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Reconstructing The Moore-Gibson-Thompson Equation

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Abstract: We are concerned with the inverse problem of recovering a third order Moore-Gibson-Thompson equation from a single observation of its solution at an arbitrary point. We show how to reconstruct its three unknown parameters and the memory kernel by using the Laplace transform.

Keywords: Moore-Gibson-Thompson equation, Inverse problem, Memory, Third order PDE

MSC: 35R30, 35Q70, 35L05, 35L35

1 Introduction

The objective of this note is to reconstruct the positive constants α, β, γ and the relaxation function g of the following Moore-Gibson-Thompson (MGT) equation

$$\begin{cases} u_{ttt} + \alpha u_{tt} - \beta \Delta u_t - \gamma \Delta u + \int_0^t g(t-s) \Delta u(s) ds = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ \partial_\nu u(x, t) = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad u_{tt}(x, 0) = u_2(x) & \text{in } \Omega, \end{cases} \quad (1)$$

from one observation of the solution at a single point, i.e. $u(b, t)$ for $t > 0$. Here $\Omega \subset \mathbb{R}^d (d \geq 1)$ is a bounded domain with a smooth boundary, say C^2 . The model in (1) describes the nonlinear propagation of acoustic waves (such as high frequency ultrasound waves) in a viscoelastic material, whose behavior and effect are governed by the convolution "memory" term $\int_0^t g(t-s) \Delta u(s) ds$. We refer to [6] for the derivation of the MGT equation and the list of references therein. These equations have been the focus of several recent research due to their wide range of applications in the medical and industrial fields of high intensity ultrasound waves.

The direct problem of (1) has received a lot attention. First Lasiecka and Wang established its well-posedness and showed exponential decay of the energy [6] and more general decay results followed, for example [7] for the subcritical case (i.e. $\alpha - \frac{\gamma}{\beta} > 0$). These have been made more precise and optimal decay rates was obtained by Liu et al. [11]. For the critical case ($\alpha - \frac{\gamma}{\beta} = 0$) Dell'Oro et al. [4] proved a polynomial decay rate under more regular initial data.

The inverse problem for (1) with $g = 0$ but α being a function of x was treated by Liu and Triggiani [9, 10], who proved uniqueness of the coefficient $\alpha(x) - \frac{\gamma}{\beta}$ and also showed stability by Carleman estimates. However no reconstruction procedure were given. We recall that the well-posedness in this case was proven

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by Kaltenbacher et al. [5] and Marchand et al. [12].

Our aim in this note is to give an explicit reconstruction procedure of the coefficients α, β, γ and the kernel function g found in (1) from a single observation of the solution at one point. To achieve this, we use the idea developed in [1], taking into consideration the nature of MGT equation under the subcritical case, and establish a simple computational method to reconstruct the sought parameters. Vespri et al. [3] considered a parabolic functional equation, where they showed that two measurements are enough to yield uniqueness.

2 Preliminaries

We now state sufficient assumptions that ensure global existence of the solution on (1).

(A1) $\beta\alpha - \gamma > 0$.

(A2) $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a C^1 decreasing function satisfying

$$0 < g(0) < \alpha(\beta\alpha - \gamma), \quad \gamma - \int_0^\infty g(s)ds = \ell > 0. \quad (2)$$

Proposition 1 ([6]). Assume that $(u_0, u_1, u_2) \in H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)$ and (A1) and (A2) are satisfied; then the problem (1) has a unique global (weak) solution

$$u \in C^1(\mathbb{R}^+; H^1(\Omega)) \cap C^2(\mathbb{R}^+; L^2(\Omega)).$$

The fact that we have global existence of the solution, means that we can observe it over $(0, \infty)$, and so the use of the Laplace transform is possible to reconstruct the function g over $(0, \infty)$. Observe that we only need to use the solution at one point only over the half line, i.e. $u(b, t)$ which provides an equivalent amount of data.

Now we can state the inverse problem:

Problem statement: Find initial functions $(u_0, u_1, u_2) \in H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)$, so to reconstruct the parameters $\{\alpha, \beta, \gamma\}$ and the memory function g from a single observation of the solution at one point $\{u(b, t)\}_{t \geq 0}$ where $b \in \Omega$.

As we shall use pseudo-spectral methods associated the Neumann Laplacian, let us denote by λ_n its eigenvalues and eigenfunctions by φ_n , i.e.

$$\begin{cases} -\Delta \varphi_n(x) = \lambda_n \varphi_n(x), & \text{in } \Omega, \\ \partial_\nu \varphi_n(x) = 0, & \text{on } \partial\Omega. \end{cases} \quad (3)$$

where ∂_ν denotes the outward normal derivative. It is known that

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_n \leq \dots,$$

$\lim_{n \rightarrow \infty} \lambda_n = \infty$, and the set $\{\varphi_n\}_{n \geq 1}$, normalized by $\|\varphi_n\| = 1$, is a basis for $L^2(\Omega)$.

As the unique global solution $u(\cdot, t) \in L^2(\Omega)$, means that it can be written as

$$u(x, t) = \sum_{n=1}^{\infty} c_n(t) \varphi_n(x), \quad (4)$$

where

$$c_n(t) = \int_{\Omega} u(x, t) \varphi_n(x) dx. \quad (5)$$

The initial conditions translate into, for all $n \geq 1$,

$$c_n^{(i)}(0) = \int_{\Omega} u_i(x, 0) \varphi_n(x) dx \quad \text{for } i = 0, 1, 2.$$

Multiply (1) by $\varphi_n(x)$ and integrate over Ω to get

$$\begin{aligned} \int_{\Omega} u_{ttt}(x, t) \varphi_n(x) dx + \alpha \int_{\Omega} u_{tt}(x, t) \varphi_n(x) dx - \beta \int_{\Omega} \varphi_n(x) \Delta u_t(x, t) dx \\ - \gamma \int_{\Omega} \varphi_n(x) \Delta u(x, t) dx + \int_0^t g(t-s) \int_{\Omega} \varphi_n(x) \Delta u(x, s) dx ds = 0. \end{aligned}$$

which reduces to the functional equation, thanks to (3) and (5),

$$c_n'''(t) + \alpha c_n''(t) + \beta \lambda_n c_n'(t) + \gamma \lambda_n c_n(t) = \lambda_n \int_0^t g(t-s) c_n(s) ds \quad (6)$$

with initial conditions

$$c_n(0) = \int_{\Omega} u_0(x) \varphi_n(x) dx, \quad c_n'(0) = \int_{\Omega} u_1(x) \varphi_n(x) dx, \quad c_n''(0) = \int_{\Omega} u_2(x) \varphi_n(x) dx.$$

Since all sought parameters appear in (6), it is easy to extract some from it. For example since $\lambda_1 = 0$, we get from (6)

$$c_1'''(t) + \alpha c_1''(t) = 0 \quad \text{for all } t \geq 0$$

as long as c_1'' is not trivial, i.e. u_2 is not orthogonal to φ_n . Hence by taking the limit, since the solution is $C^3[0, \infty)$.

$$\alpha = -\frac{c_1'''(0)}{c_1''(0)}.$$

We now show that a non trivial c_2 contains all the information required to reconstruct the sought coefficients. From (6) and by taking the limit as $t \rightarrow 0^+$, we deduce that c_2 satisfies,

$$c_2'''(0) + \alpha c_2''(0) + \beta \lambda_2 c_2'(0) + \gamma \lambda_2 c_2(0) = 0.$$

Similarly differentiating (6) and taking the limit as $t \rightarrow 0^+$ we deduce

$$\begin{aligned} \alpha c_2''(0) + \beta \lambda_2 c_2'(0) + \gamma \lambda_2 c_2(0) &= -c_2'''(0) \\ \alpha c_2'''(0) + \beta \lambda_2 c_2''(0) + \gamma \lambda_2 c_2'(0) &= \lambda_2 g(0) c_2(0) - c_2^{(4)}(0) \\ \alpha c_2^{(4)}(0) + \beta \lambda_2 c_2'''(0) + \gamma \lambda_2 c_2''(0) &= \lambda_2 (g'(0) c_2 + g(0) c_2'(0)) - c_2^{(5)}(0) \end{aligned}$$

By choosing

$$c_2(0) = c_2'(0) = 0 \quad \text{and} \quad c_2''(0) = 1, \quad (7)$$

the above system reduces to a lower triangular matrix equation, which is easily solved by the Gaussian method

$$\begin{bmatrix} 1 & 0 & 0 \\ c_2'''(0) & 1 & 0 \\ c_2^{(4)}(0) & c_2'''(0) & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \lambda_2 \\ \gamma \lambda_2 \end{bmatrix} = \begin{bmatrix} -c_2'''(0) \\ -c_2^{(4)}(0) \\ -c_2^{(5)}(0) \end{bmatrix}.$$

$A \qquad X \qquad B$

Since the determinant of A is 1, the system above has the unique solution $(\alpha, \beta\lambda_2, \gamma\lambda_2)^T$.

Remark: The solution $u(x, t)$, is known to be globally C^2 in time, and so (1) implies that u_{ttt} is also locally continuous in t . Thus its Fourier coefficients $c_n^{(3)}(t)$, are well defined and continuous in time. Therefore (6) holds in the classical sense and as the integral term is C^2 , we obtain that $c_n^{(3)} \in C^2$, i.e. $c_n \in C^5$.

Once we have obtained the values (α, β, γ) , we apply the Laplace transform, $\hat{g}(s) = \mathcal{L}(g)(s)$, to (6)

$$\hat{c}_n(s) \left\{ s^3 + \alpha s^2 + \beta\lambda_n s + \gamma\lambda_n \right\} = c_n''(0) + (s + \alpha)c_n'(0) + (s^2 + \alpha s + \beta\lambda_n)c_n(0) + \lambda_n \hat{g}(s)\hat{c}_n(s), \quad (8)$$

and so, for $n \geq 2$,

$$\hat{g}(s) = \beta s + \gamma + \frac{1}{\lambda_n} \left(s^3 + \alpha s^2 \right) - \frac{1}{\lambda_n \hat{c}_n(s)} \left(c_n''(0) + (s + \alpha)c_n'(0) + (s^2 + \alpha s + \beta\lambda_n)c_n(0) \right)$$

And if we use the boundary conditions in (7), the previous equation reduces to

$$\hat{g}(s) = \beta s + \gamma + \frac{1}{\lambda_2} \left(s^3 + \alpha s^2 \right) - \frac{1}{\lambda_2 \hat{c}_2(s)}.$$

Note that it follows from condition (A2) that $g \in L^2(0, \infty)$, since

$$\int_0^\infty g^2(t) dt \leq g(0) \int_0^\infty g(t) dt < \infty.$$

Next recall that the Laplace transform is a bounded and invertible operator in $L^2(0, \infty)$, and so not only $\hat{g} \in L^2(0, \infty)$, it is also in the range of the Laplace transform, [2]. Thus its inverse Laplace transform exists and would yield back g .

It remains to see that we can observe c_2 from the solution $u(x, t)$, see (7), by choosing the initial condition as follows

$$\begin{aligned} u(x, 0) &= c_2(0)\varphi_2(x) = 0 \\ u_t(x, 0) &= c_2'(0)\varphi_2(x) = 0 \\ u_{tt}(x, 0) &= c_2''(0)\varphi_2(x) = \varphi_2(x). \end{aligned}$$

Then pick any point $b \in \Omega$ where $\varphi_2(b) \neq 0$ to read

$$u(b, t) = c_2(t)\varphi_2(b)$$

as all the $c_n = 0$ if $n \neq 2$. Thus we proved the main result

Proposition 2. Taking $(u_0, u_1, u_2) = (0, 0, \varphi_2)$ then we can reconstruct the parameters $\{\alpha, \beta, \gamma\}$ and the function g using one observation only of the acoustic velocity potential u at a single point $b \in \Omega$, such that $\varphi_2(b) \neq 0$.

Proof. With the given initial condition, the unique solution can be observed at $x = b$ is

$$u(b, t) = c_2(t)\varphi_2(b)$$

which exists for $t \geq 0$. It remains to avoid nodal curves, so the observation is not trivial. □

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