states that one can write the underlying GNDSs in the iterative way. A simple check shows $P(f_n) = \text{Fin}(f_n) = \mathbb{R} \times \{0\}$ and $O_2 = \emptyset$ for GNDS $\{f_n\}$.

Also, for GNDS $\{g_n\}$, $P(g_n) = \text{Fin}(g_n) = \{(0, 0)\}$ and $O_2 = \mathbb{R} \times \{0\} - \{(0, 0)\}$. These facts say that periodic points, finite orbit points and the set O_2 do not remain preserved under given $\{h_n\}$ -conjugacy.

Nonetheless, under additional condition on h_n s we get below result.

Theorem 6. Let GNDS $f_{1,\infty}$ be $\{h_n\}$ -conjugate to GNDS $g_{1,\infty}$ and $h_n = h_m$ on $X_n \cap X_m$, for all $m, n \in \mathbb{N} \cup \{0\}$. Then,

- i) if x_0 is a k -periodic point for $\{f_n\}$, then $h(x_0)$ is a k -periodic point $\{g_n\}$,
- ii) if x_0 is a finite orbit point for $\{f_n\}$, then $h(x_0)$ is a finite orbit point for $\{g_n\}$.

Proof. (i) Because, x_0 is a k -periodic point for $\{f_n\}$, then $f_{mk+n}(x_0) = f_n(x_0)$, for every $m, n \in \mathbb{N} \cup \{0\}$. Thus,

$$g_{mk+n}(h_0(x_0)) = h_{mk+n}(f_{mk+n}(x_0)) = h_n(f_n(x_0)) = g_n(h_0(x_0)),$$

This means that $h_0(x_0)$ is a k -periodic point for $\{g_n\}$.

(ii) Because, x_0 is a finite orbit point, then the set $\text{orb}(x_0)$ is finite. Suppose that for indexes i and j , we have $x_i = x_j$. Therefore, $h_i(x_i) = h_j(x_j)$. This asserts that the set $\{h_n(x_n) \mid n \in \mathbb{N} \cup \{0\}\}$ is finite. This completes the proof. $\square$

By the hypothesis of Theorem 6, if $P(f_n)$ (resp. finite orbit points in GNDS $f_{1,\infty}$) is dense in X_0 , then $P(g_n)$ (resp. finite orbit points in GNDS $g_{1,\infty}$) is dense in Y_0 .

At the end of this section, let (X, d) and (Y, d') be two metric spaces and the subspace $X_0 \subset X$ and $Y_0 \subset Y$ without isolated points. At first we generalize the definition of sensitivity in [16].

Definition 5. Suppose that $f_{1,\infty}$ is a GNDS on X_0 . Also, suppose that there exists a constant δ such that for each $x_0 \in X_0$ and each open set U_0 of x_0 in X_0 , there exists $y_0 \in U_0$ and a positive integer n such that $d(x_n, y_n) > \delta$. Then $f_{1,\infty}$ is said to have sensitive dependence on initial conditions in X_0 .

In [18] it is shown that under certain conditions, transitivity and denseness of periodic points imply sensitivity. The sequence of maps $\{h_n\}_{n=1}^\infty$ is said to be *uniformly equicontinuous* in the set sequence $\{X_n\}_{n=1}^\infty$, if for any $\epsilon > 0$ there exists a $\delta > 0$ such that for all $n \in \mathbb{N}$ and for all $x, y \in X_n$ with $d(x, y) < \delta$, we have $d'(h_n(x), h_n(y)) < \epsilon$ [15].

Proposition 3. Assume that GNDS $f_{1,\infty}$ is $\{h_n\}$ -conjugate to GNDS $g_{1,\infty}$ and $\{h_n\}_{n=1}^\infty$ is uniformly equicontinuous in set $\{X_n\}_{n=1}^\infty$. If $g_{1,\infty}$ has sensitive dependence on initial conditions in Y_0 , then the same is true for $f_{1,\infty}$ in X_0 ,

Proof. Suppose that $g_{1,\infty}$ has sensitive dependence on initial conditions in Y_0 with a sensitivity constant $\tilde{\delta}$. Since, $\{h_n\}_{n=1}^\infty$ is uniformly equicontinuous in $\{X_n\}_{n=1}^\infty$, there exists a constant $\delta > 0$ such that for all $n \in \mathbb{N}$ and for all $x, y \in X_n$ with $d(x, y) < \delta$, we have $d'(h_n(x), h_n(y)) < \tilde{\delta}$. Let x_0 be any point of X_0 and U_0 be any open neighbourhood of x_0 . Take $y_0 := h_0(x_0)$ and $\tilde{U}_0 := h_0(U_0)$. Then $\tilde{U}_0$ is an open neighbourhood of y_0 since h_0 is homeomorphism. By assumption, there exist $\tilde{y}_0 \in \tilde{U}_0$ and a natural number n such that $d'(g_n(\tilde{y}_0), g_n(y_0)) \geq \tilde{\delta}$. Consequently, by letting $\tilde{x}_0 := h_0^{-1}(\tilde{y}_0)$, we have $d'(h_n(\tilde{x}_n), h_n(x_n)) \geq \tilde{\delta}$. This implies that $d(\tilde{x}_n, x_n) \geq \delta$ and implies the proof of the assertion. $\square$

Definition 5 leads us to extend Devaney's and Wiggins's chaos as follows:

Definition 6. GNDS $f_{1,\infty}$ is said to be chaotic in the sense of Devaney on X_0 if

1. the set of periodic points of $f_{1,\infty}$ is dense in X_0 ,
 2. $f_{1,\infty}$ is topologically transitive on X_0 ,
 3. $f_{1,\infty}$ has sensitive dependence on initial condition in X_0 .
- Also, $f_{1,\infty}$ said to be chaotic in the sense of Wiggins on X_0 if it satisfies the conditions (2) and (3).

Now assume that homeomorphisms $\{h_n\}_{n=1}^\infty$ and $\{h_n^{-1}\}_{n=1}^\infty$ are uniformly equicontinuous in the set $\{X_n\}_{n=1}^\infty$ and $\{Y_n\}_{n=1}^\infty$, respectively. Moreover, assume that $h_m = h_n$ on $X_n \cap X_m$ and $h_m^{-1} = h_n^{-1}$ on $Y_n \cap Y_m$ for all $m, n \in \mathbb{N} \cup \{0\}$. If GNDSs $g_{1,\infty}$ and $\{f_n\}$ are $\{h_n\}$ -conjugate, then $f_{1,\infty}$ is chaotic in the sense of Devaney (resp. Wiggins) on X_0 , if and only if $g_{1,\infty}$ is on Y_0 .

We say that the sequence of maps $\{h_n\}_{n=1}^\infty$ is *uniformly h_0 -equicontinuous* in the set sequence $\{X_n\}_{n=1}^\infty$, if for any $\epsilon > 0$ there exists a $\delta > 0$ such that for all $n \in \mathbb{N}$ and for all $x \in X_n$ and $y \in X_0$ with $d(x, y) < \delta$, we have $d'(h_n(x), h_0(y)) < \epsilon$.

Proposition 4. Assume that GNDS $\{f_n\}$ is $\{h_n\}$ -conjugate to GNDS $\{g_n\}$ and $\{h_n\}_{n=1}^\infty$ is uniformly h_0 -equicontinuous in the set $\{X_n\}_{n=1}^\infty$. Then if x_0 is a weakly hypercyclic point for $\{f_n\}$, then $h(x_0)$ is a weakly hypercyclic point for $\{g_n\}$.

Proof. Assume that $y_0 \in Y_0$ and $\epsilon > 0$ are selected arbitrarily. Since $\{h_n\}_{n=1}^\infty$ is uniformly h_0 -equicontinuous in the set sequence $\{X_n\}_{n=1}^\infty$, there exists a $\delta > 0$ such that for all $n \in \mathbb{N}$ and for all $x \in X_n$ and $y \in X_0$ with $d(x, y) < \delta$, we have $d'(h_n(x), h_0(y)) < \epsilon$. By assumption, there exists $n \in \mathbb{N}$ such that $x_n \in \mathbb{B}_d(h_0^{-1}(y_0), \delta)$. Consequently, $d'(h_n(x_n), y_0) < \epsilon$. The $\{h_n\}$ -conjugacy property gives that $g_n(h_0(x_0)) \in \mathbb{B}_{d'}(y_0, \epsilon)$ and this implies that $h_0(x_0) \in O_2$ in GNDS $\{g_n\}$. $\square$

Consequently, if GNDS $g_{1,\infty}$ is $\{h_n\}$ -conjugate to GNDS $\{f_n\}$, in addition $\{h_n\}_{n=1}^\infty$ and $\{h_n^{-1}\}_{n=1}^\infty$ are uniformly h_0 -equicontinuous in the set $\{X_n\}_{n=1}^\infty$ and $\{Y_n\}_{n=1}^\infty$, respectively. Then, $\overline{O_2^{X_0}} = X_0$, if and only if $\overline{O_2^{Y_0}} = Y_0$, since h_0 is a homeomorphism.

It is natural to look for type of $\{h_n\}$ -conjugacy to keep most of the usual dynamical properties unchanged. Above discussion leads us to pose the following definition.

Definition 7. Let X and Y be two metric spaces and $X_0 \subset X$, $Y_0 \subset Y$. Let $f_{1,\infty}$ and $g_{1,\infty}$ be two GNDSs on X_0 and Y_0 , respectively. If there exist homeomorphisms $h_0 : X_0 \rightarrow Y_0$ and $h_n : X_n \rightarrow Y_n$ such that

- i) $h_n \circ f_n = g_n \circ h_0$, for each $n \in \mathbb{N}$,
- ii) $\{h_n\}_{n=1}^\infty$ and $\{h_n^{-1}\}_{n=1}^\infty$ are uniformly equicontinuous and uniformly h_0 -equicontinuous in set $\{X_n\}_{n=1}^\infty$ and $\{Y_n\}_{n=1}^\infty$, respectively,
- iii) $h_m = h_n$ on $X_n \cap X_m$ and $h_m^{-1} = h_n^{-1}$ on $Y_n \cap Y_m$ for all $m, n \in \mathbb{N} \cup \{0\}$.

Then GNDS $f_{1,\infty}$ is said to be strongly topologically $\{h_n\}$ -conjugate to GNDS $g_{1,\infty}$.

Strongly $\{h_n\}$ -conjugacy establishes an equivalent relation on GNDSs that shares invariant sets, weakly k -periodic points, periodic points, finite orbit points, transitivity, sensitive dependence on initial conditions in X_0 and the sets O_i , $i = 1, 2$.

4 Generalized linear nonautonomous discrete systems

In this section we discuss the case that underlying space X is a Frechet space, i.e. a vector space endowed with a separating increasing sequence $(p_n)_{n \in \mathbb{N}}$ of seminorms which is complete with the following metric:

$$d(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \min(1, p_n(x - y)), \quad x, y \in X.$$

An important feature of this metric is that it is translation-invariant, i.e. $d(x, y) = d(x+z, y+z)$ for all $x, y, z \in X$ [5].

Definition 8. Let X be a Frechet space and X_0 be a Frechet subspace of X . A sequence $\{T_n\}_{n=1}^\infty$ of linear maps $T_n : X_0 \rightarrow X$ is called a generalized linear nonautonomous discrete system (in short GLNDS) on X_0 .

At first we show that in case of the GLNDSs, sensitive dependence on initial conditions follows from the topological transitivity. Hence, one may omit sensitivity assumption (third condition) from Devaney's and Wiggins's definitions of chaos in the category of the GLNDSs.

Theorem 7. Suppose that the GLNDS $T_{1,\infty}$ is topologically transitive on X_0 . Then it has sensitive dependence on initial conditions.

Proof. Let $\delta = 1$ and choose $x \in X_0$, $\epsilon > 0$. Also, let nonempty open sets $U_0 := \mathbb{B}_d(0, \epsilon)$ and $V_0 := \{z \in X_0 : d(z, 0) > 1\}$. Since $T_{1,\infty}$ is topologically transitive, there exists some $n \in \mathbb{N}$ such that

$$T_n(U_0) \cap V_0 \neq \emptyset.$$

Thus, there exists $z \in U_0$ such that $T_n(z) \in V_0$. Consider $y := x+z$. By using the translation-invariant property of the metric, we obtain

$$d(x, y) = d(y - x, 0) = d(z, 0) < \epsilon.$$

This means that $y \in \mathbb{B}_d(x, \epsilon)$. On the other hand,

$$d(T_n(x), T_n(y)) = d(T_n(y - x), 0) = d(T_n(z), 0) > 1,$$

which implies that GLNDS has sensitive dependence on initial conditions with sensitivity constant $\delta = 1$. $\square$

The next theorem states the relation between hypercyclicity and sensitivity.

Theorem 8. *Suppose $T_{1,\infty}$ is hypercyclic GLNDS. Then it has sensitive dependence on initial conditions.*

Proof. Let $\delta = 1$, and choose z_0 be a point that $\text{orb}(z_0) \cap X_0$ is dense in X_0 . Let $x_0 \in X_0$ and U_0 be an open set in X_0 containing x_0 . Then there exists a non zero number α small enough such that $\alpha z_0 + x_0$ belongs to U_0 . Now let $y_0 := \alpha z_0 + x_0$. Hence $d(x_n, y_n) = d(\alpha z_n, 0)$ for all $n \in \mathbb{N}$. We can select the natural number n large enough, such that $d(\alpha z_n, 0) > \delta$. Thus, $d(x_n, y_n) > \delta$, that implies it has sensitive dependence on initial conditions. $\square$

By a small change in the proof of Theorem 8, it can be proved that the existence of a weakly hypercyclic point always implies the sensitivity.

Now we bring two examples of GLNDSs stated in the iterative way to show that the inverse of Theorems 1 and 2 is not always true. Next example is topologically transitive while $\overline{O_1} \neq X_0$. Furthermore, $O_1 = \emptyset$ that means it is not hypercyclic.

Example 5. Let $X = \mathbb{R}^3$ and $X_0 = \mathbb{R}^2 \times \{0\} \subseteq \mathbb{R}^3$. Let $0 < \alpha < \frac{\pi}{4}$ be an irrational number, and $\{\alpha_n\}_{n=0}^\infty$ be a dense sequence in $\mathbb{R}^*$ with $\alpha_0 = 1$. Let $\{r_n\}_{n=0}^\infty$ be the dense sequence of rational numbers $r_l = \frac{m}{n}$, $(m, n) = 1$ in the interval $[0, 1)$ which are arranged in ascending order of denominators, i.e. $\{r_n\}_{n=0}^\infty = \{0, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \dots\}$. For given $n \in \mathbb{N}$, there is exactly one $k \in \mathbb{N} \cup \{0\}$ such that $2^k \leq n \leq 2^{k+1} - 1$. Hence, take X_n be the plane that includes the half plane

$$\tilde{X}_n = \{(\rho, \theta, \psi)_s \mid \theta = r_k \pi, 0 < \psi < \pi, \rho > 0\},$$

in which $(\cdot, \cdot, \cdot)_s$ denotes spherical coordinates for $\mathbb{R}^3$. Now, $\{T_n\}_{n=0}^\infty$ is defined as follows: consider the operator $T_0 : X_0 \rightarrow X_1$ defined by $(x, y, 0) \mapsto (x, 0, y)$ and let $T_n : X_n \rightarrow X_{n+1}$ to be defined by

$$T_n(x, y, z) = \begin{cases} \frac{\alpha_{n-2^{k+1}}}{\alpha_{n-2^k}}(x, y, z) & 2^k \leq n < 2^{k+1} - 2 \\ (x, y, z) & n = 2^k \\ \frac{1}{\alpha_{2^k-2}}(x, y, z) & n = 2^{k+1} - 2 \\ R_2^k \circ R_1^k(x, y, z) & n = 2^{k+1} - 1 \end{cases}$$

where $R_1^k : X_{2^{k+1}-1} \rightarrow X_{2^k}$ is the linear transformation that transfers the point $(\rho, r_k \pi, \psi)_s \in \tilde{X}_{2^{k+1}-1}$ to $(\rho, r_{k+1} \pi, \psi)_s \in \tilde{X}_{2^{k+1}}$. And also, $R_2^k : X_{2^{k+1}} \rightarrow X_{2^k}$ is the linear transformation that rotates the points counter-clockwise by the angle α . In other words, $R_2^k : \tilde{X}_{2^{k+1}} \rightarrow X_{2^k}$ is given by $(\rho, \theta, \psi)_s \mapsto (\rho, \theta, \psi - \alpha)_s$. Firstly, we

show that $\{T_n\}_{n=0}^\infty$ is topologically transitive. It is easy to check that $\{T_n\}_{n=0}^\infty$ satisfies the following equalities:

- (i) $T_{2^{k+1}-2} \circ T_{2^{k+1}-3} \circ \dots \circ T_{2^k} = \text{id}_{X_{2^k}} \quad \forall k \in \mathbb{N},$
- (ii) $T_{2^{k+m}-1} \circ T_{2^{k+m}-2} \circ \dots \circ T_{2^k} = \alpha_m \text{id}_{X_{2^k}} \quad \forall 1 \leq m < 2^k - 1.$

From relation (i), we obtain:

$$T_{2^{k-1}} \circ T_{2^k-2} \circ \dots \circ T_2 \circ T_1 \circ T_0 = T_{2^{k-1}} \circ T_{2^{k-1}-1} \circ \dots \circ T_3 \circ T_1 \circ T_0. \quad (4.1)$$

Let U_0 and V_0 be two open sets in X_0 which are bounded regions between two sectors;

$$\{(x, y, 0) \mid m_1 x < y < m_2 x, r_1 < x^2 + y^2 < r_2\}.$$

Let R_ψ be a ψ -valued counter-clockwise rotation on the points X_0 . Thus, there exist $p \in U_0$, $m_0 \in \mathbb{N}$ and $I_p \subset (0, 1)$ such that $R_{2\pi\theta}(\alpha_{m_0} \cdot p) \in V_0$ for all $\theta \in I_p$.

We identify $[0, 1] \simeq S^1 \times \{0\}$ by $\theta \mapsto (\cos 2\pi\theta, \sin 2\pi\theta, 0)$. Then there exists $k_0 \in \mathbb{N}$ such that for each $l \in \mathbb{N}$ we have

$$S^1 \subset \bigcup_{i=l}^{l+k_0} R_{i\alpha}(I_p). \quad (4.2)$$

Choose l_0 such that $m_0 < 2^{l_0}$ and

$$\{r_{2^{l_0}}, r_{2^{l_0}+1}, \dots, r_{2^{l_0}+k_0}\} \subset I_p.$$

From relation (4.2), there exists $0 \leq j_0 \leq k_0$ such that

$$T_{2^{l_0}+j_0-1} \circ T_{2^{l_0}+j_0-1-1} \circ \dots \circ T_3 \circ T_1 \circ T_0(I_p) \cap \frac{V_0}{|V_0|} \neq \emptyset.$$

Here, $\frac{V_0}{|V_0|} := \left\{ \frac{x}{|x|} \mid x \in V_0 \right\}$. By using relations (4.1), we obtain

$$\begin{aligned} & T_{2^{l_0}+j_0+m_0-1} \circ T_{2^{l_0}+j_0+m_0-2} \circ \dots \circ T_2 \circ T_1 \circ T_0(U_0) \cap V_0 \\ &= T_{2^{l_0}+j_0+m_0-1} \circ \dots \circ T_{2^{l_0}+j_0} \circ T_{2^{l_0}+j_0-1} \circ T_{2^{l_0}+j_0-1-1} \circ \dots \circ T_3 \circ T_1 \circ T_0(U_0) \cap V_0 \\ &= T_{2^{l_0}+j_0-1} \circ T_{2^{l_0}+j_0-1-1} \circ \dots \circ T_3 \circ T_1 \circ T_0(\alpha_{m_0} U_0) \cap V_0 \\ &\neq \emptyset, \end{aligned}$$

since $R_{2\pi\theta}(\alpha_{m_0} U_0) \cap V_0 \neq \emptyset$, for all $\theta \in I_p$ and $T_{2^{l_0}+j_0-1} \circ T_{2^{l_0}+j_0-1-1} \circ \dots \circ T_3 \circ T_1 \circ T_0(\alpha_{m_0} U_0)$ intersects $R_{2\pi\theta}(\alpha_{m_0} U_0)$, vertically. We emphasize that our selection of l_0 ensures that the plane $X_{2^{l_0}+j_0}$ intersects the set V_0 and $2^{l_0}+j_0 \leq 2^{l_0}+j_0 + m_0 < 2^{l_0}+j_0+1$. Therefore, $\{T_n\}_{n=0}^\infty$ is topologically transitive. On the other hand, we obtain $O_1 = \emptyset$. To see this, assume that $q \in \mathbb{R}^2 \times \{0\}$ and for any $n \in \mathbb{N}$

$$T_n \circ T_{n-1} \circ \dots \circ T_0(q) \in (\mathbb{R}^2 \times \{0\}) \cap X_{n+1}.$$

Thus when the orbit of q leaves the plane X_{n+1} , it can not again intersect $\mathbb{R}^2 \times \{0\}$ since α is an irrational number. This tells us that $\text{orb}(q) \cap (\mathbb{R}^2 \times \{0\})$ is a finite set with cardinality smaller than $2^k + 1$.

Note that, the GNDS appeared in Example 5 is chaotic in the sense of Wiggins but not in the sense of Devaney. The GLNDS appeared in the next example satisfies $\overline{O_2} = X_0$, while it is not topologically transitive, see [12].

Example 6. Let $X = \mathbb{R}^2$, $X_0 = \mathbb{R} \times \{0\}$ and $X_n = \{(x, \frac{x}{n}) : x \in \mathbb{R}\}$ for each $n \in \mathbb{N}$. Consider $T_0 : X_0 \rightarrow X_1$ given by $(x, 0) \mapsto (x, x)$, and $T_n : X_n \rightarrow X_{n+1}$ given by $(x, y) \mapsto \frac{r_{n+1}}{r_n}(x, \frac{ny}{n+1})$ where $\{r_n\}_{n=1}^\infty$ is a dense sequence of rational non zero numbers. It is not hard to show that $\{T_n\}_{n=0}^\infty$ is not topologically transitive, since $X_n \cap X_0 = \{(0, 0)\}$ for all $n \in \mathbb{N}$. Moreover, $\overline{O_2} = X_0$ since

$$T_n \circ T_{n-1} \circ \dots \circ T_0(x, 0) = \frac{r_{n+1}}{r_1}(x, \frac{1}{n+1}x),$$

for all $n \in \mathbb{N}$. Note that, $O_2 = X_0 - \{(0, 0)\}$ and so is a G_δ -set.

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