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Three weak solutions for a Neumann elliptic equations involving the $\vec{p}(x)$ -Laplacian operator

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Abstract: The aim of this paper is to establish the existence of at least three weak solutions for the following elliptic Neumann problem

$$\begin{cases} -\Delta_{\vec{p}(x)} u + \alpha(x) |u|^{p_0(x)-2} u = \lambda f(x, u) & \text{in } \Omega, \\ \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)-2} \frac{\partial u}{\partial x_i} \gamma_i = 0 & \text{on } \partial\Omega, \end{cases}$$

in the anisotropic variable exponent Sobolev spaces $W^{1, \vec{p}(\cdot)}(\Omega)$ where $\lambda > 0$ and $f(x, t) = |t|^{q(x)-2} t - |t|^{s(x)-2} t$, $x \in \Omega$, $t \in \mathbb{R}$ and $q(\cdot), s(\cdot) \in \mathcal{C}_+(\overline{\Omega})$.

Keywords: $\vec{p}(x)$ -Laplacian operator, Neumann elliptic equations, variational principle, critical point theory, anisotropic Sobolev spaces with variable exponents

MSC: 35J57, 35D30, 35J20, 35J60, 34B15, 35A15

1 Introduction

In the recent years, the anisotropic variable exponent Sobolev spaces have attracted the attention of many mathematicians, physicists and engineers. The impulse for this, mainly come from their important applications in modeling real-world problems in electrorheological, magnetorheological fluids, elastic materials and image restoration, (see for example [8, 15, 18, 38–40]).

More recently, several authors (see e.g. [4, 9, 29]) have studied the anisotropic quasi-linear elliptic equations with variable exponents, i.e. the quasi-linear elliptic equations involving the $\vec{p}(x)$ -Laplacian

$$\Delta_{\vec{p}(x)} u = \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)-2} \frac{\partial u}{\partial x_i} \right). \quad (1.1)$$

It's clear that this $\vec{p}(x)$ -Laplace operator is a generalization of the $p(\cdot)$ -Laplace operator

$$\Delta_{p(x)} u = \operatorname{div} \left(\left| \nabla u \right|^{p(x)-2} \nabla u \right). \quad (1.2)$$

We refer to [1, 22, 23, 41] for the study of the $p(x)$ -Laplacian equations and the corresponding variational problems.

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The $p(x)$ -Laplacian is a meaningful generalization of the p -Laplacian operator

$$\Delta_p u = \operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right), \quad (1.3)$$

obtained in the case when p is a positive constant.

The purpose of this paper is to prove the existence of three weak solutions in the anisotropic variable exponent Sobolev space $W^{1, \vec{p}(\cdot)}(\Omega)$ for the following problem with Neumann boundary value condition,

$$\begin{cases} -\Delta_{\vec{p}(x)} u + \alpha(x) |u|^{p_0(x)-2} u = \lambda f(x, u) & \text{in } \Omega, \\ \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)-2} \frac{\partial u}{\partial x_i} \gamma_i = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.4)$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with boundary of class $\mathcal{C}^1$, and let $\vec{\gamma}$ be the outward unit normal vector on $\partial\Omega$ and let γ_i , $i \in \{1 \dots N\}$, represent the components of the unit outer normal vector, for the reader's convenience, see section 3.

In the classical Sobolev spaces, G. Bonanno and P. Candito [12] have proved the existence of three solutions for the problem (1.4) in the case p -Laplacien, for more results see [3, 6, 7, 11, 13, 26, 37].

In the Sobolev variable exponent setting, R. A. Mashiye [25] has considered the problem (1.4) in the case of $p(x)$ -Laplacien operator, see also [19, 31, 32] for related topics.

Even though the problem (1.4) has been studied by some other authors in anisotropic variable exponent Sobolev spaces (see [4, 16, 17, 29]), the hypotheses we use in this paper are totally different from those ones and so are our results.

The main difficulties in this kind of problem that is the framework of anisotropic Sobolev spaces and the fact that we have Neumann boundary conditions that make some difficulties in the application of the theorem 1.1.

We introduce the following theorem, which will be essential to establish the existence of three weak solutions for our main problem.

Theorem 1.1. ([12]). *Let E be a separable and reflexive real Banach space; $\Psi : E \Rightarrow \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on E^* , $\Phi : E \Rightarrow \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that*

(a)

$$\lim_{\|u\|_E \rightarrow +\infty} \left(\Psi(u) + \lambda \Phi(u) \right) = +\infty \text{ for all } \lambda > 0;$$

and there are $r \in \mathbb{R}$ and $u_0, u_1 \in E$ such that

(b)

$$\Psi(u_0) < r < \Psi(u_1);$$

(c)

$$\inf_{u \in \Psi^{-1}([-\infty, r])} \Phi(u) > \frac{(\Psi(u_1) - r) \Phi(u_0) + (r - \Psi(u_0)) \Phi(u_1)}{\Psi(u_1) - \Psi(u_0)}.$$

Then there exist an open interval $\Lambda \subset]0, +\infty[$ and a positive real number ρ such that for each $\lambda \in \Lambda$ the equation

$$\Psi'(u) + \lambda \Phi'(u) = 0,$$

has at least three solutions in E whose norms are less than ρ .

This paper is organized as follows: In Section 2, we present some necessary preliminary knowledge on the anisotropic Sobolev spaces with variable exponents. We introduce in Section 3 some assumptions for which our problem has solutions and we present such improvement (see theorems 3.1 and 3.2). In section 4, we give the proof of the main results. Finally, we conclude and provide some perspectives in section 5.

2 Preliminary

In this section we summarize notation, definitions and properties of our framework. For more details we refer to [5, 20, 21, 27, 28, 30, 34]. Let Ω be a bounded domain in $\mathbb{R}^N$, we define:

$$\mathcal{C}_+(\overline{\Omega}) = \left\{ \text{measurable function } p(\cdot) : \overline{\Omega} \Rightarrow \mathbb{R} \text{ such that } 1 < p^- \leq p^+ < \infty \right\},$$

where

$$p^- = \text{ess inf } \{p(x) / x \in \overline{\Omega}\} \quad \text{and} \quad p^+ = \text{ess sup } \{p(x) / x \in \overline{\Omega}\}.$$

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \rightarrow \mathbb{R}$ for which the convex modular

$$\rho_{p(\cdot)}(u) := \int_{\Omega} |u|^{p(x)} dx,$$

is finite, then

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} = \inf \{ \lambda > 0 : \rho_{p(\cdot)}(u/\lambda) \leq 1 \},$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxemburg norm. The space $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is a separable Banach space. Moreover, the space $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. Finally, we have the Hölder type inequality:

$$\left| \int_{\Omega} uv \, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{(p^-)'} \right) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)}, \quad (2.1)$$

for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.

An important role in manipulating the generalized Lebesgue spaces is played by the modular $\rho_{p(\cdot)}$ of the space $L^{p(\cdot)}(\Omega)$. We have the following result.

Proposition 2.1. ([20, 30]). *If $u \in L^{p(\cdot)}(\Omega)$, then the following properties hold true:*

- (i) $\|u\|_{p(\cdot)} < 1 (= 1, > 1) \Leftrightarrow \rho_{p(\cdot)}(u) < 1 (= 1, > 1)$,
- (ii) $\|u\|_{p(\cdot)} > 1 \Rightarrow \|u\|_{p(\cdot)}^{p^-} < \rho_{p(\cdot)}(u) < \|u\|_{p(\cdot)}^{p^+}$,
- (iii) $\|u\|_{p(\cdot)} < 1 \Rightarrow \|u\|_{p(\cdot)}^{p^+} < \rho_{p(x)}(u) < \|u\|_{p(\cdot)}^{p^-}$.

We define the Sobolev space with variable exponent by:

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) \quad \text{and} \quad |\nabla u| \in L^{p(\cdot)}(\Omega) \right\},$$

equipped with the following norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

The space $(W^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot)})$ is a separable and reflexive Banach space. We refer to [20] for the elementary properties of these spaces.

Remark 2.1. ([20, 28]). *Recall that the definition of these spaces requires only the measurability of $p(x)$. In this work, we do not need to use Sobolev and Poincaré inequalities. Note that the sharp Sobolev inequality is proved for $p(x)$ -log-Hölder continuous, while the Poincaré inequality requires only the continuity of $p(x)$.*

Now, we present the anisotropic Sobolev space with variable exponent which is used for the study of our main problem.

Let $p_0(x), p_1(x), \dots, p_N(x)$ be $N + 1$ variable exponents in $\mathbb{C}_+(\overline{\Omega})$. We denote

$$\vec{p}(x) = \{p_0(x), p_1(x), \dots, p_N(x)\}, \quad D^0 u = u \quad \text{and} \quad D^i u = \frac{\partial u}{\partial x_i} \quad \text{for } i = 1, \dots, N.$$

We define

$$\underline{p} = \min \{p_0^-, p_1^-, \dots, p_N^-\} \quad \text{then} \quad \underline{p} > 1, \quad (2.2)$$

and

$$\overline{p} = \max \{p_0^+, p_1^+, \dots, p_N^+\}. \quad (2.3)$$

The anisotropic variable exponent Sobolev space $W^{1, \vec{p}(\cdot)}(\Omega)$ is defined as follows

$$W^{1, \vec{p}(\cdot)}(\Omega) = \left\{ u \in L^{p_0(\cdot)}(\Omega) \quad \text{and} \quad D^i u \in L^{p_i(\cdot)}(\Omega), \quad i = 1, 2, \dots, N \right\},$$

endowed with the norm

$$\|u\|_{W^{1, \vec{p}(\cdot)}(\Omega)} = \|u\|_{1, \vec{p}(\cdot)} = \|u\|_{L^{p_0(\cdot)}(\Omega)} + \sum_{i=1}^N \|D^i u\|_{L^{p_i(\cdot)}(\Omega)}. \quad (2.4)$$

(Cf. [10, 35, 36] for the constant exponent case). For the basic properties of $W^{1, \vec{p}(\cdot)}(\Omega)$, see [2, 5, 14, 21, 27, 30, 33, 34].

Proposition 2.2. ([21, 24]). *The space $(W^{1, \vec{p}(\cdot)}(\Omega), \|\cdot\|_{1, \vec{p}(\cdot)})$ is a separable and reflexive Banach space, if $p_i^- > 1$ for $i = 1, \dots, N$.*

From now on, we always assume that

$$\underline{p} > N. \quad (2.5)$$

Remark 2.2. Since $W^{1, \vec{p}(\cdot)}(\Omega)$ is continuously embedded in $W^{1, \underline{p}}(\Omega)$, and $W^{1, \underline{p}}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$ (the space of continuous functions), thus $W^{1, \vec{p}(\cdot)}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$.

Set

$$C_0 = \sup_{u \in W^{1, \vec{p}(\cdot)}(\Omega) \setminus \{0\}} \frac{\|u\|_{L^\infty(\Omega)}}{\|u\|_{1, \vec{p}(\cdot)}}. \quad (2.6)$$

Then C_0 is a positive constant.

3 Assumptions and statement of main results

Here and in the sequel:

Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$ is a bounded domain with boundary of class $\mathcal{C}^1$, and let $\vec{\gamma}$ be the outward unit normal vector on $\partial\Omega$ and let γ_i , $i \in \{1 \dots N\}$, represent the components of the unit outer normal vector.

Assume that

(A1) $\alpha(\cdot) \in L^\infty(\Omega)$, with

$$\alpha^- = \text{ess inf } \alpha(x) > 0.$$

(A2) $\lambda > 0$ is a real number.

We assume that f satisfies one of the following two conditions:

(f1) $f : \mathbb{R} \Rightarrow \mathbb{R}$ such that

$$f(t) = b|t|^{q-2}t - d|t|^{s-2}t, \quad t \in \mathbb{R},$$

where b and d are positive constants.

(f2) $f : \mathbb{R} \Rightarrow \mathbb{R}$ such that

$$f(x, t) = |t|^{q(x)-2}t - |t|^{s(x)-2}t, \quad x \in \Omega, \quad t \in \mathbb{R},$$

where $q, s \in \mathbb{C}(\overline{\Omega})$.

Case I.

Assuming that the condition (f1) is satisfied, then problem (1.4) becomes

$$\begin{cases} -\Delta_{\vec{p}(x)} u + \alpha(x)|u|^{p_0(x)-2}u = \lambda b|u|^{q-2}u - \lambda d|u|^{s-2}u & \text{in } \Omega, \\ \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)-2} \frac{\partial u}{\partial x_i} \gamma_i = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.1)$$

We introduce the functionals $\Psi, \Phi : W^{1,\vec{p}(\cdot)}(\Omega) \Rightarrow \mathbb{R}$ by

$$\Psi(u) = \sum_{i=1}^N \int_{\Omega} \frac{1}{p_i(x)} \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)} dx + \int_{\Omega} \frac{\alpha(x)}{p_0(x)} |u|^{p_0(x)} dx, \quad (3.2)$$

and

$$\Phi(u) = -\frac{b}{q} \int_{\Omega} |u|^q dx + \frac{d}{s} \int_{\Omega} |u|^s dx. \quad (3.3)$$

Definition 3.1. A measurable function $u \in W^{1,\vec{p}(\cdot)}(\Omega)$ is called a weak solution of the Neumann elliptic problem (3.1) if

$$\sum_{i=1}^N \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)-2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx + \int_{\Omega} \alpha(x)|u|^{p_0(x)-2}uv dx = \lambda b \int_{\Omega} |u|^{q-2}uv dx - \lambda d \int_{\Omega} |u|^{s-2}uv dx, \quad (3.4)$$

for all $v \in W^{1,\vec{p}(\cdot)}(\Omega)$.

The following theorem is our first main result.

Theorem 3.1. Assume that $p > N$, f satisfies (f1) and $2 < s < q < p$. Then there exist an open interval $\Lambda \subset]0, +\infty[$ and a constant $\rho > 0$ such that for any $\lambda \in \Lambda$, problem (3.1) has at least three weak solutions in $W^{1,\vec{p}(\cdot)}(\Omega)$ whose norms are less than ρ .

Case II.

Under the condition (f2), we have problem (1.4) becomes

$$\begin{cases} -\Delta_{\vec{p}(x)} u + \alpha(x)|u|^{p_0(x)-2}u = \lambda|u|^{q(x)-2}u - \lambda|u|^{s(x)-2}u & \text{in } \Omega, \\ \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)-2} \frac{\partial u}{\partial x_i} \gamma_i = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.5)$$

We define, for any $u \in W^{1,\vec{p}(\cdot)}(\Omega)$, the functional

$$\phi(u) = - \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} dx + \int_{\Omega} \frac{1}{s(x)} |u|^{s(x)} dx. \quad (3.6)$$

Definition 3.2. We say that $u \in W^{1,\vec{p}(\cdot)}(\Omega)$ is called a weak solution to the problem (3.5) if for all $v \in W^{1,\vec{p}(\cdot)}(\Omega)$, we have

$$\sum_{i=1}^N \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)-2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx + \int_{\Omega} \alpha(x)|u|^{p_0(x)-2}uv dx = \lambda \int_{\Omega} |u|^{q(x)-2}uv dx - \lambda \int_{\Omega} |u|^{s(x)-2}uv dx. \quad (3.7)$$

Our second main result is the following theorem.

Theorem 3.2. Assume that $p > N$, f satisfies (f2) and $2 < s^- \leq s^+ < q^- \leq q^+ < p$. Then there exist an open interval $\Lambda \subset]0, +\infty[$ and a constant $\rho > 0$ such that for any $\lambda \in \Lambda$, problem (3.5) has at least three weak solutions in $W^{1,\vec{p}(\cdot)}(\Omega)$ whose norms are less than ρ .

4 Proof of the main results

In this section, we are ready to prove the main result.

Step 1 : Some technical lemmas

This subsection is devoted to introducing some basic technical lemma which will be needed throughout this paper.

Lemma 4.1. ([16, 30]). The functionals Ψ , Φ and ϕ are well-defined on $W^{1,\vec{p}(\cdot)}(\Omega)$. In addition, Ψ , Φ and ϕ are of class $C^1(W^{1,\vec{p}(\cdot)}(\Omega), \mathbb{R})$ and

$$\begin{aligned}\langle \Psi'(u), v \rangle &= \sum_{i=1}^N \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)-2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx + \int_{\Omega} \alpha(x) |u|^{p_0(x)-2} uv dx, \\ \langle \Phi'(u), v \rangle &= -b \int_{\Omega} |u|^{q-2} uv dx + d \int_{\Omega} |u|^{s-2} uv dx,\end{aligned}$$

and

$$\langle \phi'(u), v \rangle = - \int_{\Omega} |u|^{q(x)-2} uv dx + \int_{\Omega} |u|^{s(x)-2} uv dx,$$

for all $u, v \in W^{1,\vec{p}(\cdot)}(\Omega)$.

Lemma 4.2. ([24, 30]). Let (A1), (A2) and (f1) hold. Then Ψ , Φ and ϕ are sequentially weakly lower semicontinuous.

Lemma 4.3. Let $\frac{1}{p_i^-} + \frac{1}{p_i^+} = 1$. Then $\Psi' : W^{1,\vec{p}(\cdot)}(\Omega) \Rightarrow W^{-1,\vec{p}(\cdot)'}(\Omega)$ is coercive, a homeomorphism and uniformly monotone, where $\vec{p}(\cdot)' = \{p_0', \dots, p_N'\}$, (cf. [10] for the constant exponent case).

Proof

$$\begin{aligned}\Psi(u) &= \sum_{i=1}^N \int_{\Omega} \frac{1}{p_i(x)} \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)} dx + \int_{\Omega} \frac{\alpha(x)}{p_0(x)} |u|^{p_0(x)} dx \\ &\geq \sum_{i=1}^N \frac{1}{p_i^+} \left(\left\| \frac{\partial u}{\partial x_i} \right\|_{p_i(\cdot)}^p - 1 \right) + \frac{\alpha^-}{p_0^+} \left(\|u\|_{p_0(\cdot)}^p - 1 \right) \quad (\text{by proposition 2.1}) \\ &\geq \frac{\min\{1, \alpha^-\}}{(N+1)^{p-1} \bar{p}} \|u\|_{1,\vec{p}(\cdot)}^p - \frac{(\alpha^- + N)}{\bar{p}},\end{aligned}$$

and thus

$$\Psi(u) \Rightarrow +\infty \quad \text{as} \quad \|u\|_{1,\vec{p}(\cdot)} \Rightarrow \infty \quad \text{for} \quad u \in W^{1,\vec{p}(\cdot)}(\Omega).$$

It is obvious that $(\Psi')^{-1} : W^{-1,\vec{p}(\cdot)'}(\Omega) \Rightarrow W^{1,\vec{p}(\cdot)}(\Omega)$ exists and continuous, because $\Psi' : W^{1,\vec{p}(\cdot)}(\Omega) \Rightarrow W^{-1,\vec{p}(\cdot)'}(\Omega)$ is a homeomorphism.

Recalling the following well-known inequality

$$\left(|a|^{\theta-2} a - |b|^{\theta-2} b \right) (a - b) \geq \frac{1}{2\theta} |a - b|^{\theta}, \quad \forall a, b \in \mathbb{R}^N, \quad \forall \theta \geq 2,$$

and for $i = 0, \dots, N$, we define the increasing function

$$\eta_i(t) = \begin{cases} t^{p_i^- - 1}, & t < 1 \\ t^{p_i^+ - 1}, & t > 1 \end{cases}.$$

which admits a positive minimum noted η_i^- , then we have $\forall u, v \in W^{1, \vec{p}(\cdot)}(\Omega)$

$$\begin{aligned} \langle \Psi'(u) - \Psi'(v), u - v \rangle &= \sum_{i=1}^N \int_{\Omega} \left[\left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)-2} \frac{\partial u}{\partial x_i} - \left| \frac{\partial v}{\partial x_i} \right|^{p_i(x)-2} \frac{\partial v}{\partial x_i} \right] \left(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i} \right) dx \\ &\quad + \int_{\Omega} \alpha(x) \left(|u|^{p_0(x)-2} u - |v|^{p_0(x)-2} v \right) (u - v) dx \\ &\geq \frac{1}{2^{\bar{p}}} \left[\sum_{i=1}^N \int_{\Omega} \left(\left| \frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i} \right|^{p_i(x)} \right) dx + \alpha^- \int_{\Omega} (|u - v|^{p_0(x)}) dx \right] \\ &\geq \frac{\min(1, \alpha^-)}{2^{\bar{p}}} \sum_{i=0}^N \eta_i \left(\left\| \frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i} \right\|_{p_i(\cdot)} \right) \left\| \frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i} \right\|_{p_i(\cdot)} \\ &\geq \frac{\min(1, \alpha^-)}{2^{\bar{p}}} \sum_{i=0}^N \eta_i^- \left\| \frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i} \right\|_{p_i(\cdot)} \\ &\geq \frac{\min(1, \alpha^-)}{2^{\bar{p}}} \min\{\eta_i^-, i = 0, \dots, N\} \|u - v\|_{1, \vec{p}(\cdot)}, \end{aligned}$$

i.e. Ψ' is uniformly monotone. We deduce that $(\Psi')^{-1}$ exists and it is continuous. $\square$

Step 2 : Proof of theorem 3.1

In order to prove this result, we apply theorem 1.1.

Since $2 < s < q < \underline{p}$, we obtain $W^{1, \vec{p}(\cdot)}(\Omega) \hookrightarrow W^{1, \underline{p}}(\Omega) \hookrightarrow L^{\underline{p}}(\Omega) \hookrightarrow L^q(\Omega)$, so we can find a constant c_1 such that

$$\|u\|_{L^q(\Omega)} \leq c_1 \|u\|_{1, \vec{p}(\cdot)}, \quad (4.1)$$

thus deduce that

$$\begin{aligned} \Psi(u) + \lambda \Phi(u) &\geq \sum_{i=1}^N \int_{\Omega} \frac{1}{p_i(x)} \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)} dx + \int_{\Omega} \frac{\alpha(x)}{p_0(x)} |u|^{p_0(x)} dx - \frac{\lambda b}{q} \int_{\Omega} |u|^q dx + \frac{\lambda d}{s} \int_{\Omega} |u|^s dx \\ &\geq \frac{\min\{1, \alpha^-\}}{(N+1)^{\underline{p}-1} \bar{p}} \|u\|_{1, \vec{p}(\cdot)}^{\underline{p}} - \frac{\lambda b}{q} \|u\|_{L^q(\Omega)}^q - \frac{(\alpha^- + N)}{\bar{p}} \\ &\geq \frac{\min\{1, \alpha^-\}}{(N+1)^{\underline{p}-1} \bar{p}} \|u\|_{1, \vec{p}(\cdot)}^{\underline{p}} - \frac{\lambda b c_1^q}{q} \|u\|_{1, \vec{p}(\cdot)}^q - \frac{(\alpha^- + N)}{\bar{p}}, \end{aligned}$$

for $\|u\|_{1, \vec{p}(\cdot)} > 1$ any $\lambda > 0$, then $\lim_{\|u\|_{1, \vec{p}(\cdot)} \rightarrow +\infty} (\Psi(u) + \lambda \Phi(u)) = +\infty$ and (a) in theorem 1.1 is verified.

In the following, we will verify the condition (b) in theorem 1.1.

We define the function $F : [0, +\infty[\rightarrow \mathbb{R}$ by

$$F(t) = \frac{b}{q} t^q - \frac{d}{s} t^s, \quad t \in [0, +\infty[.$$

It is obviously that F is of class C^1 and

$$F'(t) = d t^{s-1} \left(\frac{b}{d} t^{q-s} - 1 \right), \quad t \in [0, +\infty[.$$

So

$$F'(t) \leq 0, \quad \forall t \in \left[0, \left(\frac{d}{b} \right)^{\frac{1}{q-s}} \right] \text{ and } F'(t) \geq 0, \quad \forall t \in \left[\left(\frac{d}{b} \right)^{\frac{1}{q-s}}, +\infty \right[.$$

It follows that F is increasing for $t \in \left[\left(\frac{d}{b}\right)^{\frac{1}{q-s}}, +\infty\right[$ and decreasing for $t \in \left[0, \left(\frac{d}{b}\right)^{\frac{1}{q-s}}\right]$. Obviously

$$F(0) = 0 \quad \text{and} \quad \lim_{t \rightarrow +\infty} F(t) = +\infty.$$

Then there exists a real number $\delta > \left(\frac{dq}{bs}\right)^{\frac{1}{q-s}}$ such that

$$F(t) \geq 0 \geq F(\tau), \quad \forall t > \delta, \quad \text{and} \quad \tau \in \left[0, \left(\frac{d}{b}\right)^{\frac{1}{q-s}}\right]. \quad (4.2)$$

Let c, m be two real numbers such that $0 < c < \min\left\{\left(\frac{d}{b}\right)^{\frac{1}{q-s}}, C_0\right\}$ with C_0 given in remark 2.2 and $m > \delta$ satisfies

$$m^{\underline{p}} \|\alpha\|_{L^1(\Omega)} > 1, \quad (4.3)$$

and

$$m^{\bar{p}} \|\alpha\|_{L^1(\Omega)} > 1. \quad (4.4)$$

By (4.2) we obtain

$$\frac{\sup_{0 \leq t \leq c} F(t)}{c^{\bar{p}}} \leq 0 < \frac{1}{c_0^{\bar{p}}} \frac{F(m)}{m^{\underline{p}}}.$$

Consider $u_0, u_1 \in W^{1, \bar{p}(\cdot)}(\Omega)$, with $u_0(x) = 0$ and $u_1(x) = m$ for any $x \in \Omega$. We define $r = \frac{\min\{1, \alpha^-\}}{(N+1)^{\underline{p}-1} \bar{p}} \left(\frac{c}{C_0}\right)^{\bar{p}}$. Clearly, $r \in]0, 1[$. A simple computation implies

$$\Psi(u_0) = \Phi(u_0) = 0.$$

Let $m > 1$. Then, if we consider formula (4.3) we get

$$\begin{aligned} \Psi(u_1) &= \int_{\Omega} \frac{1}{p_0(x)} \alpha(x) m^{p_0(x)} dx \\ &\geq \frac{1}{\bar{p}} m^{\underline{p}} \int_{\Omega} \alpha(x) dx \\ &= \frac{1}{\bar{p}} m^{\underline{p}} \|\alpha\|_{L^1(\Omega)} \\ &> \frac{1}{\bar{p}} \\ &> \frac{\min\{1, \alpha^-\}}{(N+1)^{\underline{p}-1} \bar{p}} \left(\frac{c}{C_0}\right)^{\bar{p}} \\ &= r. \end{aligned} \quad (4.5)$$

Similarly for $m < 1$, by help of (4.4), we get the desired result.

Thus, we deduce that

$$\Psi(u_0) < r < \Psi(u_1),$$

and (b) in theorem 1.1 is verified.

Finally, we will verify that condition (c) of theorem 1.1 is fulfilled.

Moreover, we have

$$\begin{aligned} \Phi(u_1) &= -\frac{b}{q} \int_{\Omega} |m|^q dx + \frac{d}{s} \int_{\Omega} |m|^s dx \\ &= - \int_{\Omega} F(m) dx \\ &= -F(m)|\Omega|, \end{aligned} \quad (4.6)$$

and

$$\begin{aligned}
 \frac{(\Psi(u_1) - r)\Phi(u_0) + (r - \Psi(u_0))\Phi(u_1)}{\Psi(u_1) - \Psi(u_0)} &= r \frac{\Phi(u_1)}{\Psi(u_1)} \\
 &= -r \frac{F(m)|\Omega|}{\int_{\Omega} \frac{\alpha(x)}{p_0(x)} m^{p_0(x)} dx} \\
 &< 0.
 \end{aligned} \tag{4.7}$$

Next, we consider the case $u \in W^{1, \vec{p}(\cdot)}(\Omega)$ such that $\|u\|_{1, \vec{p}(\cdot)} \leq 1$ with $\Psi(u) \leq r < 1$. Since

$$\begin{aligned}
 \frac{\min\{1, \alpha^-\}}{(N+1)^{\frac{p-1}{p}} \bar{p}} \|u\|_{1, \vec{p}(\cdot)}^{\bar{p}} &\leq \frac{1}{\bar{p}} \left(\sum_{i=1}^N \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)} dx + \int_{\Omega} |u|^{p_0(x)} dx \right) \\
 &\leq \Psi(u) \\
 &\leq r \\
 &< 1.
 \end{aligned}$$

Thus, using remark 2.2, we have

$$\|u\|_{L^\infty(\Omega)} \leq C_0 \left(\bar{p} r \right)^{\frac{1}{\bar{p}}} \leq c.$$

The above inequality shows that

$$\begin{aligned}
 - \inf_{u \in \Psi^{-1}([-\infty, r])} \Phi(u) &= \sup_{u \in \Psi^{-1}([-\infty, r])} -\Phi(u) \\
 &\leq \int_{\Omega} \sup_{0 \leq t \leq c} F(t) dx \\
 &\leq 0.
 \end{aligned}$$

It follows that

$$- \inf_{u \in \Psi^{-1}([-\infty, r])} \Phi(u) < r \frac{\int_{\Omega} F(m) dx}{\int_{\Omega} \frac{1}{p_0(x)} \alpha(x) m^{p_0(x)} dx}.$$

That is

$$\inf_{u \in \Psi^{-1}([-\infty, r])} \Phi(u) > \frac{(\Psi(u_1) - r)\Phi(u_0) + (r - \Psi(u_0))\Phi(u_1)}{\Psi(u_1) - \Psi(u_0)}.$$

which means that condition (c) in theorem 1.1 is verified. Then the proof of theorem 3.1 is achieved.

Step 3 : Proof of theorem 3.2

Next, we will verify that condition (a) of theorem 1.1 is fulfilled. In fact, by proposition 2.1 and 4.1, we have

$$\begin{aligned}\Psi(u) + \lambda\phi(u) &= \sum_{i=1}^N \int_{\Omega} \frac{1}{p_i(x)} \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)} dx + \int_{\Omega} \frac{\alpha(x)}{p_0(x)} |u|^{p_0(x)} dx \\ &\quad - \lambda \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} dx + \lambda \int_{\Omega} \frac{1}{s(x)} |u|^{s(x)} dx \\ &\geq \frac{\min\{1, \alpha^-\}}{(N+1)^{\frac{p}{p-1}} \bar{p}} \|u\|_{1, \bar{p}(\cdot)}^{\frac{p}{p-1}} - \frac{(\alpha^- + N)}{\bar{p}} - \frac{\lambda}{\bar{p}} \left(\int_{\Omega} (|u|^{q^+} + |u|^{q^-}) dx \right) \\ &\geq \frac{\min\{1, \alpha^-\}}{(N+1)^{\frac{p}{p-1}} \bar{p}} \|u\|_{1, \bar{p}(\cdot)}^{\frac{p}{p-1}} - \frac{(\alpha^- + N)}{\bar{p}} - \frac{\lambda}{\bar{p}} \left(c_1^{q^+} \|u\|_{1, \bar{p}(\cdot)}^{q^+} + c_1^{q^-} \|u\|_{1, \bar{p}(\cdot)}^{q^-} \right) \\ &\geq \frac{\min\{1, \alpha^-\}}{(N+1)^{\frac{p}{p-1}} \bar{p}} \|u\|_{1, \bar{p}(\cdot)}^{\frac{p}{p-1}} - \frac{(\alpha^- + N)}{\bar{p}} - \frac{\lambda}{\bar{p}} \max\{c_1^{q^-}, c_1^{q^+}\} \|u\|_{1, \bar{p}(\cdot)}^{q^+},\end{aligned}$$

for $\|u\|_{1, \bar{p}(\cdot)}$ and any $\lambda > 0$. Since $q^+ < \bar{p}$, then $\lim_{\|u\|_{1, \bar{p}(\cdot)} \rightarrow +\infty} (\Psi(u) + \lambda\phi(u)) = +\infty$ and (a) is verified.

In the following, we will verify the conditions (b) and (c) in theorem 1.1.

We define the function $F : \Omega \times [0, +\infty[\rightarrow \mathbb{R}$ by

$$F(x, t) = \frac{1}{q(x)} t^{q(x)} - \frac{1}{s(x)} t^{s(x)}, \quad \forall t \in [0, +\infty[, \quad \forall x \in \Omega,$$

It follows that F is increasing for $t \in [1, +\infty[$ and decreasing for $t \in [0, 1]$. Obviously

$$F(x, 0) = 0 \quad \text{and} \quad \lim_{t \rightarrow +\infty} F(x, t) = +\infty.$$

Then there exists a real number $\delta > t_0$ such that

$$F(x, t) \geq 0 = F(x, 0) \geq F(x, \tau), \quad \forall x \in \Omega, \quad \forall t > \delta, \quad \text{and} \quad \tau \in [0, 1]. \quad (4.8)$$

Let c, m be two real numbers such that $0 < c < \min\{1, C_0\}$ with C_0 given in remark 2.2 and $m > \delta$ satisfies

$$m^{\frac{p}{p-1}} \|\alpha\|_{L^1(\Omega)} > 1, \quad (4.9)$$

and

$$m^{\bar{p}} \|\alpha\|_{L^1(\Omega)} > 1. \quad (4.10)$$

By (4.8) we obtain

$$\int_{\Omega} \sup_{0 \leq t \leq c} F(x, t) dx \leq 0 < \frac{1}{m^{\frac{p}{p-1}}} \left(\frac{c}{C_0} \right)^{\bar{p}} \int_{\Omega} F(x, m) dx.$$

Consider $u_0, u_1 \in W^{1, \bar{p}(\cdot)}(\Omega)$, with $u_0(x) = 0$ and $u_1(x) = m$ for any $x \in \Omega$. We define $r = \frac{\min\{1, \alpha^-\}}{(N+1)^{\frac{p}{p-1}} \bar{p}} \left(\frac{c}{C_0} \right)^{\bar{p}}$.

Clearly, $r \in]0, 1[$. Thus, by (4.5) we deduce that

$$0 = \Psi(u_0) < r < \Psi(u_1),$$

and (b) in theorem 1.1 is verified.

Moreover, we have

$$\begin{aligned}\phi(u_1) &= - \int_{\Omega} \frac{1}{q(x)} |m|^{q(x)} dx + \int_{\Omega} \frac{1}{s(x)} |m|^{s(x)} dx \\ &= - \int_{\Omega} F(x, m) dx \\ &= -F(x, m)|\Omega|,\end{aligned} \quad (4.11)$$

and

$$\begin{aligned}
 \frac{(\Psi(u_1) - r)\phi(u_0) + (r - \Psi(u_0))\phi(u_1)}{\Psi(u_1) - \Psi(u_0)} &= r \frac{\phi(u_1)}{\Psi(u_1)} \\
 &= -r \frac{F(x, m)|\Omega|}{\int_{\Omega} \frac{\alpha(x)}{p_0(x)} m^{p_0(x)} dx} \\
 &< 0.
 \end{aligned} \tag{4.12}$$

Next, we consider the case $u \in W^{1, \bar{p}(\cdot)}(\Omega)$ such that $\|u\|_{1, \bar{p}(\cdot)} \leq 1$ with $\Psi(u) \leq r < 1$. Since

$$\begin{aligned}
 \frac{\min\{1, \alpha^-\}}{(N+1)^{\frac{p-1}{p}} \bar{p}} \|u\|_{1, \bar{p}(\cdot)}^{\bar{p}} &\leq \frac{1}{\bar{p}} \left(\sum_{i=1}^N \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)} dx + \int_{\Omega} |u|^{p_0(x)} dx \right) \\
 &\leq \Psi(u) \\
 &\leq r \\
 &< 1.
 \end{aligned}$$

Thus, using remark 2.2, we have

$$\|u\|_{L^\infty(\Omega)} \leq C_0 \left(\bar{p} r \right)^{\frac{1}{\bar{p}}} \leq c.$$

The above inequality shows that

$$\begin{aligned}
 - \inf_{u \in \Psi^{-1}([-\infty, r])} \phi(u) &= \sup_{u \in \Psi^{-1}([-\infty, r])} -\phi(u) \\
 &\leq \int_{\Omega} \sup_{0 \leq t \leq c} F(x, t) dx \\
 &\leq 0.
 \end{aligned}$$

It follows that

$$- \inf_{u \in \Psi^{-1}([-\infty, r])} \phi(u) < r \frac{\int_{\Omega} F(x, b) dx}{\int_{\Omega} \frac{1}{p_0(x)} \alpha(x) m^{p_0(x)} dx}.$$

That is

$$\inf_{u \in \Psi^{-1}([-\infty, r])} \phi(u) > \frac{(\Psi(u_1) - r)\phi(u_0) + (r - \Psi(u_0))\phi(u_1)}{\Psi(u_1) - \Psi(u_0)}.$$

which means that condition (c) in theorem 1.1 is verified.

So, all the assumptions of theorem 1.1 are satisfied and the conclusion follows.

5 Conclusion and perspective

Through this paper, we have studied the existence of three weak solutions of a nonlinear elliptic partial differential equation of Neumann type in the anisotropic variable exponent Sobolev spaces, and without using the log-Hölder continuity.

So we are aware of a lot of open questions about this works for example the question of uniqueness, with totally different conditions, is very important and remains as an open question, therefore our future works will be devoted to this question.

On the other hand, we will try to show the existence of three weak solutions for the problem (1.4) in Orlicz-Sobolev space and Musielak-Orlicz-Sobolev space.

Finally, note that in the literature, very few parabolic problems have been treated. It seems that this difficulty is related to the understanding and the definition of adapted functional spaces. Therefore, interesting questions open up research tracks in this area.

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